mathcentre

Pythagoras' theorem

Pythagoras' theorem is well-known from schooldays. In this unit we revise the theorem and use it to solve problems involving right-angled triangles. We will also meet a less-familiar form of the theorem.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- state Pythagoras' theorem
- use Pythagoras' theorem to solve problems involving right-angled triangles.

Contents

1. Introduction

The Theorem of Pythagoras is a well-known theorem. It is also a very old one, not only does it bear the name of Pythagoras, an ancient Greek, but it was also known to the ancient Babylonians and to the ancient Egyptians. Most school students learn of it as $a^2 + b^2 = c^2$. The actual statement of the theorem is more to do with areas. So, let's have a look at the statement of the theorem.

2. The Theorem of Pythagoras

The theorem makes reference to a right-angled triangle such as that shown in Figure 1. The side opposite the right-angle is the longest side and is called the **hypotenuse**.

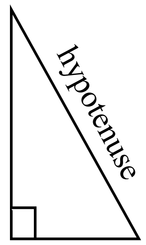


Figure 1. A right-angled triangle with hypotenuse shown.

What the theorem says is that the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the two shorter sides. Figure 2 shows squares drawn on the hypotenuse and on the two shorter sides. The theorem tells us that area A + area B = area C.

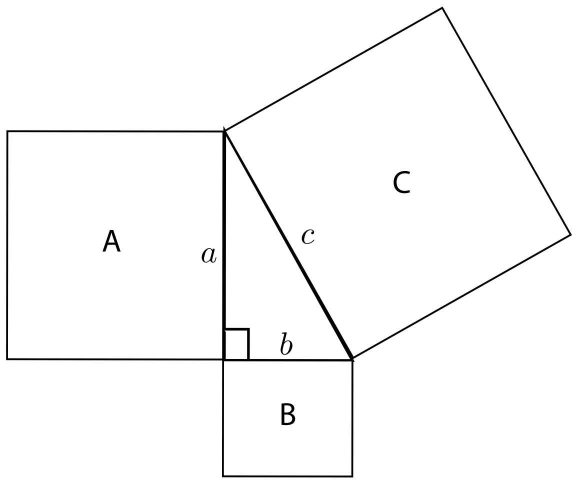


Figure 2. A right-angled triangle with squares drawn on each side.

An excellent demonstration of this is available on the accompanying video. If we denote the lengths of the sides of the triangle as a, b and c, as shown, then area A $= a^2$, area B $= b^2$ and area C $= c^2$. So, using Pythagoras' theorem

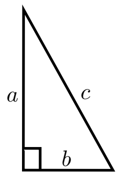
$$\begin{aligned} \text{area A} + \text{area B} &= \text{area C} \\ a^2 + b^2 &= c^2 \end{aligned}$$

This is the traditional result.

Key Point

Pythagoras' theorem:

$$a^2 + b^2 = c^2$$



Example
Suppose we wish to find the length of the hypotenuse of the right-angled triangle shown in Figure 4. We have labelled the hypotenuse c.

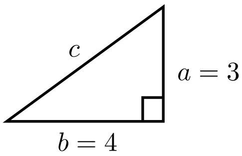


Figure 4.

Using the theorem:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= c^2 \\ 9 + 16 &= c^2 \\ 25 &= c^2 \\ 5 &= c \end{aligned}$$

So 5 is the length of the hypotenuse, the longest side of the triangle.

Example

In this Example we will assume we know the length of the hypotenuse.

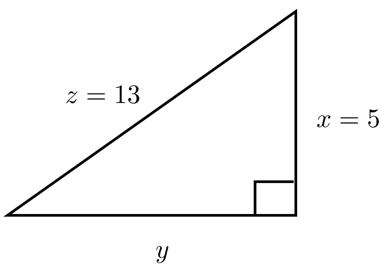


Figure 5.

The corresponding statement of Pythagoras' theorem is $x^2 + y^2 = z^2$. So,

$$\begin{aligned} x^2 + y^2 &= z^2 \\ 5^2 + y^2 &= 13^2 \\ 25 + y^2 &= 169 \\ y^2 &= 144 \\ y &= \sqrt{144} = 12 \end{aligned}$$

So, 12 is the length of the unknown side.

Example

Suppose we wish to find the length q in Figure 6. The statement of the theorem is now

$$p^2 + q^2 = r^2$$

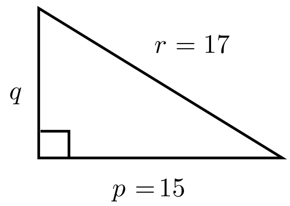


Figure 6.

$$\begin{aligned} p^2 + q^2 &= r^2 \\ 15^2 + q^2 &= 17^2 \\ 225 + q^2 &= 289 \\ q^2 &= 64 \\ q &= 8 \end{aligned}$$

So, 8 is the length of the unknown side.

In each of these three Examples, the answers have been exact and they have been whole number values. These whole number triples, ((3,4,5), (5,12,13), (8,15,17)), or **Pythagorean triples**, as we call them, occur quite frequently. When you do questions like these, you probably won't be so lucky to get exact answers as we have done here; you almost certainly will have to use a calculator for many of them and you have to decide to approximate the answer to a given number of decimal places or a given number of significant figures.

Exercise 1

1. Determine, to 2 decimal places, the length of the hypotenuses of the right-angled triangles whose two shorter sides have the lengths given below.

   a) 5 cm, 12 cm b) 1 cm, 2 cm c) 3 cm, 4 cm

   d) 1 cm, 1 cm e) 1.73 cm, 1 cm f) 2 cm, 5 cm

2. Determine, to 2 decimal places, the length of the third sides of the right-angled triangles where the hypotenuse and other side have the lengths given below.

   a) 8 cm, 2 cm b) 5 cm, 4 cm c) 2 cm, 1 cm

   d) 6 cm, 5 cm e) 10 cm, 7 cm f) 1 cm, 0.5 cm

3. A further application of the theorem

Let's have a look at another application of Pythagoras' theorem.
Look at the cuboid shown in Figure 7. Suppose we wish to find the length, y, of the diagonal of this cuboid. This is the bold line in Figure 7. Note that ABC is a right-angled triangle with the right-angle at C. Note also, that ACD is a right-angled triangle with hypotenuse AC. Let AC have length x, as shown.

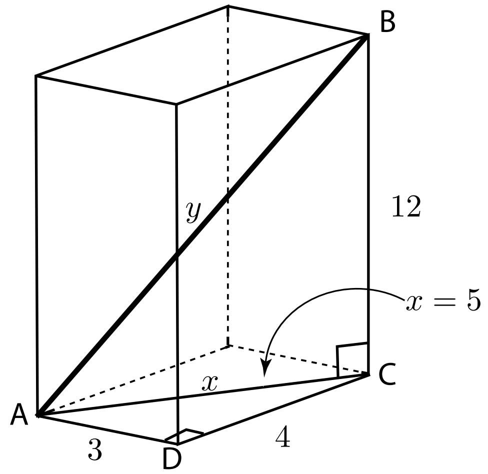


Figure 7.

Referrring to triangle ACD and using Pythagoras' theorem:

$$\begin{aligned} 3^2 + 4^2 &= x^2 \\ 9 + 16 &= x^2 \\ x^2 &= 25 \\ x &= 5 \end{aligned}$$

Now let the length of the diagonal AB be y. Using Pythagoras' theorem in triangle ABC:

$$\begin{aligned} 5^2 + 12^2 &= y^2 \\ 25 + 144 &= y^2 \\ 169 &= y^2 \\ y &= \sqrt{169} = 13 \end{aligned}$$

So we can use the theorem of Pythagoras in 3-dimensions; we can use it to solve problems that are set up in 3-dimensional objects.

Exercise 2

1. Calculate the length of the diagonals of the following cuboids
   a) $2 \times 3 \times 4$ b) $1 \times 1 \times 3$ c) $5 \times 5 \times 5$
   Give your answers to 2 decimal places.

4. Applications in cartesian geometry

Cartesian geometry is geometry that is set out on a plane that uses cartesian co-ordinates. These are the familiar (x, y) coordinates you will have seen before. We have already seen that Pythagoras' theorem gives us a relationship which is satisfied between the lengths of the sides of a right-angled triangle. However, we can turn this theorem around. If we study any triangle, and find that the area of the square on the longest side is equal to the sum of the areas of the squares on the two shorter sides of the triangle, then the triangle must be right-angled. So we can use Pythagoras' theorem to tell whether a triangle is right-angled or not.

Example
Suppose we have the co-ordinates of three points:

$$(3, 4) \quad (2, 6) \quad (1, 0)$$

Can we use Pythagoras' theorem to find out whether they form the corners of a right-angled triangle ? The points are plotted in Figure 8. Looking at Figure 8 we might guess that the triangle does contain a right-angle, but we can't be sure. The scales on the two axes are not quite the same and so appearances can be deceptive.

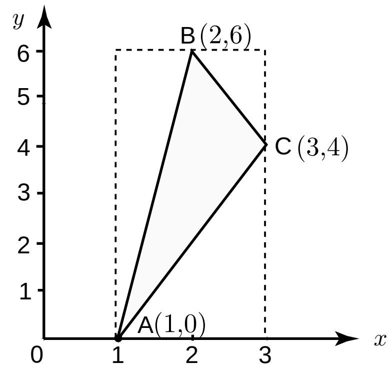


Figure 8.

From Figure 8 note that

$$\begin{aligned}(AB)^2 &= 1^2 + 6^2 = 1 + 36\\ &= 37\\ (BC)^2 &= 2^2 + 1^2 = 4 + 1\\ &= 5\\ \\ (AC)^2 &= 2^2 + 4^2 = 4 + 16\\ &= 20\end{aligned}$$

Now ask the question. Do the squares of the shorter sides add together to give the square of the longer side ?

$$(AC)^2 + (BC)^2 = 5 + 20 = 25 \neq 37$$

So we do not have a right-angled triangle. In fact because the longest side, AB, is greater than that which would be required to form a right-angle (i.e. $\sqrt{25} = 5$) we can deduce that the angle at C is in fact greater than $90°$. Thus C is an obtuse angle. This is not immediately obvious from a sketch.

Exercise 3

1. The lengths of the sides of a number of different triangles are given below. In each case, determine whether the largest angle is obtuse, right angle or acute.

   a) 1, 1, 1 b) 1, 1, 2 c) 4, 3, 2 d) 5, 7, 9

   e) 5, 12, 13 f) 2, 6, 4 g) 8, 10, 6 h) 1, 2, 2

2. The co-ordinates of the vertices of a number of triangles are given below. In each case, determine if the triangle is right angled.
   a) (1,2), (2,3), (5,0) b) (2,1), (4,2), (2,6) c) (3,2), (5,1), (2,5)

5. A final result

Finally, the fact that Pythagoras' Theorem is about squares is fairly well-known. What is not so well-known, although it is fairly obvious once you have seen it, is that if you take any regular figure or similar figures and place them on the sides of a right-angled triangle, then the area of the figure on the hypotenuse is equal to the sum of the areas of the figures on the two shorter sides. Consider, for example, Figure 9 where we have drawn semi-circles on each side of the right-angled triangle.

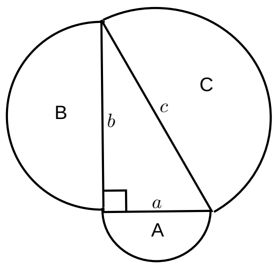


Figure 9. Semi-circles drawn on the sides of the right-angled triangle.

From the formula for the area of a circle:

$$\text{Area of } A = \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 = \frac{\pi}{8}a^2$$
$$\text{Area of } B = \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 = \frac{\pi}{8}b^2$$
$$\text{Area of } C = \frac{1}{2}\pi\left(\frac{c}{2}\right)^2 = \frac{\pi}{8}c^2$$

and so

$$\begin{aligned}\text{Area of } A + \text{Area of } B &= \frac{\pi}{8}a^2 + \frac{\pi}{8}b^2 \\ &= \frac{\pi}{8}(a^2 + b^2)\end{aligned}$$

But from Pythagoras' theorem $a^2 + b^2 = c^2$ and so

$$\begin{aligned}\text{Area of } A + \text{Area of } B &= \frac{\pi}{8}c^2 \\ &= \text{Area of } C\end{aligned}$$

and so it is true that Area of $A +$ Area of $B =$ Area of C.

Answers

Exercise 1

1. a) 13 cm b) 2.24 cm c) 5 cm d) 1.41 cm e) 2.00 cm f) 5.39 cm
2. a) 7.75 cm b) 3 cm c) 1.73 cm d) 3.32 cm e) 7.14 cm f) 0.87 cm

Exercise 2

1. a) 5.39 b) 3.32 c) 8.66

Exercise 3

1. a) acute b) obtuse c) obtuse d) obtuse e) right angle f) obtuse g) right angle h) acute
2. a) Yes b) Yes c) No