PHY 202 – Quantum Mechanics

TUTORIAL SHEET 1 – ANSWERS

PLANE WAVES

1. a) Plane wave moving in +ve x direction

$$\Psi(x,t) = \Psi_0 \exp\left(-\tfrac{i}{\hbar}Et\right)\exp\left(\tfrac{i}{\hbar}px\right)$$

$x \rightarrow -x$

$$\Psi(x,t) = \Psi_0 \exp\left(-\tfrac{i}{\hbar}Et\right)\exp\left(-\tfrac{i}{\hbar}x\right)$$

This is a trivial excercise but its purpose is to teach the students manipulate expressions.

Points to stress:

- the time dependence is the same : $e^{-\frac{i}{\hbar}Et}$
- for a plane wave moving in the $\pm$ve x dir : $e^{\pm\frac{i}{\hbar}px}$

b) $$\Psi(x,t) = \Psi_0\, e^{\frac{i}{\hbar}Et}\, e^{\frac{i}{\hbar}px}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = i\hbar\left(\frac{iE}{\hbar}\right)\Psi = -E\Psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = -\frac{\hbar^2}{2m}\left(\frac{ip}{\hbar}\right)^2\Psi = \frac{p^2}{2m}\Psi$$

$\Rightarrow$ TDSE for a free particle $i\hbar\dfrac{\partial\Psi(x,t)}{\partial t} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi(x,t)}{\partial x^2}$

gives $$E = -\frac{p^2}{2m} = KE < 0 \text{ (for } V=0)$$

which gives a wrong sign for KE
$\Rightarrow$ unphysical solution

Note: One can mention here that in Dirac theory this leads to antiparticles

INFINITE POTENTIA WELLS

2. $V(x) = \begin{cases} \infty & x<0 \\ 0 & 0 \le x \le L \\ \infty & x>L \end{cases}$

$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$, $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$= \sqrt{\frac{2}{L}} \sin(k_n x)$ $\quad n = 1, 2, 3, \ldots$

$= \sqrt{\frac{2}{L}} \sin\left(\frac{p_n}{\hbar}x\right)$ $\quad k_n = \frac{n\pi}{L}$

$p_n = \hbar k_n$

$$\psi_n(x) = \sqrt{\frac{2}{L}} \frac{e^{\frac{i}{\hbar}p_n x} - e^{-\frac{i}{\hbar}p_n x}}{2i}$$

$$= \underbrace{\frac{-i}{\sqrt{2L}} e^{\frac{i}{\hbar}p_n x}}_{\text{plane wave moving } \rightarrow} + \underbrace{\frac{i}{\sqrt{2L}} e^{-\frac{i}{\hbar}p_n x}}_{\text{plane wave moving } \leftarrow}$$

$\Leftarrow$ STANDING WAVE as a sum of two plane waves moving in the opposite directions

Point to stress:

- the waves have to have the same properties (amplitude, wavelength, speed) but opposite direction
- the amplitudes have the same magnitude: $\left|\frac{i}{\sqrt{2L}}\right| = \frac{1}{\sqrt{2L}}$

---

3. a) $-\frac{\hbar^2}{2m}\psi''(x) + V_0\psi(x) = E\psi(x)$ $\quad V(x) = \begin{cases} \infty & x<0 \\ V_0 & 0 \le x \le L \\ \infty & x>L \end{cases}$

$\Rightarrow \psi''(x) = -\frac{2m}{\hbar^2}(E - V_0)\psi$ or $\boxed{\psi'' = -\tilde{k}^2\psi}$ $\quad \boxed{\tilde{k}^2 = \frac{2m}{\hbar^2}(E - V_0)}$

This eq. has exactly the same form as the S. eq. solved in class except $k \to \tilde{k}$ (or $E \to E - V_0$). Therefore the form of the wavefn's is the same: $\boxed{\psi(x) = \sqrt{\frac{2}{L}} \sin \tilde{k}x}$ but the corresponding energies are shifted $\boxed{E_n = V_0 + \frac{n^2\pi^2\hbar^2}{2mL^2}}$ $n = 1, 2, 3, \ldots$

b) $-\frac{\hbar^2}{2m}\psi''(x) + (V(x) + V_0)\psi(x) = E\psi(x)$

is the same as TISE for $V(x)$ but with $E \to E - V_0$

$\Rightarrow$ the form of the wavefn's remains the same but the energy is shifted up by V_0.

4. $$V(x)=\begin{cases}\infty & x<-L/2\\ 0 & -L/2\leq x\leq L/2\\ \infty & x>L/2\end{cases} \qquad \psi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

$$x\to x+L/2$$

$$\sin\left(\frac{n\pi}{L}(x+L/2)\right)=\sin\left(\frac{n\pi}{L}x+\frac{n\pi}{2}\right)=$$

$$=\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{2}\right)+\cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{2}\right)$$

$$=\begin{cases}\sin\left(\frac{n\pi}{L}x\right) & n\text{ - even}\\ \cos\left(\frac{n\pi}{L}x\right) & n\text{ - odd}\end{cases}$$

Points to stress:

- these functions are the solutions of the symmetric well because we've started with the solutions ψ_n. All we have done was to move the co-ordinate system by $L/2$.
  $\Rightarrow$ No need to plug them into the TISE to verify that they indeed are the solutions.
- sin and cos solutions correspond to antisymmetric and symmetric standing waves (Please sketch them.)