Mathematics for Physics Students

Introduction

Ever since Newton, maths has been the language in which we do physics. This is why we insist on A level maths for potential physics students. However, we are only too aware that over the past year your schooling has been badly disrupted by the pandemic, and so you may not have had the chance to develop your mathematical skills as much as you would normally have done. In this resource pack, we are offering you the chance to get some more practice in two aspects of maths that are particularly important for your first year physics course: vectors and calculus.

This is not compulsory: we're not going to set you a test on it in week 1 or anything like that. But our experience is that one of the most difficult things to adapt to in first year physics is that maths and physics, which until now you have studied as separate subjects, need to be brought together to tackle university-level physics problems. We hope that the material we cover here will help you to make this transition more smoothly.

Topic 1: vectors

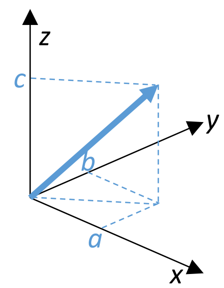
What is a vector?

A vector is a mathematical entity that has both **magnitude** and **direction**. Many important variables in physics are vectors. For example, **momentum** is a vector (it has magnitude and direction), whereas **energy** is a scalar (it has only magnitude). In diagrams, we usually represent vectors by drawing an arrow representing the magnitude and direction of the vector.

> Question 1: which of the following physical quantities are vectors?
> (a) position; (b) temperature; (c) heat; (d) magnetic field; (e) force; (f) acceleration.

Vectors in component form

Because a vector has both magnitude and direction, it requires more than one number to describe a vector. Most commonly, we express a vector using Cartesian coordinates. We might write the vector on the right in a number of different forms:



- $a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + c\hat{\boldsymbol{k}}$
- $a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}} + c\hat{\boldsymbol{z}}$
- (a, b, c)
- $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

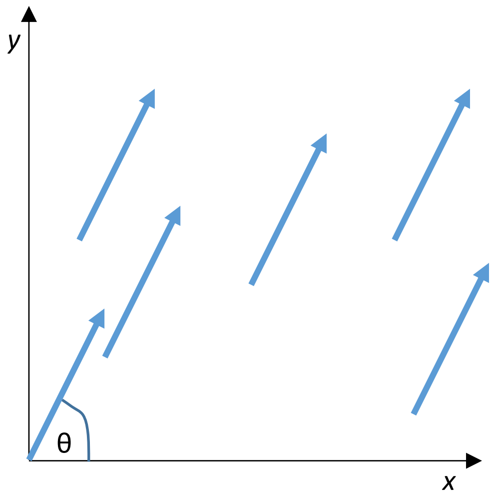
All of these are equivalent, and different books use different conventions. In the first two, $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ represent **unit vectors** (that is, vectors of magnitude 1) oriented along the *x*, *y* and *z* axes respectively. Most books use the circumflex or "hat", ^, to indicate a unit vector.

In equations, vectors are usually written in **bold** or ***bold italic***. In handwriting, one can either put a squiggly line under the symbol, $\underset{\sim}{\mathrm{v}}$ (in the days before computer typesetting, this was an instruction to a printer to set the letter in bold), or an arrow above it, $\vec{\mathrm{v}}$. It is **very important** to distinguish vectors from scalars, because the vector $\boldsymbol{p}$ is different from its magnitude, p:

$$\boldsymbol{p} = \vec{p} = (p_x, p_y, p_z); \quad p = |\boldsymbol{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}.$$

Question 2: write each of the following in the form (x, y, z), and calculate their magnitudes.
(a) $\boldsymbol{v_1} = -2\hat{\boldsymbol{j}}$; (b) $\boldsymbol{v_2} = 3\hat{\boldsymbol{z}}$; (c) $\boldsymbol{v_3} = \hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{k}}$; (d) $\boldsymbol{v_4} = 6\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}$; (e) $\boldsymbol{v_5} = 0.83\hat{\boldsymbol{\imath}} + 0.89\hat{\boldsymbol{j}} + 0.45\hat{\boldsymbol{k}}$.

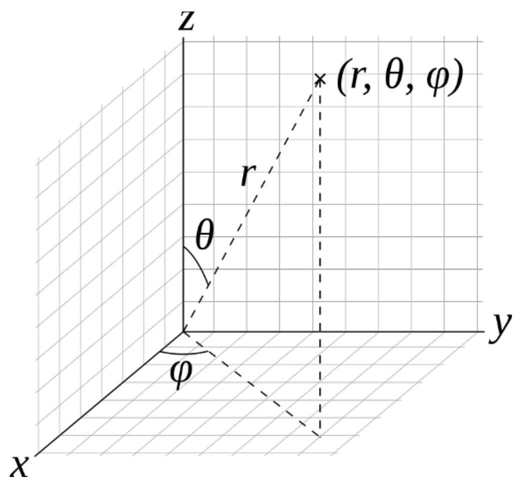
Note that what defines a vector is its magnitude and its direction, not its starting point. We can move a vector around in space without changing it. All of the blue arrows in the diagram on the right represent the same two-dimensional vector.



We can also describe vectors in different coordinate systems. The rule is that an *n*-dimensional vector always needs *n* numbers to describe it, but you have a choice of what those *n* numbers are. We could describe the vector in this diagram by its *x* and *y* components as (1,2), but we could also describe it by its length $r = \sqrt{5}$ and the angle it makes with the *x*-axis, $\theta = \arctan(2) = 63.4°$. Both descriptions are equivalent.

For three dimensions, we need two angles. In physics, these are defined as shown on the right[1]. You should be able to see that the relationships between these angles and the coordinates (*x,y,z*) are

$$x = r\sin\theta\cos\varphi\,;$$
$$y = r\sin\theta\sin\varphi\,;$$
$$z = r\cos\theta\,.$$



These **spherical polar coordinates** can be used to describe vectors or anything else that needs a three-dimensional label. They are particularly useful in dealing with systems with spherical symmetry.

Question 3: calculate *r*, *θ* and *φ* for the vectors in question 2.

Addition and subtraction of vectors

To add or subtract vectors, we simply add or subtract their components:

$$(a_x, a_y, a_z) + (b_x, b_y, b_z) = (a_x + b_x, a_y + b_y, a_z + b_z).$$

This is obvious if we write the vectors in the form $a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{j}} + a_z\hat{\boldsymbol{k}}$: this is already a sum, and when we add on $b_x\hat{\boldsymbol{\imath}} + b_y\hat{\boldsymbol{j}} + b_z\hat{\boldsymbol{k}}$ all we have to do is rearrange the terms of the sum to get $(a_x + b_x)\hat{\boldsymbol{\imath}} + (a_y + b_y)\hat{\boldsymbol{j}} + (a_z + b_z)\hat{\boldsymbol{k}}$.

The usual rules of addition apply:

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a};$$
$$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}.$$

We can think of subtraction as a special case of addition:

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b}),$$

where the vector –***b*** has the same magnitude as ***b***, but opposite direction (in component form, multiply all components by −1).

Question 4: For each of the following pairs of vectors ***a, b***, calculate ***a*** + ***b*** and ***a*** – ***b***.
(a) ***a*** = (5,4,9), ***b*** = (5,2,3)
(b) ***a*** = (−3,7,6), ***b*** = (6,−2,−6)

[1] In maths textbooks, you will sometimes find that the angle labelled θ here is labelled φ and vice versa. This is more logical, because it means that the angle called θ is the same angle that is called θ when working in two dimensions. But in physics the system shown in the diagram is always used, as specified in ISO standard 80000-2: 2019.

(c) $\boldsymbol{a}$ = (−9,3,0), $\boldsymbol{b}$ = (−3,−7,8)
(d) $\boldsymbol{a}$ = (0.209,−0.638,−0.053), $\boldsymbol{b}$ = (0.284,0.912,0.281)

Multiplication by scalars

Any vector can be multiplied by a scalar. This multiplies the length of the vector by the scalar, but does not change its direction. In component form, we multiply each component by the scalar: for example, a velocity vector (v_x, v_y, v_z) can be converted into a momentum vector (assuming $v \ll c$) by multiplying by mass, which is a scalar: $\boldsymbol{p} = m\boldsymbol{v} = (mv_x, mv_y, mv_z)$.

Question 5: A spacecraft of mass 450 kg travelling at velocity (−63,89,−16) m/s collides with another craft of mass 800 kg travelling at velocity (−94,24,57) m/s. After the collision the wreckage all sticks together. What is its final momentum? What is its final velocity?

Incidentally, there are also ways of multiplying vectors by vectors. You have not met these in your A-level maths course (unless you did Further Maths), but you will meet them in your first year at university, because they have important applications in physics.

Differentiation and integration of vectors with respect to scalars

We can generalise the idea that vectors can be multiplied or divided by scalars to differentiation and integration with respect to scalars. If we consider a vector in its component form, $\boldsymbol{v} = (v_x, v_y, v_z)$, its derivative with respect to a scalar t is simply

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \left(\frac{\mathrm{d}v_x}{\mathrm{d}t}, \frac{\mathrm{d}v_y}{\mathrm{d}t}, \frac{\mathrm{d}v_z}{\mathrm{d}t}\right).$$

It follows from the Fundamental Theorem of Calculus that integration of a vector with respect to a scalar works in the same way.

Example: An object is in a circular orbit around the Earth. Its velocity is $\boldsymbol{V} = (V_0 \cos \omega t, V_0 \sin \omega t, 0)$. What are (a) its position, (b) its acceleration, and (c) the magnitude of its acceleration, as a function of t? At time $t = 0$, the position of the object is $\boldsymbol{r}_0 = (0, -V_0/\omega, 0)$.

To answer this question we use the basic facts that velocity is the rate of change of position, $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, and acceleration is the rate of change of velocity, $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$. Therefore, for our satellite,

$$\boldsymbol{r} = \left(\int V_0 \cos \omega t \, \mathrm{d}t, \int V_0 \sin \omega t \, \mathrm{d}t, \int 0 \, \mathrm{d}t\right)$$
$$= \left(\frac{V_0}{\omega} \sin \omega t + C_x, -\frac{V_0}{\omega} \cos \omega t + C_y, C_z\right).$$

To evaluate the constants of integration we note that at $t = 0$

$$\boldsymbol{r}_0 = \left(0, -\frac{V_0}{\omega}, 0\right) = \left(C_x, -\frac{V_0}{\omega} + C_y, C_z\right)$$

and thus all the constants must be zero. (This is an example of using a boundary condition to evaluate an integration constant—see page 10.)

This gives us

$$\boldsymbol{r} = \frac{V_0}{\omega}(\sin \omega t, -\cos \omega t, 0).$$

Note that the magnitude of $\boldsymbol{r}$ is $\frac{V_0}{\omega}(\sin^2 \omega t + \cos^2 \omega t)^{1/2} = \frac{V_0}{\omega}$, which is constant, as we would expect for circular motion.

For the acceleration

$$\boldsymbol{a} = \left(\frac{\mathrm{d}}{\mathrm{d}t}(V_0 \cos \omega t), \frac{\mathrm{d}}{\mathrm{d}t}(V_0 \sin \omega t), 0\right)$$
$$= (-\omega V_0 \sin \omega t\,, \omega V_0 \cos \omega t\,, 0)$$
$$= \omega V_0(-\sin \omega t\,, \cos \omega t\,, 0).$$

Note that $\boldsymbol{a} = -\omega^2 \boldsymbol{r}$. The magnitude of the acceleration is

$$a = \sqrt{\omega^2 V_0^2(\sin^2 \omega t + \cos^2 \omega t)} = \omega V_0.$$

Again, this is independent of time. If we compare it with r, we see that $a = V_0^2/r$. You may recognise this as the standard expression for acceleration in uniform circular motion.

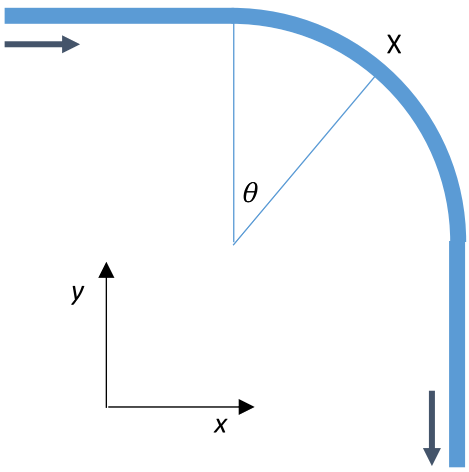
This is a classic example of how you will apply calculus to physics problems in your university studies. By using differentiation and integration to relate position, velocity and acceleration, we can apply the laws of motion to situations where the acceleration is not constant, thereby greatly increasing our ability to analyse a range of different physical systems.

Problems for vectors

Here are some practice problems you might like to try. We'll distribute the solutions in Intro Week.

1. Sheffield has latitude 53.4°N and longitude 1.5°W. Define a coordinate system such that the *z* axis points from the centre of the Earth to the North Pole, the *x* axis points from the centre of the Earth to the equator at the Greenwich meridian (i.e. longitude 0°), and the *y* axis points from the centre of the Earth to the equator at longitude 90°E. Using this coordinate system, and assuming that the Earth is a sphere of radius 6370 km, calculate the position vector of Sheffield in the form (x, y, z). [Hint: remember that latitude is measured north and south from the equator, whereas the θ coordinate in spherical polar coordinates is measured from the positive *z* axis, so in this case southwards from the North Pole.]

2. A car travelling at speed *V* enters the right-angled bend shown on the right. The bend is a 90° arc of a circle of radius *R*. Expressed in component form, what is



    a. the velocity of the car before entering the bend;

    b. the velocity of the car after exiting the bend;

    c. the velocity of the car at point X, when it has turned through angle θ as shown;

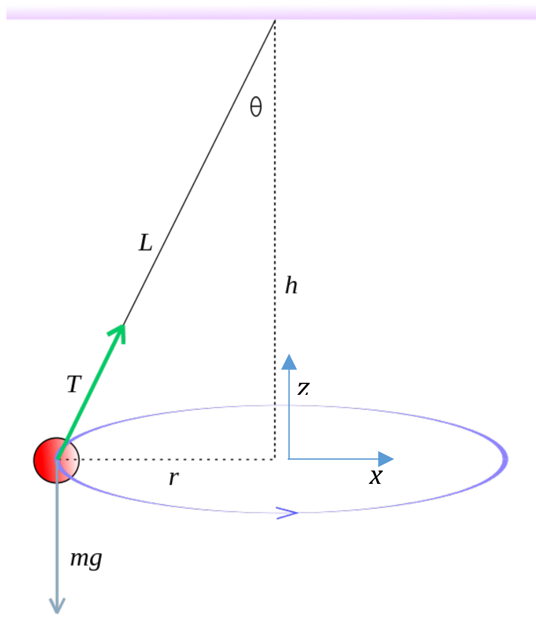
    d. the acceleration of the car at point X?

   Assume that the car maintains a constant speed *V* throughout, and that the road is completely level (i.e. the *z* components of velocity and acceleration are zero throughout).

    e. By integrating the acceleration from $\theta = 0°$ to $\theta = 90°$, calculate the change in the car's velocity as a result of negotiating the bend. Check that this is consistent with your answers to parts a and b.

3. A spacecraft heading to Mars has mass 650 kg and velocity $(-20.8, -12.0, 0.7)$ km/s in some appropriate coordinate system. In a mid-course correction, the spacecraft fires its rocket engine, delivering an impulse $(130, -455, -520)$ kN·s. What is the velocity vector of the spacecraft after the mid-course correction?

4. A conical pendulum is a pendulum whose bob describes a circle, as shown on the right. The *z* and *x* axes are defined as shown, and the *y* axis is directed into the page (so the bob is moving in the direction of increasing φ). The bob moves at a constant speed *V*.



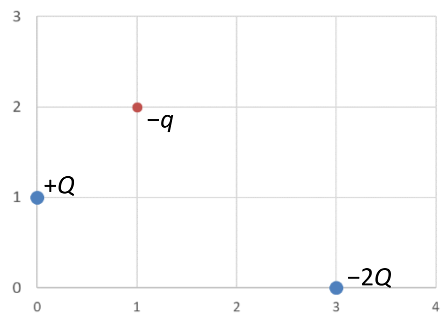
   a. At time *t* = 0 the position of the bob is $(-r, 0, 0)$ as shown in the diagram. In terms of *r* and φ, what is the position of the bob at time t, when it has rotated through an angle $\varphi = \omega t$ (where $\omega = \mathrm{d}\varphi/\mathrm{d}t$, and is constant)?
   b. What is the velocity of the bob at time t, in terms of *r* and φ?
   c. What is the acceleration of the bob at time t?
   d. Verify that the magnitude of the acceleration is constant. Also check that your answers to parts a and c are consistent with the relation $\boldsymbol{a} = -\omega^2 \boldsymbol{r}$.
   e. By solving the force diagram shown above, show that for small angles θ the period P of the conical pendulum (defined as the time the bob takes to complete a full circle, so $P = 2\pi r/V$) is given by $P \simeq 2\pi\sqrt{L/g}$, exactly the same as an ordinary pendulum of the same length. (This isn't a vector problem!)

5. Coulomb's law can be written in vector form as

$$\boldsymbol{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}},$$

where, if we are calculating the force on Q_2, $\boldsymbol{r}$ is the vector running from Q_1 to Q_2 and $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$ is the unit vector in the direction of $\boldsymbol{r}$. (If we are calculating the force on Q_1, $\boldsymbol{r}$ runs in the opposite direction, from Q_2 to Q_1.)

   a. We fix two charges +*Q* and −2*Q* to a flat table at positions (0,1) and (3,0) as shown. We then place another charge −*q* on the table at position (1,2). The charge −*q* is free to slide around the table. Assuming negligible friction, in what direction will the charge −*q* start to slide? Express your answer as a unit vector.



   b. In what direction will the charge −*q* start to slide if we replace the fixed charge −2*Q* by a charge +2*Q* in the same location?
   c. In case b, where both fixed charges are positive, is there anywhere on the table where we could place charge −*q* and it would not start to slide if released? If so, what are the coordinates of that position?

Topic 2: Calculus

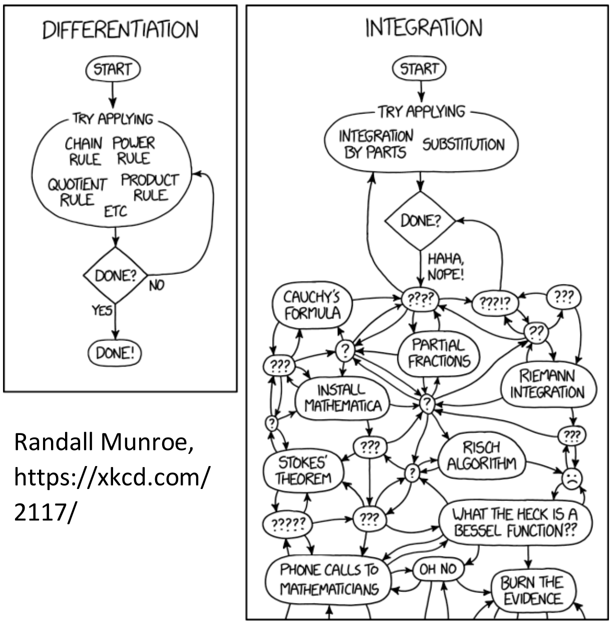
Why is calculus important?

Calculus is one of the most important tools in physics. It was his invention of calculus that allowed Isaac Newton to develop his laws of motion and of gravity, which in many respects were the beginning of modern physics. The main reason for this is that calculus enables us to deal effectively with change.

Much of physics is about change. Velocity is the rate of change of position, and acceleration is the rate of change of velocity. The applied force is equal to the rate of change of momentum. Differentiation is a technique for measuring instantaneous rates of change, so we do not have to restrict ourselves to situations where the rate of change is constant over long periods (e.g. constant acceleration). Conversely, integration allows us to calculate the cumulative effect of change, even when the rate of change has varied over the period of interest.

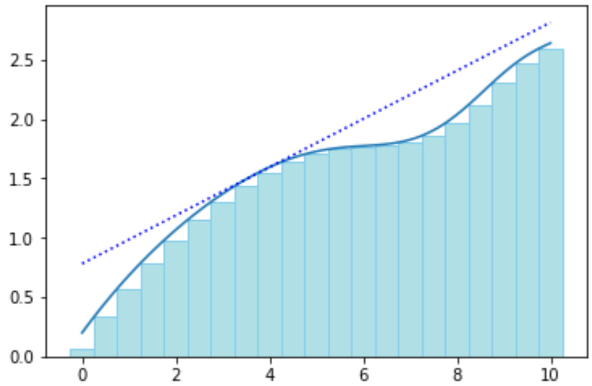
The ideas of differentiation and integration

Integration is generally acknowledged to be a more difficult problem than differentiation (see right), but it is conceptually simpler and was more frequently anticipated by ancient mathematicians. Integration consists, in essence, of adding up lots of small quantities to obtain the value of something that is hard to calculate directly. For example, one might imagine calculating the volume of a cone by slicing it into numerous thin discs. The volume of each disc can be approximated by $\pi r^2 \mathrm{d}h$, where r is the radius of the disc and $\mathrm{d}h$ is its (small) thickness, and we can find the volume of the cone by adding up all the disc volumes. Calculations similar to this were done by the ancient Greek mathematician Archimedes in the 3rd century BCE, but he did not have a general method—he produced a different construction for every area or volume he calculated using this method of splitting into smaller pieces—and, crucially, he always regarded his small pieces as finite. The Greeks did not develop the concept of a limit, so Archimedes' sums never quite became true integrals.



Randall Munroe, https://xkcd.com/2117/

In modern calculus, the simplest visualisation of differentiation and integration is that the integral of a function $f(x)$ is the area under the curve, and the derivative of $f(x)$ is the gradient of the line tangent to the curve. An example of this is shown on the right: the histogram is an approximation to the area under the curve, and will become the area under the curve if we shrink the histogram bin widths towards zero, while the dotted line is the tangent to the curve at the point *x* = 4, and represents the derivative of the function at *x* = 4.



Note that the area $\mathrm{d}A$ of the bar at x is given by $\mathrm{d}A = f(x)\mathrm{d}x$, which can be rearranged to $\mathrm{d}A/\mathrm{d}x = f(x)$. Thus, if $A(x)$ is the area under the curve measured between some reference value (say 0) and x, we have

$$A(x) = \int_0^x f(x)\mathrm{d}x\,; \qquad \frac{\mathrm{d}A}{\mathrm{d}x} = f(x).$$

This relationship between differentiation and integration is known as the **Fundamental Theorem of Calculus.**

Differentiation

The derivative of a function $f(x)$ is **defined by**

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

For finite Δx, this is the mean gradient of the function between x and $x + \Delta x$, so we think of the derivative as the gradient of the function at the point x (or as the gradient of a line which is tangent to the function at point x).

Example: the derivative of x^4

Using the definition of the derivative, we can write

$$\frac{(x+\Delta x)^4 - x^4}{\Delta x} = \frac{1}{\Delta x}(x^4 + 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4) = 4x^3 + 6x^2\Delta x + 4x(\Delta x)^2,$$

and when we let $\Delta x \to 0$ this will clearly give us $\mathrm{d}(x^4)/\mathrm{d}x = 4x^3$. In fact, since

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{1}{2}n(n-1)x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n,$$

we can easily see that the general rule must be

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

All derivatives can in principle be evaluated explicitly from the definition, as in this example. However, in practice we use standard rules, such as the one we just derived for x^n.

Differentiation of more complicated functions

Many functions that arise in physics are more complicated than simple power laws or trigonometric functions. It's not reasonable to tabulate the derivatives of every function you could possibly meet: instead, we develop rules for obtaining the derivatives of more complicated functions from simpler basic functions that we *can* reasonably tabulate (and learn).

The three basic ways in which we can combine functions are:

1. **Addition**: $f(x) = u(x) + v(x)$.
2. **Multiplication**: $f(x) = u(x)v(x)$.
3. **Composition**: $f(x) = f\big(u(x)\big)$.

Division can be seen as a combination of composition and multiplication: if we have $f(x) = u(x)/v(x)$, we can define $w(x) = 1/v(x)$ (a composition: we're writing $w = 1/v$ where v happens to be a function of x) and then write $f(x) = u(x)w(x)$ (a product).

Differentiating a sum

The derivative of a sum follows easily from the definition of the derivative. If $f(x) = u(x) + v(x)$, it follows that $f(x+\Delta x) = u(x+\Delta x) + v(x+\Delta x)$, and so

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x}\right] = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x}.\end{aligned}$$

In other words, the derivative of the sum is the sum of the derivatives.

Differentiating a product

The product rule also follows fairly naturally from the definition of the derivative. From the definition, we know that

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \mathcal{O}(\Delta x^2),$$

where $u'(x) = \mathrm{d}u/\mathrm{d}x$ and $\mathcal{O}(\Delta x^2)$ (i.e. "terms of order Δx^2") stands for any terms that include squares or higher powers of Δx and will therefore go to zero when we take the limit $\Delta x \to 0$. It follows that

$$u(x+\Delta x)\cdot v(x+\Delta x) = u(x)v(x) + u(x)v'(x)\Delta x + u'(x)\Delta x \cdot v(x) + \mathcal{O}(\Delta x^2),$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \lim_{\Delta x\to 0}\frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x} = u(x)v'(x) + u'(x)v(x).$$

This is just the standard product rule, $(uv)' = uv' + u'v$.

Differentiating a composite function (the chain rule)

If a function is composite, $f(x) = u\big(v(x)\big)$, we can calculate the derivative using a series of steps:

1. Make a small change Δx in x.
2. Compute the resulting small change in u, $\Delta u = u(x+\Delta x) - u(x)$.
3. Compute the resulting small change in f, $\Delta f = f(u+\Delta u) - f(u)$.
4. Take the limit of $\Delta f/\Delta x$ as $\Delta x \to 0$ (note that for any well-behaved function u, this also implies $\Delta u \to 0$).

Before we take the limit, we can write

$$\frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta u}\frac{\Delta u}{\Delta x}.$$

From the definitions,

$$\lim_{\Delta u\to 0}\frac{\Delta f}{\Delta u} = \frac{\mathrm{d}f}{\mathrm{d}u} \quad \text{and} \quad \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = \frac{\mathrm{d}u}{\mathrm{d}x}.$$

It follows that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x,\Delta u\to 0}\frac{\Delta f}{\Delta x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

This is **the chain rule**. It is one of the most commonly applied mathematical tools in physics, so it is essential that you understand this rule and can use it quickly and accurately.

Example: Use the product rule and the chain rule to derive the quotient rule.

Consider $f(x) = u(x)/v(x)$. Define $w(x) = 1/v(x)$. Then by the chain rule

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{d}w}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{v^2}v'(x).$$

From the definition of w, $f(x) = u(x)w(x)$, so we can use the product rule to write

$$\frac{\mathrm{d}f}{\mathrm{d}x} = u\frac{\mathrm{d}w}{\mathrm{d}x} + w\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{v^2}uv' + \frac{u'}{v} = \frac{u'v - uv'}{v^2},$$

which is the standard form of the quotient rule.

Higher derivatives

The derivative of a function is another function, so we can also take the derivative of that. This is known as the **second derivative**, and denoted $\mathrm{d}^2 f/\mathrm{d}x^2$. This should *not*, of course, be confused with the square of the derivative, $(\mathrm{d}f/\mathrm{d}x)^2$. It is possible to extend this indefinitely (or at least up to the point where the derivative is identically zero), but in physics we aren't usually interested in much beyond the second derivative.

Example: A car's position after time t is given by $s = bt^2 - ct^3$. What is its acceleration?
We know that velocity is the rate of change of position, $v = \mathrm{d}s/\mathrm{d}t$, and acceleration is the rate of change of velocity, $a = \mathrm{d}v/\mathrm{d}t$. Therefore $a = \mathrm{d}^2s/\mathrm{d}t^2$. Differentiating twice, we find $v = 2bt - 3ct^2$, and therefore $a = 2b - 6ct$.

Maxima and minima

An important application of differentiation is finding **maxima and minima** of a function. If the derivative of a function $f(x)$ is zero at some point x_0, then the curve of the function is horizontal at that point. This implies one of three possibilities:

- $f(x)$ has a **maximum** at that point (the gradient of the function has been positive but decreasing, is now zero, and will subsequently be negative);
- $f(x)$ has a **minimum** at that point (the gradient of the function has been negative but increasing, is now zero, and will subsequently be positive);
- this is a **point of inflection** for $f(x)$ (the gradient is zero at x_0, but has the same sign (either positive or negative) either side of x_0).

Which case applies can usually be found by taking the second derivative, $f''(x) = \mathrm{d}^2f/\mathrm{d}x^2$:

- if $f''(x) < 0$ (i.e. $f'(x)$ is decreasing), x_0 is a maximum of $f(x)$;
- if $f''(x) > 0$ (i.e. $f'(x)$ is increasing), x_0 is a minimum of $f(x)$;
- if $f''(x) = 0$, x_0 is usually, but not always, a point of inflection. This needs checking, since $f''(x) = 0$ is a *necessary but not sufficient* condition for an inflection point (i.e. all inflection points have $f''(x) = 0$, but not all points with $f''(x) = 0$ are inflection points). For example, if $f(x) = x^4$, then $f'(x) = 4x^3$, which is zero at $x = 0$. The second derivative $f''(x) = 12x^2$, which is also zero at $x = 0$, but this is a minimum, not an inflection point.

It is worth noting that the definition of an inflection point is that it is a point at which the curvature of the function changes sign: the function goes from concave up to concave down or vice versa. Therefore, inflection points can occur when $f'(x) \neq 0$: for example, $\sin x$ has inflection points at 0, π, 2π, …

Problems for differentiation

Here are some practice problems you might like to try. We'll distribute the solutions in Intro Week.

1. Differentiate the following with respect to x (all other symbols are constants):
   a. x^3 b. $4/x$ c. $x^4 - x^{1/3}$ d. $\sin(2x)$ e. e^{-2x} f. 3^x g. $\ln(5x)$ h. $ax^2 + bx + c$.

2. Differentiate the following with respect to t (all other symbols are constants):
   a. $\sin(at^2)$ b. $\sqrt{at^3 + bt^2}$ c. $t^2\cos(2t)$ d. e^{at^2+bt} e. $(at+b)^t$ f. $A\sin(\omega t - kx)$

3. Differentiate the following with respect to x (all other symbols are constants):
   a. $3x\cos(2x^2 + \phi)$ b. xe^{-x} c. $\frac{A\sin(kx)}{x}$ d. $\frac{x}{ax^2+b}$ e. $1/\cos(x^2)$ f. $x\ln(x^2)$

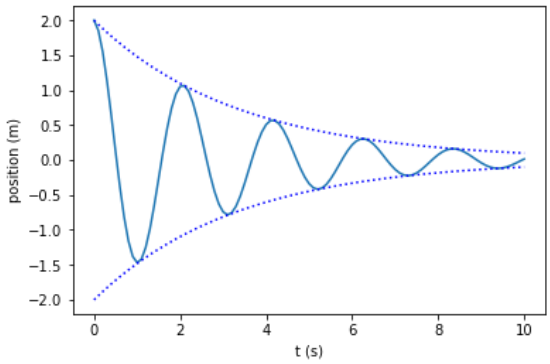
4. Find the maxima and minima of the functions:
   a. $4x^2 - 3x + 2$ b. $\sin(3x^2 - 5x + 2)$ (in the range $0 \le x \le \pi$) c. $A\sin(\omega t) + B\cos(\omega t)$
   d. $Ax^2 - B/x^3$ e. $6x^4 - 3x^2 + 4$ f. $(x^2 - 3x + 2)e^x$

5. An important function in statistics is $f(x) = e^{-x^2/2}$. Find the first and second derivatives of $f(x)$. Hence find the inflection points of $f(x)$.

6. A **damped harmonic oscillator** is an oscillator whose amplitude decreases with time, such as a pendulum with non-negligible friction in the pivot. The equation for the displacement of an object undergoing damped harmonic oscillation is

$$x = A\,e^{-\alpha t}\cos(\omega t)\,,$$

where A, α and ω are constants and we assume that $x = A$ at $t = 0$, as shown in the plot.



   a. Find the acceleration of the object at time t.

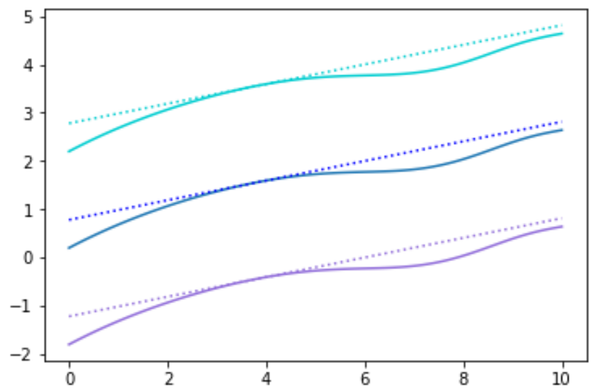
   b. In the plot, A = 2.0 m, α = 0.3 s^{-1}, and ω = 3.0 s^{-1}. Find the first three times t at which the velocity is zero, and check that these correspond to times at which the position is a maximum or a minimum.

Integration

By the Fundamental Theorem of Calculus, integration is the inverse of differentiation. This means that it applies to the same sort of physics problems as differentiation does, but with different unknowns: if we know the position of an object as a function of time, we can use differentiation to find its velocity and acceleration, while if we know the acceleration, we can use integration to find the velocity and position. Integration can also be used to find physical quantities such as the mass or volume of an object, and—perhaps less obviously—the average of some quantity.

Definite and indefinite integrals

If we think of integration as the inverse of differentiation—that is, the integral of $f(x)\mathrm{d}x$ is some function $F(x)$ whose derivative is $f(x)$—it is apparent that $F(x)$ is not completely defined. Because the derivative of a constant is 0, we can add any constant to $F(x)$ without changing its derivative: the three curves in the figure have the same derivative at any point (the three dotted tangent lines have the same gradient). This **constant of integration** can be eliminated in two ways:



1. If we know the value of the integral at any point, we can explicitly evaluate the constant. For example, $\int \cos\theta\ \mathrm{d}\theta = \sin\theta + C$, but if we know that the integral is 0 at $\theta = \pi/2$, we can establish that $C = 0 - \sin(\pi/2) = -1$. This is known as a **boundary condition** for the integral. See page 3 for an example of this.
2. If we are evaluating the integral between two specified limits, the constant will cancel out: for example, $\int_0^{\pi/2} \cos\theta\ \mathrm{d}\theta = [\sin\theta + C]_0^{\pi/2} = 1$ independent of the value of C. An integral evaluated between two limits is called a **definite integral**, and in this case we generally don't bother to include the constant, because it will always cancel. An integral without limits is an **indefinite integral**, and in this case we do need the constant.

In physics problems, it is often the case that one or both of the limits of integration is infinite. This is not technically legitimate, since infinity is not a real number, and such integrals are known as **improper integrals**. However, they can often be evaluated quite sensibly, and yield finite answers. Formally, what we need to do is integrate to some finite limit, say t, and then allow t to tend to infinity, for example

$$\int_0^\infty e^{-x}\,\mathrm{d}x = \lim_{t\to\infty}\int_0^t e^{-x}\,\mathrm{d}x = \lim_{t\to\infty}[-e^{-x}]_0^t = \lim_{t\to\infty}[1 - e^{-t}] = 1.$$

Physicists, however, are typically sloppy about such things and would simply write

$$\int_0^\infty e^{-x}\,\mathrm{d}x = [-e^{-x}]_0^\infty = 1.$$

It is, of course, entirely possible that taking the limit as $t \to \infty$ does not yield a nice finite number (consider replacing e^{-x} with e^{+x} in the above example). This can also happen with finite limits: for example,

$$\int_0^1 \frac{\mathrm{d}x}{x^2} = \left[-\frac{1}{x}\right]_0^1,$$

and clearly we have a problem at the lower limit. Such integrals are said to be **divergent**. In physics problems, if you have wound up with a divergent integral (and there was no clue in the question to suggest that all might not be well, such as "Why does your answer indicate a problem with this approximation?"), the chances are that you have set the problem up incorrectly.

Integration techniques

Differentiation is relatively straightforward: repeated applications of the chain rule and the product rule will produce the desired result (provided the function is differentiable in the first place), and it is generally clear which rule needs to be applied where. The techniques used for integration are essentially the inverse of the differentiation rules (substitution is the chain rule run backwards, and integration by parts is the product rule run backwards), but it tends to be more difficult to identify which rule needs to be applied to a particular integral, and if we need repeated applications it can get very messy.

Substitution

Substitution is the inverse of the chain rule. The chain rule states that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(F\big(u(x)\big)\Big) = \frac{\mathrm{d}F}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Integrating this gives

$$F\big(u(x)\big) = \int \frac{\mathrm{d}F}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{\mathrm{d}F}{\mathrm{d}u}\,\mathrm{d}u.$$

What this says is that if we have a function that can be expressed in the form

$$f(x) = \frac{\mathrm{d}F\big(u(x)\big)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x},$$

we can find the integral with respect to x by doing the easier integral with respect to u.

Example: Evaluate $\int xe^{-x^2}\,\mathrm{d}x$.

We can see here that $x = \frac{1}{2}\frac{\mathrm{d}(x^2)}{\mathrm{d}x}$. Therefore if we write $u = x^2$, we have $\mathrm{d}u = 2x\,\mathrm{d}x$. Substituting into the equation gives

$$\int xe^{-x^2}\,\mathrm{d}x = \frac{1}{2}\int e^{-u}\,\mathrm{d}u = -\frac{1}{2}e^{-u} + C = -\frac{1}{2}e^{-x^2} + C.$$

The key challenge in using substitution is identifying the right choice for u. The best way to do this is to identify the "inner function", i.e. the function that is acting as the argument of another function, such as x^2 in the above example, $3x + 5$ in $\sin(3x + 5)$ or $\sqrt{3x + 5}$, $\cos x$ in $\cos^5 x$, and see if you can identify its derivative elsewhere in the integrand. There are some simple cases that are worth looking for:

- If the inner function has the form $ax + b$, then its derivative with respect to x is just a constant. In this case it is *always* worth trying $u = ax + b$. For example,

$$\int \sqrt{3x + 5}\,\mathrm{d}x = \frac{1}{3}\int \sqrt{u}\,\mathrm{d}u = \frac{1}{3}\left[\frac{2}{3}u^{3/2} + C\right] = \frac{2}{9}(3x + 5)^{3/2} + C.$$

- If the integral is an odd power of $\cos x$ (or $\sin x$), writing $u = \sin x$ (or $u = \cos x$) will work, because you can use $\cos^2 x + \sin^2 x = 1$ to convert any even power of $\cos x$ into a polynomial in $\sin x$ or vice versa. For example,
$$\int \cos^5 x \ \mathrm{d}x = \int (1 - \sin^2 x)^2 \cos x \ \mathrm{d}x\,,$$
and if we write $u = \sin x$, $\mathrm{d}u = \cos x \ \mathrm{d}x$, this becomes
$$\int (1 - u^2)^2 \,\mathrm{d}u = \int (1 - 2u^2 + u^4)\mathrm{d}u = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C,$$
and you can substitute back to get this in terms of $\sin x$.
- Integrals of the form
$$\int \frac{\mathrm{d}x}{\sqrt{a - bx^2}},$$
which show up surprisingly often in physics problems, can also be dealt with using $\cos^2 x + \sin^2 x = 1$. If we write $\sin u = x\sqrt{b/a}$, we have $\mathrm{d}x = \sqrt{a/b}\cos u \ \mathrm{d}u$ and $\sqrt{a - bx^2} = \sqrt{a - a\sin^2 u} = \sqrt{a}\cos u$. This gives us
$$\int \frac{\mathrm{d}x}{\sqrt{a - bx^2}} = \int \frac{\sqrt{a/b}\cos u \,\mathrm{d}u}{\sqrt{a}\cos u} = \frac{1}{\sqrt{b}}\int \mathrm{d}u = \frac{u}{\sqrt{b}} + C.$$
Substituting back for u gives us
$$\int \frac{\mathrm{d}x}{\sqrt{a - bx^2}} = \frac{1}{\sqrt{b}}\arcsin\left(x\sqrt{b/a}\right) + C$$
(using $\arcsin x$ rather than $\sin^{-1} x$ to avoid any possible confusion with $1/\sin x$).
- A similar trick can be played for integrals of the form
$$\int \frac{\mathrm{d}x}{a + bx^2},$$
using the closely related trig identity $1 + \tan^2 x = \sec^2 x$.

Choosing the right substitution is an art form! There is no shortcut to developing your skills in this area: you just have to do lots of practice problems. Your ability to solve them will grow with experience.

Example: Evaluate $\int \frac{3x+5}{x^2+7}\,\mathrm{d}x$.

This looks like a case where a substitution is probably necessary, but it is not obvious what to do. The derivative of $x^2 + 7$ is not equal to a multiple of $3x + 5$, so the obvious approach won't work. In fact, what we need to do is split it into two integrals and use a different substitution in each case. We have
$$\int \frac{3x + 5}{x^2 + 7}\,\mathrm{d}x = \int \frac{3x}{x^2 + 7}\,\mathrm{d}x + \int \frac{5}{x^2 + 7}\,\mathrm{d}x.$$
The first of these we can do by letting $u = x^2 + 7$; $\mathrm{d}u = 2x\mathrm{d}x$. Then
$$\int \frac{3x}{x^2 + 7}\,\mathrm{d}x = \frac{3}{2}\int \frac{\mathrm{d}u}{u} = \frac{3}{2}\ln u + C.$$
The second integral can be written as
$$\frac{5}{7}\int \frac{\mathrm{d}x}{1 + x^2/7}\,.$$
If we write $x = \sqrt{7}\tan\theta$, then $\mathrm{d}x = \sqrt{7}\sec^2\theta \ \mathrm{d}\theta$ and $1 + x^2/7 = 1 + \tan^2\theta = \sec^2\theta$. With these substitutions, we have
$$\frac{5}{7}\int \frac{\mathrm{d}x}{1 + x^2/7} = \frac{5}{\sqrt{7}}\int \frac{\sec^2\theta \,\mathrm{d}\theta}{\sec^2\theta} = \frac{5}{\sqrt{7}}\int \mathrm{d}\theta = \frac{5}{\sqrt{7}}\theta + C.$$
We now add these together to get $\frac{3}{2}\ln(x^2 + 7) + \frac{5}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right) + C$.

Integration by parts

Integration by parts is the inverse of the product rule. The product rule says

$$(uv)' = uv' + u'v.$$

If we integrate both sides of this we get

$$uv = \int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x + \int \frac{\mathrm{d}u}{\mathrm{d}x} v \, \mathrm{d}x + C.$$

This can be rearranged to give

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = uv - \int \frac{\mathrm{d}u}{\mathrm{d}x} v \, \mathrm{d}x + C,$$

which is sometimes written as

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u + C.$$

This is particularly useful when u is some power of x, because differentiating u with respect to x will eventually make that term disappear altogether.

Example: Evaluate $\int x^2 e^{-kx} \, \mathrm{d}x$.

We put $u = x^2$, so $\mathrm{d}u/\mathrm{d}x = 2x$, and $\mathrm{d}v/\mathrm{d}x = e^{-kx}$, so $v = -e^{-kx}/k$. Then we have

$$\int x^2 e^{-kx} \, \mathrm{d}x = -\frac{x^2 e^{-kx}}{k} + \frac{2}{k} \int x e^{-kx} \, \mathrm{d}x.$$

Repeating the process, we put $u = x$, so $\mathrm{d}u/\mathrm{d}x = 1$, and $\mathrm{d}v/\mathrm{d}x = e^{-kx}$ again. This gives

$$\int x e^{-kx} \, \mathrm{d}x = -\frac{x e^{-kx}}{k} + \frac{1}{k} \int e^{-kx} \, \mathrm{d}x = -\frac{x e^{-kx}}{k} - \frac{1}{k^2} e^{-kx}.$$

Substituting this into our previous expression gives

$$\int x^2 e^{-kx} \, \mathrm{d}x = -\frac{x^2 e^{-kx}}{k} - \frac{2x e^{-kx}}{k^2} - \frac{2e^{-kx}}{k^3} + C.$$

(We only add the integration constant at the end, just to make the calculation less messy.)

Exercise: Differentiate the above answer to check that it does indeed give $x^2 e^{-kx}$.

Integration by parts is a useful technique if the integrand obviously factorises into two simpler functions, one of which you know how to integrate. However, it can also be applied in less obvious situations.

Example: Evaluate $\int \ln x \, \mathrm{d}x$.

There is no obvious substitution that will help here, so we should consider integration by parts. Although the integrand doesn't seem to consist of two factors, we can make it so by multiplying by 1 to get $\int 1 \cdot \ln x \, \mathrm{d}x$.

We don't know how to integrate $\ln x$ (obviously!), but we do know how to differentiate it, and $\int 1 \, \mathrm{d}x = x$. So we put $\ln x = u$ and $x = v$, giving

$$\int 1 \cdot \ln x \, \mathrm{d}x = x \ln x - \int \frac{x}{x} \, \mathrm{d}x = x \ln x - x + C.$$

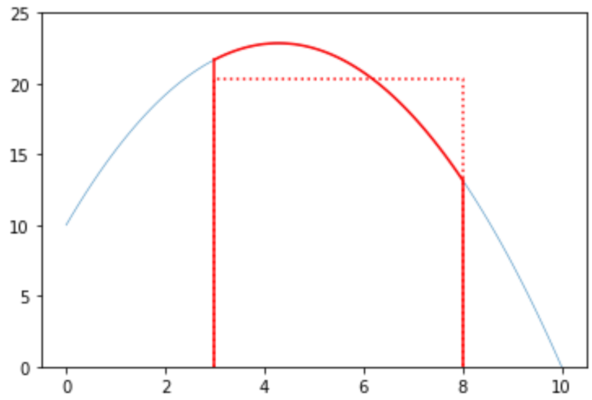
Differentiating the answer gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \ln x - x) = \ln x + \frac{x}{x} - 1 = \ln x,$$

confirming that our integration is correct.

Integration and averaging

An important application of integration, especially in quantum mechanics, is in finding the average of some variable.



The relationship between integration and averaging can be seen in the plot on the right. The integral of the function between $x = 3$ and $x = 8$ is the area under the solid red curve. The value of the function averaged over the range $x = 3$ to $x = 8$ is the height of the dotted rectangle that will make its area equal to the area under the curve. Therefore, since the area of the rectangle is $\bar{f} \cdot (8 - 3)$, where $\bar{f}$ is the average, we must have

$$\bar{f}(x_1 \leq x \leq x_2) = \frac{\int_{x_1}^{x_2} f(x)\mathrm{d}x}{x_2 - x_1},$$

where $\bar{f}(x_1 \leq x \leq x_2)$ is the average of $f(x)$ over the range from x_1 to x_2.

This makes intuitive sense. We would calculate the average of N discrete points by writing

$$\bar{f} = \frac{1}{N}\sum_{i=0}^{N} f(x_i).$$

If our N points are equally spaced and span the range x_1 to x_2, then the spacing between them is $\mathrm{d}x = (x_2 - x_1)/N$, which means that we can write our average as

$$\bar{f} = \frac{1}{x_2 - x_1}\sum_{i=0}^{N} f(x_i)\mathrm{d}x\,,$$

and if we let $\mathrm{d}x \to 0$ this will become the integral expression we derived above.

Problems for integration

Here are some practice problems you might like to try. We'll distribute the solutions in Intro Week.

1. Evaluate the following indefinite integrals:
   a. $\int (x-4)^2 \,\mathrm{d}x$ b. $\int \sin 3\theta \,\mathrm{d}\theta$ c. $\int 2^x \mathrm{d}x$ d. $\int \left(x + \frac{1}{x}\right)^2 \mathrm{d}x$ e. $\int e^{-4x}\mathrm{d}x$ f. $\int 3x^{-5/2}\mathrm{d}x$

2. Use appropriate substitutions to evaluate the following indefinite integrals:
   a. $\int(\tan\theta + \cot\theta)\mathrm{d}\theta$ b. $\int \frac{\mathrm{d}x}{x\ln x}$ c. $\int (2x+5)^{3/2}\,\mathrm{d}x$ d. $\int \frac{t}{2t+7}\,\mathrm{d}t$ e. $\int \frac{t}{2t^2+7}\,\mathrm{d}t$ f. $\int \frac{\mathrm{d}t}{2t+7}$

3. Use integration by parts to evaluate the following indefinite integrals:
   a. $\int xe^{-x}\mathrm{d}x$ b. $\int x^2 \ln x \,\mathrm{d}x$ c. $\int e^{-x}\cos x \,\mathrm{d}x$ d. $\int \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$ e. $\int (x^2 + x - 2)\sin x \,\mathrm{d}x$

4. Evaluate the following definite integrals:
   a. $\int_0^5 \frac{\mathrm{d}x}{\sqrt{x+5}}$ b. $\int_0^1 (3-2x)^{-3/2}\mathrm{d}x$ c. $\int_0^\pi \cos^2\theta \,\mathrm{d}\theta$ d. $\int_0^1 \frac{\mathrm{d}x}{\sqrt{4-x^2}}$ e. $\int_2^5 \frac{x\mathrm{d}x}{(3x^2-2)^4}$ f. $\int_0^\pi \sin^3\theta \,\mathrm{d}\theta$

5. Consider the following improper integrals. State whether or not they are divergent, and evaluate them if they are not.
   a. $\int_{-\infty}^1 e^x \,\mathrm{d}x$ b. $\int_{-\infty}^\infty xe^{-x^2}\mathrm{d}x$ c. $\int_0^\infty e^x \,\mathrm{d}x$ d. $\int_1^\infty \frac{\mathrm{d}x}{x}$ e. $\int_1^\infty \frac{\mathrm{d}x}{x^2}$

6. Over 10 seconds a car's velocity is given by $v_0 t(t_0 - t)$, where $v_0 = 2$ m/s and $t_0 = 10$ s. What is the distance that the car has travelled (a) over the entire period of 10 s and (b) as a function of t? (c) Find the average speed of the car over the period 3.0 s < t < 5.0 s.

7. A bar of material 1.700 m long and 1.00 cm^2 in cross-sectional area has been constructed so that its density is given by $\rho(x) = \rho_0/(5.0 + x)$, where x is the distance from the denser end of the rod in cm and $\rho_0 = 25.0$ g cm^{-2}.

    a. Calculate the mass of the bar.

    b. If I wish to cut this bar into two pieces of equal mass, at what value of x do I cut?

    c. Recall that the position of the centre of mass of a system of masses is given by

$$M\boldsymbol{R} = \sum_{i=1}^{N} m_i \boldsymbol{r}_i \,,$$

    where the $\boldsymbol{r}_i$ are the position vectors of the masses m_i, M is the total mass, and $\boldsymbol{R}$ is the position vector of the centre of mass. This can be generalised to an integral

$$M\boldsymbol{R} = \int_{\text{body}} \boldsymbol{r}\, \mathrm{d}m \,,$$

    where $\mathrm{d}m$ is the small element of mass at position $\boldsymbol{r}$. Use this to find the centre of mass of the bar. Explain why this is different from your answer to part b.