Quantum Mechanics - PHY202
Tutorial Questions 3: Answers

1. Time Dependence in Quantum Mechanics

A particle of mass m is confined to an infinite potential well, $V(x) = 0$ for $-L/2 \le x \le L/2$ and $V(x) = \infty$ otherwise. The wavefuncton at time $t = 0$ is $\Psi(x,0) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x))$, where $\psi_1(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right)$ with energy $E_1 = \hbar^2\pi^2/2mL^2$, and $\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$ with energy $E_2 = 4E_1$.

1.1 Write down the equation for the wavefunction $\Psi(x,t)$ at all subsequent times.

$$\Psi(x,t) = \sqrt{\frac{1}{L}}\cos\frac{\pi x}{L}\exp\left(-\frac{iE_1 t}{\hbar}\right) + \sqrt{\frac{1}{L}}\sin\frac{2\pi x}{L}\exp\left(-\frac{iE_2 t}{\hbar}\right)$$

1.2 Does $\Psi(x,t)$ have a definite parity?

No, since it is a superposition of two states with opposite parities.

1.3 Derive an expression for the probability density $P(x,t)$.

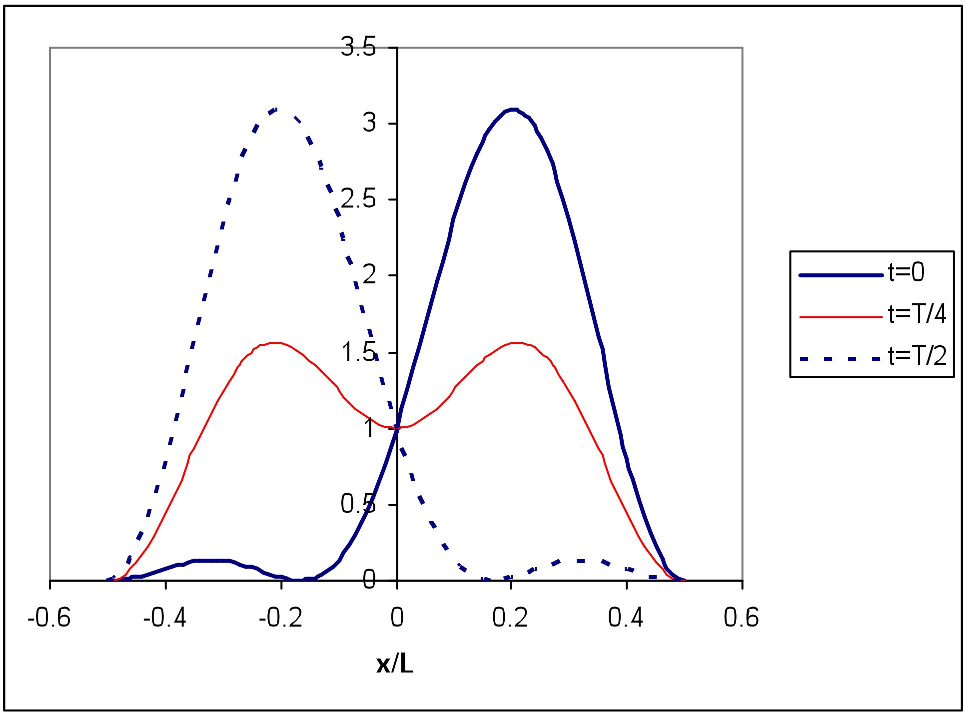
$$P(x,t) = \Psi^*\Psi = \frac{1}{L}\left(\cos^2\frac{\pi x}{L} + \sin^2\frac{2\pi x}{L} + \cos\frac{\pi x}{L}\sin\frac{2\pi x}{L}\left(\exp\frac{i(E_2 - E_1)t}{\hbar} + \exp\frac{-i(E_2 - E_1)t}{\hbar}\right)\right)$$

$$= \frac{1}{L}\left(\cos^2\frac{\pi x}{L} + \sin^2\frac{2\pi x}{L} + 2\cos\frac{\pi x}{L}\sin\frac{2\pi x}{L}\cos\frac{(E_2 - E_1)t}{\hbar}\right)$$

1.4 What is the period of oscillation, T?

$h/(E_1 - E_2)$ (note: not $\hbar$!)

1.5 Sketch $P(x,t)$ at time $t = 0$, $t = T/4$ and $t = T/2$.



(Note: ignore y axis, since plot is not normalised (no 1/L term).)

2. Classical Currents

2.1 An observer stands beside a road and sees one car, one bicycle and one pedestrian pass him every second. If they are travelling at 16 m s^{-1}, 7 m s^{-1} and 1.5 m s^{-1} respectively, what is the distance between two travellers of the same type?
16, 7 and 1.5 m, respectively.

2.2 What is the rate at which lorries would pass the observer if there is 4 m between the fronts of the lorries and they are travelling at 2 m s^{-1}?
1 every 2 seconds (presumably this is a multilane motorway chock full of lorries on several lanes!)

3. Quantum Mechanical Currents

A beam of particles of mass m and energy E is incident from the left on a step potential given by $V = 0$ for $x < 0$ and $V = V_0$ for $x > 0$, where $E > V_0$. The general solutions of the Schrödinger equation valid on the two sides of the step are of the forms

$$\Psi_L = A\exp(ikx - iEt/\hbar) + R\ \exp(-ikx - iEt/\hbar) \text{ and}$$

$$\Psi_R = T\exp(ik'x - iEt/\hbar), \text{ respectively.}$$

3.1 Show that $k = \sqrt{\dfrac{2mE}{\hbar^2}}$ and $k' = \sqrt{\dfrac{2m(E - V_0)}{\hbar^2}}$.

Schrödinger equation is $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + V\psi = E\psi$. Rewrite this as $\dfrac{d^2\psi}{dx^2} = -\dfrac{2m}{\hbar^2}(E - V)\psi$, which has solution $\psi = A\exp(\pm ikx)$ where $k^2 = \dfrac{2m}{\hbar^2}(E - V)$. Substitute in the relevant values of *V* to get the required values of *k* and *k'*.

3.2 Show that the probability of a particle being reflected is given by

$$P_R = \left|\frac{R}{A}\right|^2 = \left|\frac{k - k'}{k + k'}\right|^2.$$

Boundary conditions are A + R = T (equality of ψ) and ik(A – R) = ik'T (equality of dψ/dx).
Therefore, eliminating T, k'(A + R) = k(A – R), so (k – k')A = (k + k')R, and R/A = (k – k')/ (k + k').
Flux is $\hbar k|A|^2/m$ or $\hbar k|R|^2/m$, so the $\hbar k/m$ terms cancel and the flux ratio is just $|R|^2/|A|^2$.

3.3 Are the particles travelling faster or slower on the RHS than on the LHS?
Slower on the right, since KE = $p^2/2m$ is less on the right.

3.4 Show that the probability of transmission is given by

$$P_T = \frac{k'}{k}\left|\frac{T}{A}\right|^2 = \frac{4kk'}{(k + k')^2}.$$

Eliminate R from the boundary conditions: 2kA = (k + k')T, hence $|T|^2/|A|^2 = 4k^2/(k + k')^2$ (note that k and k' are real by construction).
Incident flux is $\hbar k|A|^2/m$; transmitted flux is $\hbar k'|T|^2/m$, so flux ratio is $k'|T|^2/k|A|^2$. Substitute in above expression for $|T|^2/|A|^2$ and all will be well.

3.5 Show that the sum of the probabilities for transmission and reflection add up to unity.
$(k - k')^2 + 4kk' = k^2 + 2kk' + k'^2 = (k + k')^2$, so $P_R + P_T = 1$.