Cone Based Hypervolume Indicator

Construction, Properties, and Efficient Computation

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Brief Summary

What are (convex, pointed) cones?

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Definition (cone)

A subset $\mathcal{C} \subseteq \mathbb{R}^m$ is called a cone, iff $\alpha \mathbf{p} \in \mathcal{C}$ for all $\mathbf{p} \in \mathcal{C}$ and for all $\alpha \in \mathbb{R}, \alpha > \mathbf{0}$.

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Definition (convex cone)

A cone $\mathcal{C}$ in $\mathbb{R}^m$ is convex, iff $\alpha \mathbf{p}^{(1)} + (1 - \alpha)\mathbf{p}^{(2)} \in \mathcal{C}$ for all $\mathbf{p}^{(1)} \in \mathcal{C}$ and $\mathbf{p}^{(2)} \in \mathcal{C}$ and for all $0 \leq \alpha \leq 1$.

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Definition (pointed cone)

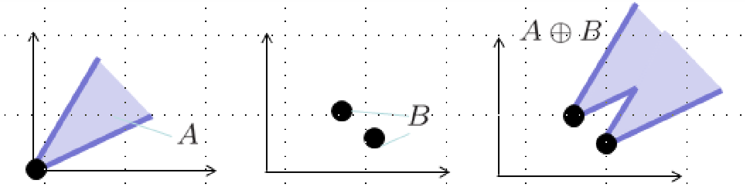
A cone $\mathcal{C}$ in $\mathbb{R}^m$ is pointed, iff $\mathcal{C} \cap -\mathcal{C} \subseteq \{0\}$.

What are cone orders?

Definition (Minkowski sum)

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ denote sets of vectors in $\mathbb{R}^{\boldsymbol{m}}$. Then

$$\boldsymbol{A} \oplus \boldsymbol{B} = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in \boldsymbol{A} \text{ and } \mathbf{b} \in \boldsymbol{B}\}.$$



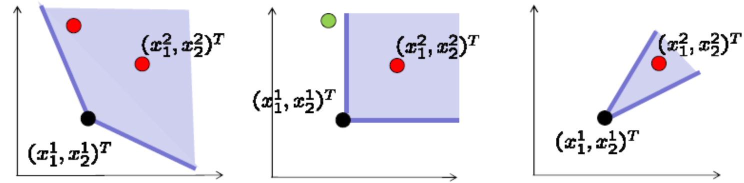
Example for the Minkowski sum.

What are cone orders?

Definition (cone order)

Let C denote a pointed convex cone:

$$\mathbf{x}^1 \preceq_C \mathbf{x}^2 \Leftrightarrow \mathbf{x}^2 \in \{\mathbf{x}\}^1 \oplus C.$$



Pareto order is a special case of a cone order.

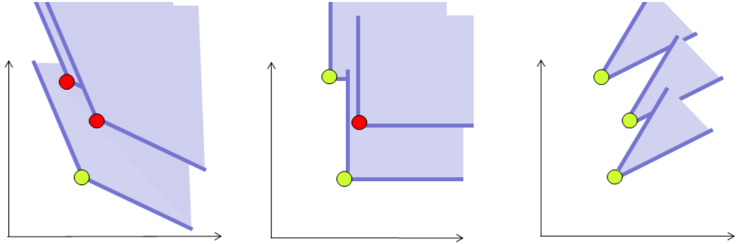
Minimal sets

Lemma (Minima for acute cones)

$\boldsymbol{C} \subset \boldsymbol{C}_{\text{Pareto}} \Rightarrow \forall \boldsymbol{A} \subset \mathbb{R}^{\boldsymbol{m}} : \text{Minima}_{\boldsymbol{Pareto}}(\boldsymbol{A}) \supseteq \text{Minima}_{\boldsymbol{C}}(\boldsymbol{A})$

Lemma (Minima for obtuse cones)

$\boldsymbol{C} \supset \boldsymbol{C}_{\text{Pareto}} \Rightarrow \forall \boldsymbol{A} \subset \mathbb{R}^{\boldsymbol{m}} : \text{Minima}_{\boldsymbol{Pareto}}(\boldsymbol{A}) \subseteq \text{Minima}_{\boldsymbol{C}}(\boldsymbol{A})$



Cone Based Hypervolume Indicator

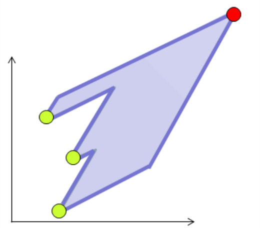
Cone Based Hypervolume Indicator (CHI)

Definition (Cone-based hypervolume (CHI))

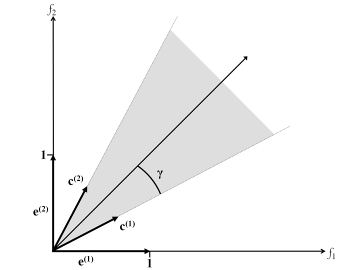
For $\boldsymbol{P} \in \mathbb{R}^m$ and a reference point $\mathbf{r}$ with $\forall \mathbf{p} \in \boldsymbol{P} : \mathbf{p} \preceq_C \mathbf{r}$:

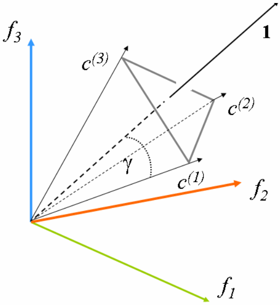
$$\mathrm{CHI}(\boldsymbol{P}) = \mathrm{LebesgueMeasure}(\underbrace{(\boldsymbol{P} \oplus \boldsymbol{C})}_{(1)} \cap \underbrace{(\{\mathbf{r}\} \ominus \boldsymbol{C})}_{(2)}).$$

(1) cone-dominated subspace
(2) anti-cone for $\boldsymbol{r}$.



Definition of γ cones





Definition (γ-cone)

A cone spanned by m base vectors, $\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(m)}$:

1. the angle between the and each of the base vectors $\mathbf{c}^{(i)}$ is γ
2. each base vector $\mathbf{c}^{(i)}$ is a unit vector in the plane spanned by $\mathbf{1}$ and $\mathbf{e}^{(i)}$.

Construction of γ cones

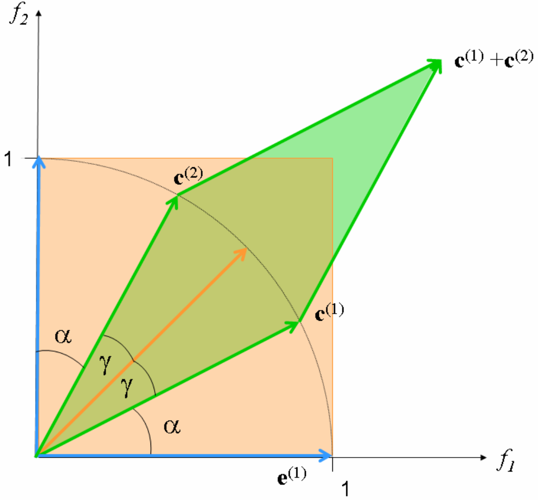
Theorem (Base vectors of pyramidal γ cone)

$$c_j^{(i)} = \begin{cases} (1/\sqrt{m-1})\sin(\alpha) & i \neq j \\ \cos(\alpha) & i = j \end{cases}, \alpha = \underbrace{\arccos(1/\sqrt{m})}_{\theta} - \gamma$$

Proof in coordinate-free geometric algebra: Rotate $\mathbf{e}^{(i)}$ in the plane determined by the normalized bivector $B = (\mathbf{e}^{(i)} \wedge \mathbf{a})/\sin(\theta)$ over an angle α to get $\mathbf{c}^{(i)}$:

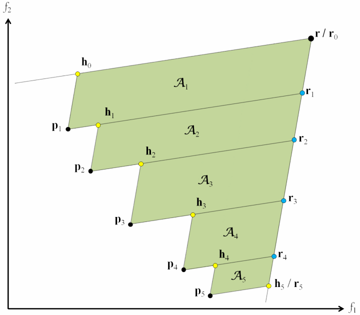
$$\mathbf{c}^{(i)} = \exp\left(-\frac{\alpha}{2}\,\frac{\mathbf{e}^{(i)} \wedge \mathbf{a}}{\sin(\theta)}\right) \mathbf{e}^{(i)} \, \exp\left(\frac{\alpha}{2}\,\frac{\mathbf{e}^{(i)} \wedge \mathbf{a}}{\sin(\theta)}\right).$$

Fundamental Transformation



Efficient Computation of CHI

In two dimensions we can use a simple partitioning scheme:



Computation of CHI in m dimensions

Lemma (Billingsley: Probability and Measure, 1995)

We denote the Lebesgue measure by $\lambda(.)$. Let $F(\mathbf{x}) = \mathbf{T}\mathbf{x} + \mathbf{x_0}$ denote an non-singular affine transformation, then

$$\lambda(\mathbf{F}A) = \det(\mathbf{T})\lambda(A).$$

Algorithm: m-dimensional CHI computation

Input: Cone base $\mathbf{C}$, $P \subset \mathbb{R}^m$, reference point $\mathbf{r}$

1. Let $Q = \{\mathbf{q}^{(1)}, \dots, \mathbf{q}^{(\mu)}\}$, with $\mathbf{q}^{(i)} = \mathbf{C}^{-1}\mathbf{p}^{(i)}$, and $\mathbf{r}' = \mathbf{C}^{-1}\mathbf{r}$.
2. Compute the standard hypervolume HI$(Q, \mathbf{r}')$.
3. Return CHI$(P) = (1/\det \mathbf{C}^{-1}) \cdot$ HI$(Q, \mathbf{r}')$.

Efficient Computation of CHI

Lemma

For any fixed dimension $m > 1$, the computational complexity of CHI in the size of an approximation set $|A|$ is equal to that of HI.

Proof.

- Recall: The computational complexity of HI is in $\Omega(|A| \log |A|)$ (Beume et al. 2009).
- The complexity of the reduction of CHI to HI is in $O(|A|)$.

□

Hence, for $m = 2, 3$ CHI has complexity $\Theta(n \log n)$.

Efficiently computing all hypervolume contributions

Lemma

Computing ***all*** contributions

$$\Delta CHI(a, A) = CHI(A) - CHI(A - \{a_i\})$$

can be reduced in linear time to computing all contributions to the standard hypervolume.

- Asymptotically optimal algorithm [Emmerich and Fonseca, EMO 2011] with complexity $O(|A| \log |A|)$ can be applied for $m = 2, 3$.
- Makes efficient implementation of steady state evolutionary algorithms such as SMS-EMOA, Steady-state IBEA, and MOO-CMA possible.

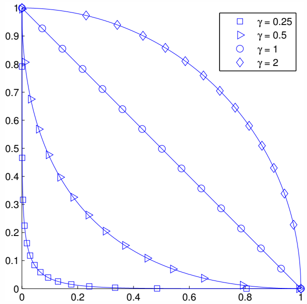
Definition of optimal μ-distribution

Definition (Auger, Bader, Brockhoff and Zitzler, FOGA09)

For a Pareto front $\mathcal{Y}$ the optimal μ-distribution is defined as:

$$P^*_\mu \in \arg\max_{P \subseteq \mathcal{Y}, |P| \leq \mu} \mathrm{HI}(P)$$

optimal μ-distribution of HI for $(|y|^\gamma)^{1/\gamma} \equiv 1$, $\gamma \in \{\frac{1}{4}, \frac{1}{2}, 1, 2\}$:



For the CHI optimal μ-distributions we proved two lemmas:

Lemma

For a compact, connected linear Pareto front in $\mathbb{R}^2$ the optimal μ-distribution is evenly spaced for $\gamma > 0$.

Lemma

For a compact and connected Pareto front in $\mathbb{R}^2$ the optimal μ-distribution is evenly spaced in the Manhattan distance for $\gamma \to 0$.

Both proofs exploit that a point has only a local influence on the CHI. This was also used in similar proofs for the HI in (Auger, Bader, Brockhoff and Zitzler [2009]).

CHI EMOA Results in 2 dimensions

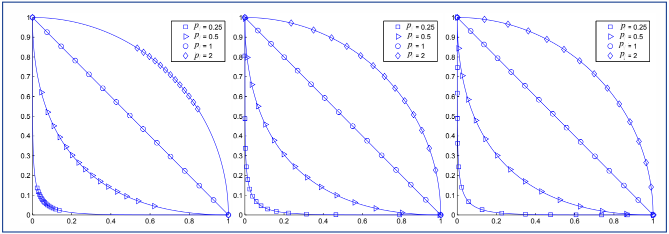


Figure: Left: an obtuse cone, $\gamma = \pi/3$. Center: a Pareto cone, $\gamma = \pi/4$. Right: an acute cone $\gamma = \pi/8$.

CHI-EMOA

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  - CHI contributions replace HI contributions as a secondary selection criterion.

CHI EMOA Results in 2 dimensions

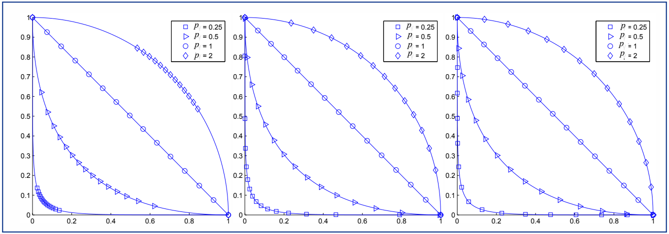
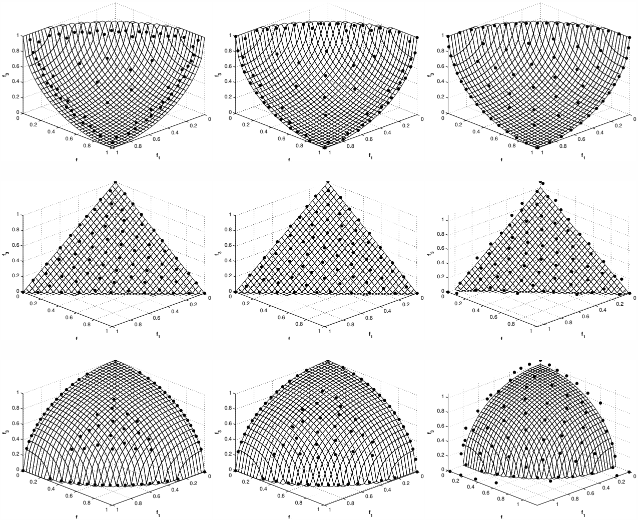


Figure: L: obtuse: $\gamma = \pi/3$. C: $\gamma = \pi/4$. R: acute $\gamma = \pi/8$.

- Generalized Schaffer problems with scalable curvature (Emmerich and Deutz, EMO 2007)
- The number of function evaluations is 50000.
- 10 D test problems.

Results on 3-D superspheres problem



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- γ-cones: Efficient construction and computation of CHI
- CHI allows scaling from knee point focused to evenly spaced via cone-parameter γ
- CHI EMOA: sort by contributions and strictest order; Search gets more difficult for small γ

Outlook

- CHI can also be used for curvature based preference formulation*

*[Pradyumn Kumar Shukla, Michael Emmerich, and André Deutz: A Theoretical Analysis of Curvature Based Preference Models, EMO 2013]

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- Implementations of CHI EMOA and CHI in MATLAB: `rodeolib (sourceforge)` (by J. Kruisselbrink) and jmetal (by Pradyumn Shukla)

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  - Algorithm design aspects: can we do better?

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End of our presentation

Questions?