Formulae and Transposition

1.5

Introduction

Formulae are used frequently in almost all aspects of engineering in order to relate a physical quantity to one or more others. Many well-known physical laws are described using formulae. For example, you may have already seen Ohm's law, $v = iR$, or Newton's second law of motion, $F = ma$.

In this Section we describe the process of evaluating a formula, explain what is meant by the **subject** of a formula, and show how a formula is rearranged or transposed. These are basic skills required in all aspects of engineering.

Prerequisites

Before starting this Section you should ...

- be able to add, subtract, multiply and divide algebraic fractions

Learning Outcomes

On completion you should be able to ...

- transpose a formula

1. Using formulae and substitution

In the study of engineering, physical quantities can be related to each other using a formula. The formula will contain variables and constants which represent the physical quantities. To evaluate a formula we must **substitute** numbers in place of the variables.

For example, Ohm's law provides a formula for relating the voltage, v, across a resistor with resistance value, R, to the current through it, i. The formula states

$$v = iR$$

We can use this formula to calculate v if we know values for i and R. For example, if $i = 13$ A, and $R = 5\,\Omega$, then

$$\begin{aligned} v &= iR \\ &= (13)(5) \\ &= 65 \end{aligned}$$

The voltage is 65 V.

Note that it is important to pay attention to the units of any physical quantities involved. Unless a consistent set of units is used a formula is not valid.

Example 62

The kinetic energy, K, of an object of mass M moving with speed v can be calculated from the formula, $K = \frac{1}{2}Mv^2$.

Calculate the kinetic energy of an object of mass 5 kg moving with a speed of 2 m s^{-1}.

Solution

In this example $M = 5$ and $v = 2$. Substituting these values into the formula we find

$$\begin{aligned} K &= \frac{1}{2}Mv^2 \\ &= \frac{1}{2}(5)(2^2) \\ &= 10 \end{aligned}$$

In the SI system the unit of energy is the joule. Hence the kinetic energy of the object is 10 joules.

Task

The area, A, of the circle of radius r can be calculated from the formula $A = \pi r^2$.

If we know the diameter of the circle, d, we can use the equivalent formula $A = \dfrac{\pi d^2}{4}$. Find the area of a circle having diameter 0.1 m. Your calculator will be preprogrammed with the value of π.

Your solution

$A =$

Answer

$\dfrac{\pi(0.1)^2}{4} = 0.0079$ m^2

Example 63

The volume, V, of a circular cylinder is equal to its cross-sectional area, A, times its length, h.

Find the volume of a cylinder having diameter 0.1 m and length 0.3 m.

Solution

We can use the result of the previous Task to obtain the cross-sectional area $A = \dfrac{\pi d^2}{4}$. Then

$$\begin{aligned} V &= Ah \\ &= \frac{\pi(0.1)^2}{4} \times 0.3 \\ &= 0.0024 \end{aligned}$$

The volume is 0.0024 m^3.

2. Rearranging a formula

In the formula for the area of a circle, $A = \pi r^2$, we say that A is the **subject** of the formula. A variable is the subject of the formula if it appears by itself on one side of the formula, usually the left-hand side, and **nowhere else in the formula**. If we are asked to **transpose** the formula for r, or **solve** for r, then we have to make r the subject of the formula. When transposing a formula *whatever is done to one side is done to the other.* There are five rules that must be adhered to.

Key Point 22

Rearranging a formula

You may carry out the following operations

- add the same quantity to both sides of the formula
- subtract the same quantity from both sides of the formula
- multiply both sides of the formula by the same quantity
- divide both sides of the formula by the same quantity
- take a 'function' of both sides of the formula: for example, find the reciprocal of both sides (i.e. invert).

Example 64

Transpose the formula $p = 5t - 17$ for t.

Solution

We must obtain t on its own on the left-hand side. We do this in stages by using one or more of the five rules in Key Point 22. For example, by adding 17 to both sides of $p = 5t - 17$ we find

$$p + 17 = 5t - 17 + 17$$

so that $$p + 17 = 5t$$

Dividing both sides by 5 we obtain t on its own:

$$\frac{p+17}{5} = t$$

so that $t = \dfrac{p+17}{5}$.

Example 65

Transpose the formula $\sqrt{2q} = p$ for q.

Solution

First we square both sides to remove the square root. Note that $(\sqrt{2q})^2 = 2q$. This gives

$$2q = p^2$$

Second we divide both sides by 2 to get $q = \frac{p^2}{2}$.

Note that in general by squaring both sides of an equation may introduce extra solutions not valid for the original equation. In Example 65 if $p = 2$ then $q = 2$ is the only solution. However, if we transform to $q = \dfrac{p^2}{2}$, then if $q = 2$, p can be $+2$ or -2.

Task

Transpose the formula $v = \sqrt{t^2 + w}$ for w.

You must obtain w on its own on the left-hand side. Do this in several stages.

First square both sides to remove the square root:

Your solution

Answer

$v^2 = t^2 + w$

Then, subtract t^2 from both sides to obtain an expression for w:

Your solution

Answer

$v^2 - t^2 = w$

Finally, write down the formula for w:

Your solution

Answer

$w = v^2 - t^2$

Example 66

Transpose $x = \dfrac{1}{y}$ for y.

Solution

We must try to obtain an expression for y. Multiplying both sides by y has the effect of removing this fraction:

$$
\begin{array}{lrcll}
\text{Multiply both sides of} & x & = & \dfrac{1}{y} & \text{by } y \text{ to get} \\
 & yx & = & y \times \dfrac{1}{y} & \\
\text{so that} & yx & = & 1 &
\end{array}
$$

Divide both sides by x to leaves y on its own, $y = \dfrac{1}{x}$.

Alternatively: simply invert both sides of the equation $x = \dfrac{1}{y}$ to get $\dfrac{1}{x} = y$.

Example 67

Make R the subject of the formula

$$\frac{2}{R} = \frac{3}{x+y}$$

Solution

In the given form R appears in a fraction. Inverting both sides gives

$$\frac{R}{2} = \frac{x+y}{3}$$

Thus multiplying both sides by 2 gives

$$R = \frac{2(x+y)}{3}$$

Task Make R the subject of the formula $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$.

(a) Add the two terms on the right:

Your solution

Answer

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{R_2 + R_1}{R_1 R_2}$$

(b) Write down the complete formula:

Your solution

Answer

$$\frac{1}{R} = \frac{R_2 + R_1}{R_1 R_2}$$

(c) Now invert both sides:

Your solution

Answer

$$R = \frac{R_1 R_2}{R_2 + R_1}$$

Engineering Example 2

Heat flow in an insulated metal plate

Introduction

Thermal insulation is important in many domestic (e.g. central heating) and industrial (e.g cooling and heating) situations. Although many real situations involve heat flow in more than one dimension, we consider only a one dimensional case here. The flow of heat is determined by temperature and thermal conductivity. It is possible to model the amount of heat Q (J) crossing point x in one dimension (the heat flow in the x direction) from temperature T_2 (K) to temperature T_1 (K) (in which $T_2 > T_1$) in time t s by

$$\frac{Q}{t} = \lambda A\left(\frac{T_2 - T_1}{x}\right),$$

where λ is the thermal conductivity in W m^{-1} K.

Problem in words

Suppose that the upper and lower sides of a metal plate connecting two containers are insulated and one end is maintained at a temperature T_2 (K) (see Figure 7).

The plate is assumed to be infinite in the direction perpendicular to the sheet of paper.

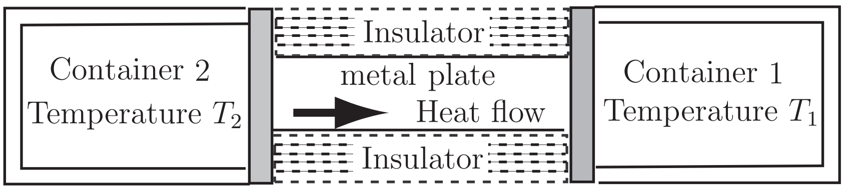

Figure 7: A laterally insulated metal plate

(a) Find a formula for T.

(b) If $\lambda = 205$ (W m^{-1} K^{-1}), $T_1 = 300$ (K), $A = 0.004$ (m^2), $x = 0.5$ (m), calculate the value of T_2 required to achieve a heat flow of 100 J s^{-1}.

Mathematical statement of the problem

(a) Given $\dfrac{Q}{t} = \lambda A\left(\dfrac{T_2 - T_1}{x}\right)$ express T_2 as the subject of the formula.

(b) In the formula found in part (a) substitute $\lambda = 205,\ T_1 = 300,\ A = 0.004$, $x = 0.5$ and $\dfrac{Q}{t} = 100$ to find T_2.

Mathematical analysis

(a) $\dfrac{Q}{t} = \lambda A \left(\dfrac{T_2 - T_1}{x}\right)$

Divide both sides by λA

$$\frac{Q}{t\lambda A} = \frac{T_2 - T_1}{x}$$

Multiply both sides by x

$$\frac{Qx}{t\lambda A} = T_2 - T_1$$

Add T_1 to both sides

$$\frac{Qx}{t\lambda A} + T_1 = T_2$$

which is equivalent to

$$T_2 = \frac{Qx}{t\lambda A} + T_1$$

(b) Substitute $\lambda = 205,\ T_1 = 300,\ A = 0.004,\ x = 0.5$ and $\dfrac{Q}{t} = 100$ to find T_2:

$$T_2 = \frac{100 \times 0.5}{205 \times 0.004} + 300 \approx 60.9 + 300 = 360.9$$

So the temperature in container 2 is 361 K to 3 sig.fig.

Interpretation

The formula $T_2 = \dfrac{Qx}{t\lambda A} + T_1$ can be used to find a value for T_2 that would achieve any desired heat flow. In the example given T_2 would need to be about 361 K ($\approx 78°$C) to produce a heat flow of 100 J s^{-1}.

Exercises

1. The formula for the volume of a cylinder is $V = \pi r^2 h$. Find V when $r = 5$ cm and $h = 15$ cm.

2. If $R = 5p^2$, find R when (a) $p = 10$, (b) $p = 16$.

3. For the following formulae, find y at the given values of x.

   (a) $y = 3x + 2$, $x = -1$, $x = 0$, $x = 1$.

   (b) $y = -4x + 7$, $x = -2$, $x = 0$, $x = 1$.

   (c) $y = x^2$, $x = -2$, $x = -1$, $x = 0$, $x = 1$, $x = 2$.

4. If $P = \dfrac{3}{QR}$ find P if $Q = 15$ and $R = 0.300$.

5. If $y = \sqrt{\dfrac{x}{z}}$ find y if $x = 13.200$ and $z = 15.600$.

6. Evaluate $M = \dfrac{\pi}{2r + s}$ when $r = 23.700$ and $s = -0.2$.

7. To convert a length measured in metres to one measured in centimetres, the length in metres is multiplied by 100. Convert the following lengths to cm. (a) 5 m, (b) 0.5 m, (c) 56.2 m.

8. To convert an area measured in m^2 to one measured in cm^2, the area in m^2 is multiplied by 10^4. Convert the following areas to cm^2. (a) 5 m^2, (b) 0.33 m^2, (c) 6.2 m^2.

9. To convert a volume measured in m^3 to one measured in cm^3, the volume in m^3 is multiplied by 10^6. Convert the following volumes to cm^3. (a) 15 m^3, (b) 0.25 m^3, (c) 8.2 m^3.

10. If $\eta = \dfrac{4Q_P}{\pi d^2 L n}$ evaluate η when $Q_P = 0.0003$, $d = 0.05$, $L = 0.1$ and $n = 2$.

11. The moment of inertia of an object is a measure of its resistance to rotation. It depends upon both the mass of the object and the distribution of mass about the axis of rotation. It can be shown that the moment of inertia, J, of a solid disc rotating about an axis through its centre and perpendicular to the plane of the disc, is given by the formula

$$J = \frac{1}{2}Ma^2$$

    where M is the mass of the disc and a is its radius. Find the moment of inertia of a disc of mass 12 kg and diameter 10 m. The SI unit of moment of inertia is kg m^2.

12. Transpose the given formulae to make the given variable the subject.

    (a) $y = 3x - 7$, for x, (b) $8y + 3x = 4$, for x, (c) $8x + 3y = 4$ for y,

    (d) $13 - 2x - 7y = 0$ for x.

13. Transpose the formula $PV = RT$ for (a) V, (b) P, (c) R, (d) T.

14. Transpose $v = \sqrt{x + 2y}$, (a) for x, (b) for y.

15. Transpose $8u + 4v - 3w = 17$ for each of u, v and w.

16. When a ball is dropped from rest onto a horizontal surface it will bounce before eventually coming to rest after a time T where

$$T = \frac{2v}{g}\left(\frac{1}{1-e}\right)$$

where v is the speed immediately after the first impact, and g is a constant called the acceleration due to gravity. Transpose this formula to make e, the coefficient of restitution, the subject.

17. Transpose $q = A_1\sqrt{\dfrac{2gh}{(A_1/A_2)^2 - 1}}$ for A_2.

18. Make x the subject of (a) $y = \dfrac{r + x}{1 - rx}$, (b) $y = \sqrt{\dfrac{x-1}{x+1}}$.

19. In the design of orifice plate flowmeters, the volumetric flowrate, Q (m^3 s^{-1}), is given by

$$Q = C_d A_o \sqrt{\frac{2g\Delta h}{1 - A_o^2/A_p^2}}$$

where C_d is a dimensionless discharge coefficient, Δh (m) is the head difference across the orifice plate and A_o (m^2) is the area of the orifice and A_p (m^2) is the area of the pipe.

(a) Rearrange the equation to solve for the area of the orifice, A_o, in terms of the other variables.

(b) A volumetric flowrate of 100 cm^3 s^{-1} passes through a 10 cm inside diameter pipe. Assuming a discharge coefficient of 0.6, calculate the required orifice diameter, so that the head difference across the orifice plate is 200 mm.

[Hint: be very careful with the units!]

Answers

1. 1178.1 cm^3
2. (a) 500, (b) 1280
3. (a) $-1, 2, 5$, (b) $15, 7, 3$, (c) 5,3,1,0,
4. P=0.667
5. $y = 0.920$
6. $M = 0.067$
7. (a) 500 cm, (b) 50 cm, (c) 5620 cm.
8. (a) 50000 cm^2, (b) 3300 cm^2, (c) 62000 cm^2.
9. (a) 15000000 cm^3, (b) 250000 cm^3, (c) 8200000 cm^3.
10. $\eta = 0.764$.
11. 150 kg m^2
12. (a) $x = \dfrac{y+7}{3}$, (b) $x = \dfrac{4-8y}{3}$, (c) $y = \dfrac{4-8x}{3}$, (d) $x = \dfrac{13-7y}{2}$
13. (a) $V = \dfrac{RT}{P}$, (b) $P = \dfrac{RT}{V}$, (c) $R = \dfrac{PV}{T}$, (d) $T = \dfrac{PV}{R}$
14. (a) $x = v^2 - 2y$, (b) $y = \dfrac{v^2 - x}{2}$
15. $u = \dfrac{17 - 4v + 3w}{8}$, $v = \dfrac{17 - 8u + 3w}{4}$, $w = \dfrac{8u + 4v - 17}{3}$
16. $e = 1 - \dfrac{2v}{gT}$
17. $A_2 = \pm\sqrt{\dfrac{A_1^2 q^2}{2A_1^2 gh + q^2}}$
18. (a) $x = \dfrac{y-r}{1+yr}$, (b) $x = \dfrac{1+y^2}{1-y^2}$
19.

(a) $A_0 = \dfrac{QA_p}{\sqrt{Q^2 + 2g\Delta h A_p^2 C_d^2}}$

(b) $Q = 100$ $\text{cm}^3\ \text{s}^{-1}$$= 10^{-4}$ $\text{m}^2\ \text{s}^{-1}$

$A_p = \pi\dfrac{0.1^2}{4} = 0.007854$ m^2

$C_d = 0.6$

$\Delta h = 0.2$ m

$g = 9.81$ m s^{-2}

Substituting in answer (a) gives

$A_o = 8.4132 \times 10^{-5}$ m^2

so diameter $= \sqrt{\dfrac{4A_o}{\pi}} = 0.01035$ m $= 1.035$ cm