RMIT UNIVERSITY

IN1.2: INTEGRATION OF POLYNOMIALS

Antidifferentiation

Antidifferentiation is the reverse process from differentiation. Given a derivative $f'(x)$ the task is to find the original function $f(x)$.

If $f(x)=\left(\frac{x^3}{3}\right)$ then $f'(x)=x^2$, therefore $\left(\frac{x^3}{3}\right)$ is an antiderivative of x^2

If $f(x)=\left(\frac{x^3}{3}+1\right)$ then $f'(x)=x^2$ $\left(\frac{x^3}{3}\right)+1$ is an antiderivative of x^2

If $f(x)=\left(\frac{x^3}{3}+2\right)$ then $f'(x)=x^2$ $\left(\frac{x^3}{3}\right)+2$ is an antiderivative of x^2

…….and so on.

Rule for powers of x

In general:

If $\frac{dy}{dx}=x^n$, $y=\frac{1}{n+1}x^{n+1}+c$ where c is a constant $(n \neq -1)$

This rule applies for **positive, negative** and **fractional** values of n **except** $n=-1$

Examples

1.

Given $\frac{dy}{dx}=2x$ find the antiderivative

$y=\frac{2x^{1+1}}{1+1}+c$ add one to the power of x, divide by the new power, add a constant

$y=x^2+c$ is the antiderivative

2. Given $\frac{dy}{dx}=3$ find the antiderivative.

$$y=\frac{3x^{0+1}}{0+1}+c$$

$$y=3x+c$$

$3 = 3 \times x^0$, add one to the power of x, divide by the new power, add a constant

3. If $\dfrac{dy}{dx} = x^{-3}$ find y

$$y = \frac{1}{-3+1} x^{-3+1} + c$$

$$y = \frac{1}{-2x^{2}} + c$$

add one to the power of x
divide by the new power
add a constant

4. If $f'(x) = x^{\frac{1}{2}}$ find $f(x)$

$$f(x) = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$f(x) = \frac{2x^{\frac{3}{2}}}{3} + c$$

add one to the power of x
divide by the new power
add a constant

See Exercise 1

The indefinite integral

The symbol $\int$ stands for "the integral of" and may be used to indicate that we wish to find an antiderivative.

For example

$\int x^{4} dx$ reads

the integral of x^{4} with respect to x,

Operational rules

$$\int \left(f(x) \pm g(x) \right) dx = \int f(x) dx \pm \int g(x) dx$$

$$\int kf(x) dx = k \int f(x) dx$$

Polynomials

Using the rule for finding the antiderivative of x^{n}

$$\int x^{n} dx = \frac{1}{n+1} x^{n+1} + c \qquad (n \neq -1)$$

- add one to the power
- divide by the new power
- add a constant

c is called the **constant of integration**.

Examples

1.

$$\int \left(x^{3} + 4 \right) dx = \int x^{3} dx + \int 4 dx$$

Integrate each term separately

$$= \frac{x^{4}}{4} + 4x + c$$

Only one constant of integration is needed

2.

$$\int\left(\frac{1}{3\sqrt{x}}\right)dx = \frac{1}{3}\int\left(\frac{1}{\sqrt{x}}\right)dx$$ Use $\int kf(x)dx = k\int f(x)\,dx$

$$\frac{1}{3}\int x^{-\frac{1}{2}}dx = \frac{1}{3}\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)+c$$ Write in index form

Integrate

$$=\frac{2}{3}\sqrt{x}+c$$ Simplify

3.

$$\int\frac{3s^4-s^3+7}{s^2}ds = \int\left(3s^2-s+7s^{-2}\right)ds$$ Simplify – divide each term by s^2 $(s \neq 0)$

$$=\frac{3s^3}{3}-\frac{s^2}{2}-\frac{7}{s}+c$$ Integrate

$$=s^3-\frac{s^2}{2}-\frac{7}{s}+c$$ Simplify

4.

$$\int\left(x^3+\frac{2}{x^2}+7\right)dx = \int\left(x^3+2x^{-2}+7\right)dx$$ Write $\frac{2}{x^2}$ as $2x^{-2}$

$$=\frac{x^4}{4}-2x^{-1}+7x+c$$ Integrate

$$=\frac{x^4}{4}-\frac{2}{x}+7x+c$$

See Exercise 2

Integrals of the form $(ax+b)^n$

$$\int(ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1}+c \qquad (n \neq -1)$$

The **function in brackets must be linear**. This rule cannot be used with expressions such as $\left(ax^2+b\right)^n$.

Examples

1.

$$\int(8x-5)^9dx = \frac{(8x-5)^{10}}{8\times10}+c$$ $a=8$, $b=-5$, $n=9$

$$=\frac{(8x-5)^{10}}{80}+c$$

2.

$$\int(7-2x)^3dx = \frac{(7-2x)^4}{-2\times4}+c$$ $a=-2$, $b=7$, $n=3$

$$=\frac{(7-2x)^4}{-8}+c$$

See Exercise 3

Exercises

Exercise 1

Find an antiderivative of

(a) x^3 (b) s^8 (c) $\sqrt{x}$ (d) x^{-5} (e) $a^{\frac{2}{3}}$ (f) s^{-2} (g) $p^{-\frac{1}{2}}$

Exercise 2

Find the following integrals.

(a) $\int 3x^2 dx$ (b) $\int \left(4x^5 - 2x^3 + 9\right) dx$ (c) $\int \left(x^2 + \frac{1}{x^2}\right) dx$

(d) $\int \left(5\sqrt{x} + 1\right) dx$ (e) $\int \left(\frac{4x^5 - 2x^3 + 9}{2}\right) dx$ (f) $\int \left(s + \frac{2}{3\sqrt{s^3}}\right) ds$

(g) $\int \left(\frac{3x+1}{\sqrt{x}}\right) dx$ (h) $\int \left(\frac{t^4 - 2t^3 + 1}{2t^2}\right) dt$

Answers

1(a) $\frac{x^4}{4}$ (b) $\frac{s^9}{9}$ (c) $\frac{2x\sqrt{x}}{3}$ (d) $\frac{x^{-4}}{-4} = \frac{1}{-4x^4}$

(e) $\frac{3a^{\frac{5}{3}}}{5}$ (f) $-s^{-1} = -\frac{1}{s}$ (g) $2p^{\frac{1}{2}}$

2.(a) $x^3 + c$ (b) $\frac{2x^6}{3} - \frac{x^4}{2} + 9x + c$ (c) $\frac{x^3}{3} - \frac{1}{x} + c$

(d) $\frac{10x\sqrt{x}}{3} + x + c$ (e) $\frac{x^6}{3} - \frac{x^4}{4} + \frac{9x}{2} + c$ (f) $\frac{s^2}{2} - \frac{4}{3\sqrt{s}} + c$

(g) $2x\sqrt{x} + 2\sqrt{x} + c = 2\sqrt{x}\left(x+1\right) + c$ (h) $\frac{t^3}{6} - \frac{t^2}{2} - \frac{1}{2t} + c$

3 (a) $\frac{(5x+1)^5}{25} + c$ (b) $\frac{2\left(x-9\right)\sqrt{\left(x-9\right)}}{3} + c$ (c) $\frac{(8-4x)^4}{-4} + c$

(d) $\frac{1}{-\left(x+3\right)^2} + c$ (e) $\frac{3\left(5x+8\right)^{\frac{4}{3}}}{20} + c = \frac{3\left(5x+8\right)\sqrt[3]{\left(5x+8\right)}}{20}$ (f) $\frac{(3x-2)^8}{12} + c$

(g) $\sqrt{\left(2x+3\right)} + c$