A Theoretical Analysis of Curvature Based Preference Models

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Multi-objective Optimization Problem

Given $m, n \in \mathbb{N}$ such that $m \geq 2$, a multi-objective optimization problem (MOP) is a 3-tuple $(\mathbf{F}, X, \mathcal{C})$ such that

- $\mathbf{F}(x) := (F_1(x), F_2(x), \ldots, F_m(x))$ is a vector valued objective function
- $X \subseteq \mathbb{R}^n$ a feasible set
- $\mathcal{C}$ is a cone (set) that induces a partial ordering on $\mathbb{R}^m$

Definition

A point $\hat{x} \in X$ *is* $\mathcal{C}$***-optimal*** *if* $(\{\mathbf{F}(\hat{\mathbf{x}})\} - \mathcal{C}) \cap \mathbf{F}(X) = \{\mathbf{F}(\hat{\mathbf{x}})\}$.

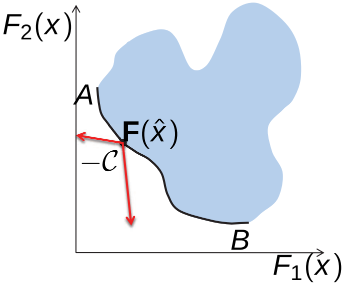


Image of $\mathcal{C}$-optimal points of a bicritetia problem

Preference Models

Let $X_p(\mathbf{F}, X, \mathcal{C})$ be the set of $\mathcal{C}$-optimal optimal points.

Definition

A preferred solution set, denoted by $X_{\mathcal{P}}(\mathbf{F}, X)$, is a proper subset of $X_p(\mathbf{F}, X, \mathbb{R}^m_+)$. The set $X_{\mathcal{P}}(\mathbf{F}, X)$ is said to be induced by a preference model $\mathcal{P}$.

Characteristics of Preference Models

Domination transformation: A (convex) cone $\mathcal{C} \supset \mathbb{R}^m_+$ exists such that

$$X_{\mathcal{P}}(\mathbf{F}, X) = X_p(\mathbf{F}, X, \mathcal{C})$$

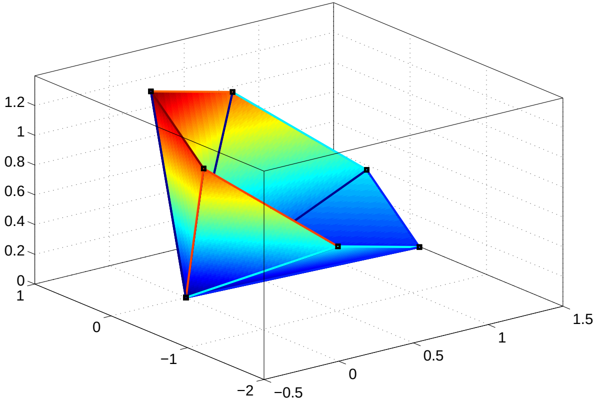
Objective transformation: A function $\mathbf{T} : \mathbb{R}^m \to \mathbb{R}^k$ exists such that

$$X_{\mathcal{P}}(\mathbf{F}, X) = X_p(\mathbf{T} \circ \mathbf{F}, X, \mathbb{R}^m_+)$$

Lemma

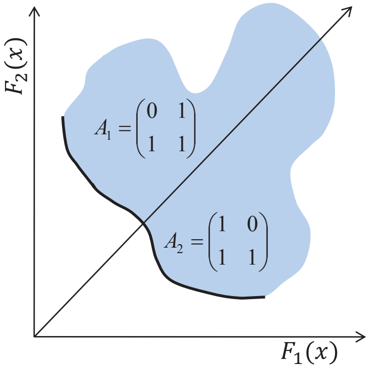
If $\mathbf{T} := \mathcal{A}$, where $\mathcal{A}$ is a m by k matrix, then the above two transformations are equivalent and, $\mathcal{C}$ is the polyhedral cone $\{d \in \mathbb{R}^m | Ad \geq \mathbf{0}\}$. In general, these do not imply each other.

A Polyhedral Model



A polyhedral domination cone. The k can be much larger than m. Some real world applications use $k = m(m-1)$.

A Piecewise Polyhedral Model



Piecewise polyhedral transformations of the objectives in the case of equitable efficiency. There are $m!$ polyhedral transformation for an m-objective problem.

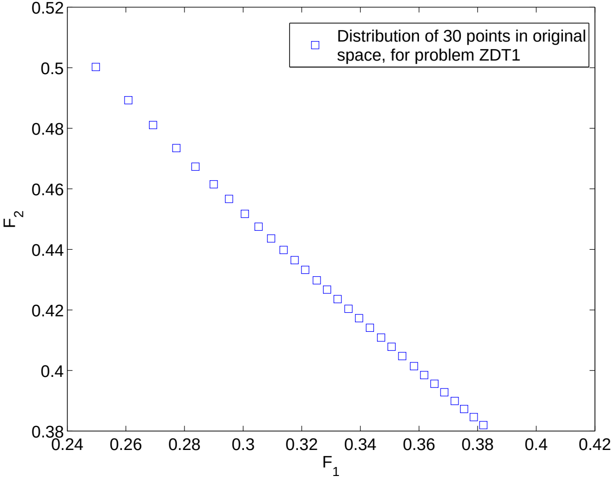
An Algorithm

Algorithm: General cone-based hypervolume computation

Input: An m by k matrix $\mathcal{A}$, points $S \subset \mathbb{R}^m$, and a reference point $\mathbf{r}$

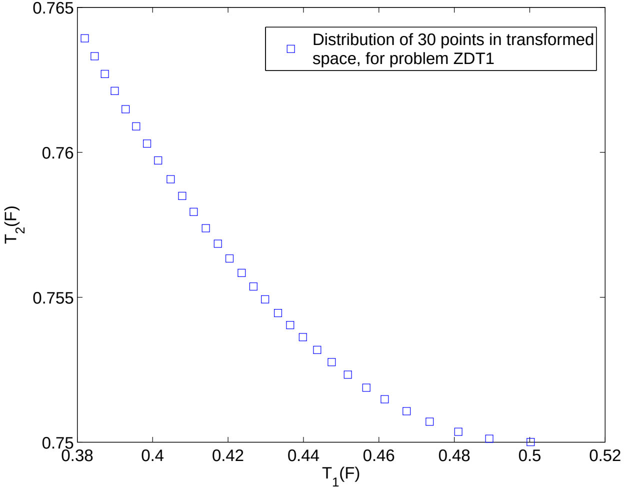
1. Let $\mathbf{r}' = \mathcal{A}\mathbf{r}$.
2. For all $i = 1, \ldots, |S|$, let $Q = \{\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{|S|}\}$, with $\mathbf{q}^{(i)} = \mathcal{A}\mathbf{s}^{(i)}$.
3. Compute the standard hypervolume $\mathrm{HI}(Q, \mathbf{r}')$.
4. Return $\mathrm{CHI}(S) = (1/\det(\mathcal{A}^\top \mathcal{A})) \cdot \mathrm{HI}(Q, \mathbf{r}')$.

An Application in Equitable Efficiency



Distribution of points in the original ZDT1 objective space

An Application in Equitable Efficiency



Distribution of points in the transformed ZDT1 objective space

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- Presented new theoretical results

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- Many other details in the paper