

# Calculating Centres of Mass

# 15.2

## Introduction

In this Section we show how the idea of integration as the limit of a sum can be used to find the centre of mass of an object such as a thin plate (like a sheet of metal). Such a plate is also known as a **lamina**. An understanding of the term **moment** is necessary and so this concept is introduced.



### Prerequisites

Before starting this Section you should ...

- understand integration as the limit of a sum
- be able to calculate definite integrals



### Learning Outcomes

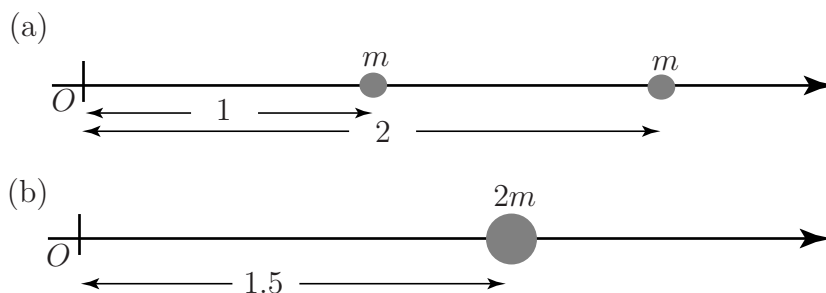
On completion you should be able to ...

- calculate the position of the centre of mass of a variety of simple plane shapes

# 1. The centre of mass of a collection of point masses

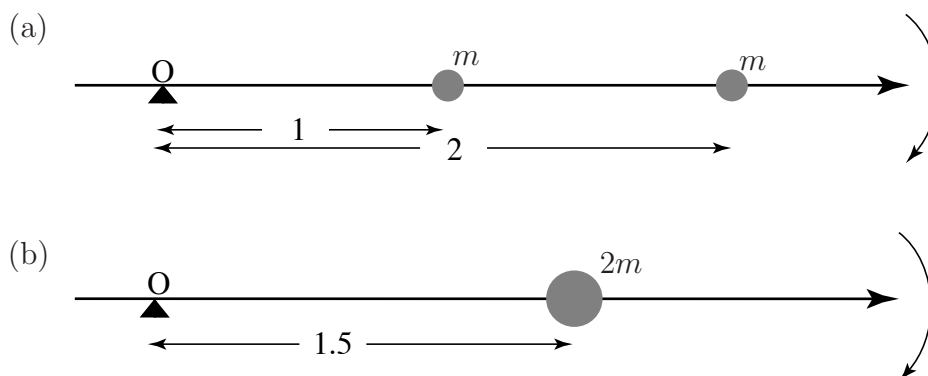
Suppose we have a collection of masses located at a number of known points along a line. The **centre of mass** is the point where, for many purposes, all the mass can be assumed to be located.

For example, if two objects each of mass  $m$  are placed at distances 1 and 2 units from a point  $O$ , as shown in Figure 2a, then the total mass,  $2m$ , might be assumed to be concentrated at distance 1.5 units as shown in Figure 2b. This is the point where we could imagine placing a pivot to achieve a perfectly balanced system.



**Figure 2:** Equivalent position of the centre of mass of the objects in (a) is shown in (b)

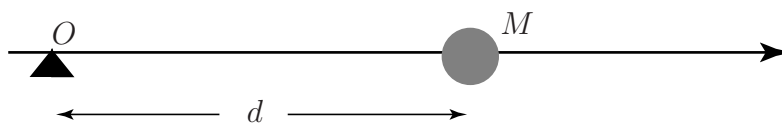
To think of this another way, if a pivot is placed at the origin  $O$ , as on a see-saw, then the two masses at  $x = 1$  and  $x = 2$  together have the same **turning effect** or **moment** as a single mass  $2m$  located at  $x = 1.5$ . This is illustrated in Figure 3.



**Figure 3:** The single object of mass  $2m$  has the same turning effect as the two objects each of mass  $m$

Before we can calculate the position of the centre of mass of a collection of masses it is important to define the term 'moment' more precisely. Given a mass  $M$  located a distance  $d$  from  $O$ , as shown in Figure 4, its moment about  $O$  is defined to be

$$\text{moment} = M \times d$$

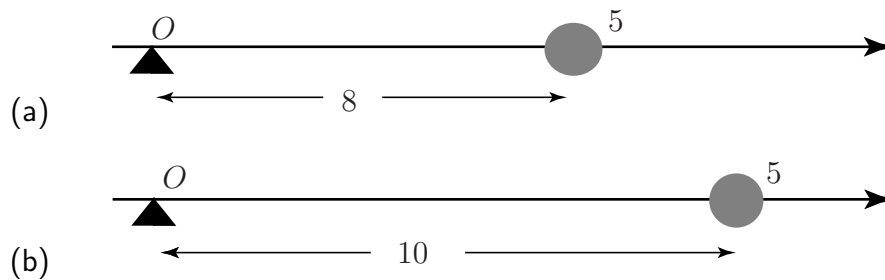


**Figure 4:** The moment of the mass  $M$  about  $O$  is  $M \times d$

In words, the moment of the mass about  $O$ , is the mass multiplied by its distance from  $O$ . The units of moment will therefore be kg m if the mass is measured in kilogrammes and the distance in metres. (N.B. Unless specified otherwise these will be the units we shall always use.)



Calculate the moment of the mass about  $O$  in each of the following cases.



#### Your solution

(a)

(b)

#### Answer

(a) 40 kg m (b) 50 kg m

Intuition tells us that a large moment corresponds to a large turning effect. A mass placed 8 metres from the origin has a smaller turning effect than the same mass placed 10 metres from the origin. This is, of course, why it is easier to rock a see-saw by pushing it at a point further from the pivot. Our intuition also tells us the side of the pivot on which the masses are placed is important. Those placed to the left of the pivot have a turning effect opposite to those placed to the right.

For a collection of masses the moment of the total mass located at the centre of mass is equal to the sum of the moments of the individual masses. This definition enables us to calculate the position of the centre of mass. It is conventional to label the  $x$  coordinate of the centre of mass as  $\bar{x}$ , pronounced 'x bar'.

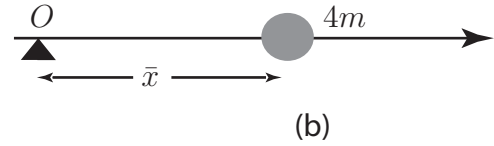
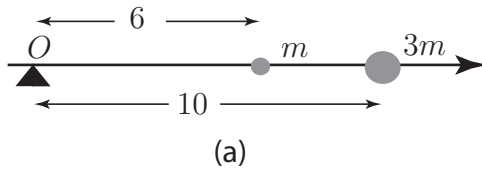


#### Key Point 1

The **moment** of the total mass located at the centre of mass is equal to the sum of the moments of the individual masses.



Objects of mass  $m$  and  $3m$  are placed at the locations shown in diagram (a). Find the distance  $\bar{x}$  of the centre of mass from the origin  $O$  as illustrated in diagram (b).



First calculate the sum of the individual moments:

**Your solution**

**Answer**

$$6 \times m + 10 \times 3m = 36m$$

The moment of the total mass about  $O$  is  $4m \times \bar{x}$ .

The moment of the total mass is equal to the sum of the moments of the individual masses. Write down and solve the equation satisfied by  $\bar{x}$ :

**Your solution**

**Answer**

$$36m = 4m\bar{x}, \quad \text{so} \quad \bar{x} = 9$$

So the centre of mass is located a distance 9 units along the  $x$ -axis. Note that it is closer to the position of the  $3m$  mass than to the position of the  $1m$  mass (actually in the ratio 3 : 1).



### Example 2

Obtain an equation for the location of the centre of mass of two objects of masses  $m_1$  and  $m_2$ :

(a) located at distances  $x_1$  and  $x_2$  respectively, as shown in Figure 5(a)

(b) positioned on opposite sides of the origin as shown in Figure 5(b)

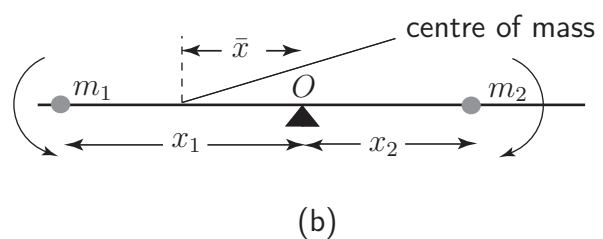
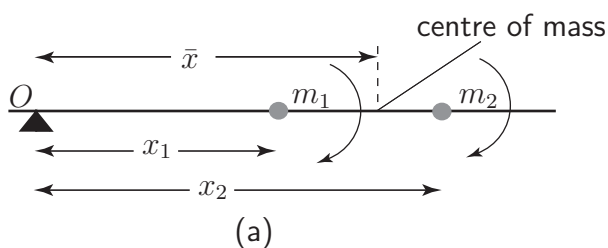


Figure 5

Referring to Figure 5(a) we first write down an expression for the sum of the individual moments:

$$m_1x_1 + m_2x_2$$

The total mass is  $m_1 + m_2$  and the moment of the total mass is  $(m_1 + m_2) \times \bar{x}$ .

The moment of the total mass is equal to the sum of the moments of the individual masses. The equation satisfied by  $\bar{x}$  is

$$(m_1 + m_2)\bar{x} = m_1x_1 + m_2x_2 \quad \text{so} \quad \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

For the second case, as depicted in Figure 5(b), the mass  $m_1$  positioned on the left-hand side has a turning effect **opposite** to that of the mass  $m_2$  positioned on the right-hand side. To take account of this difference we use a minus sign when determining the moment of  $m_1$  about the origin. This gives a total moment

$$-(m_1x_1) + (m_2x_2)$$

leading to

$$(-m_1x_1 + m_2x_2) = (m_1 + m_2)\bar{x} \quad \text{so} \quad \bar{x} = \frac{-m_1x_1 + m_2x_2}{m_1 + m_2}$$

However, this is precisely what would have been obtained if, when working out the moment of a mass, we use its **coordinate** (which takes account of sign) rather than using its **distance** from the origin.

The formula obtained in the Task can be generalised very easily to deal with the general situation of  $n$  masses,  $m_1, m_2, \dots, m_n$  located at **coordinate positions**  $x_1, x_2, \dots, x_n$  and is given in Key Point 2.



### Key Point 2

The **centre of mass** of individual masses  $m_1, m_2, \dots, m_n$  located at positions  $x_1, x_2, \dots, x_n$  is

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}$$



Calculate the centre of mass of the 4 masses distributed as shown below.



Use Key Point 2 to calculate  $\bar{x}$ :

**Your solution**

$$\bar{x} =$$

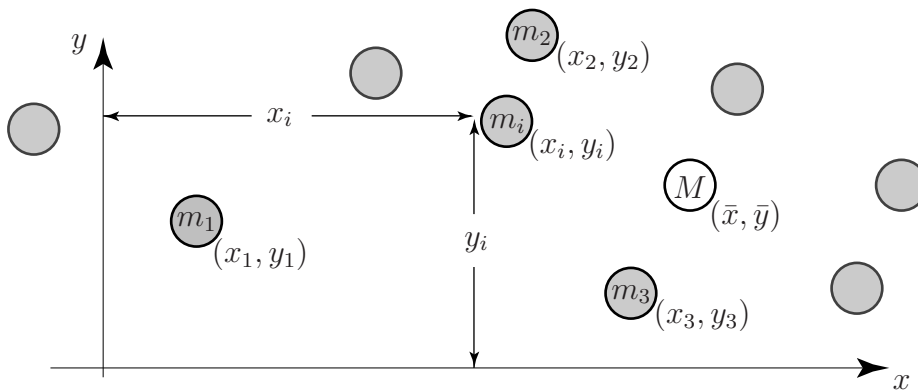
**Answer**

$$\bar{x} = \frac{(9)(-1) + (1)(2) + (5)(6) + (2)(8)}{9 + 1 + 5 + 2} = \frac{39}{17}$$

The centre of mass is located a distance  $\frac{39}{17} \approx 2.29$  units along the  $x$ -axis from  $O$ .

### Distribution of particles in a plane

If the particles are distributed in a plane then the position of the centre of mass can be calculated in a similar way.



**Figure 6:** These masses are distributed throughout the  $xy$  plane

Now we must consider the moments of the individual masses taken about the  $x$ -axis **and** about the  $y$ -axis. For example, in Figure 6, mass  $m_i$  has a moment  $m_i y_i$  about the  $x$ -axis and a moment  $m_i x_i$  about the  $y$ -axis. Now we define the centre of mass at that point  $(\bar{x}, \bar{y})$  such that the total mass  $M = m_1 + m_2 + \dots + m_n$  placed at this point would have the same moment about each of the axes as the sum of the individual moments of the particles about these axes.



### Key Point 3

The **centre of mass** of  $m_1, m_2, \dots, m_n$  located at  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  has coordinates  $(\bar{x}, \bar{y})$  where

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}, \quad \bar{y} = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i}$$



Masses of 5 kg, 3 kg and 9 kg are located at the points with coordinates  $(-1, 1)$ ,  $(4, 3)$ , and  $(8, 7)$  respectively. Find the coordinates of their centre of mass.

Use Key Point 3:

**Your solution**

$$\bar{x} =$$

$$\bar{y} =$$

**Answer**

$$\bar{x} = \frac{\sum_{i=1}^3 m_i x_i}{\sum_{i=1}^3 m_i} = \frac{5(-1) + 3(4) + 9(8)}{5 + 3 + 9} = \frac{79}{17} \approx 4.65$$

$$\bar{y} = \frac{5(1) + 3(3) + 9(7)}{17} = 4.53.$$

Hence the centre of mass is located at the point  $(4.65, 4.53)$ .

### Exercises

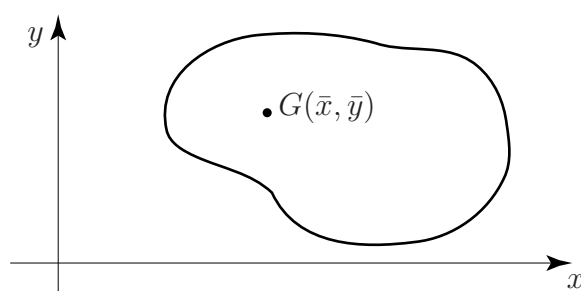
1. Find the  $x$  coordinate of the centre of mass of 5 identical masses placed at  $x = 2$ ,  $x = 5$ ,  $x = 7$ ,  $x = 9$ ,  $x = 12$ .
2. Derive the formula for  $\bar{y}$  given in Key Point 3.

**Answer** 1.  $\bar{x} = 7$

## 2. Finding the centre of mass of a plane uniform lamina

In the previous Section we calculated the centre of mass of several individual point masses. We are now interested in finding the centre of mass of a thin sheet of material, such as a plane sheet of metal, called a **lamina**. The mass is not now located at individual points. Rather, it is distributed continuously over the lamina. In what follows we assume that the mass is distributed **uniformly** over the lamina and you will see how integration as the limit of a sum is used to find the centre of mass.

Figure 6 shows a lamina where the centre of mass has been marked at point  $G$  with coordinates  $(\bar{x}, \bar{y})$ . If the total mass of the lamina is  $M$  then the moments about the  $y$ - and  $x$ -axes are respectively  $M\bar{x}$  and  $M\bar{y}$ . Our approach to locating the position of  $G$ , i.e. finding  $\bar{x}$  and  $\bar{y}$ , is to divide the lamina into many small pieces, find the mass of each piece, and calculate the moment of each piece about the axes. The sum of the moments of the individual pieces about the  $y$ -axis must then be equal to  $M\bar{x}$  and the sum of the moments of the individual pieces about the  $x$ -axis must equal  $M\bar{y}$ .



**Figure 6:** The centre of mass of the lamina is located at  $G(\bar{x}, \bar{y})$

There are no formulae which can be memorized for finding the centre of mass of a lamina because of the wide variety of possible shapes. Instead you should be familiar with the general technique for deriving the centre of mass.

An important preliminary concept is 'mass per unit area' which we now introduce.

### Mass per unit area

Suppose we have a uniform lamina and select a piece of the lamina which has area equal to **one unit**. Let  $\rho$  stand for the mass of such a piece. Then  $\rho$  is called the **mass per unit area**. The mass of any other piece can be expressed in terms of  $\rho$ . For example, an area of 2 units must have mass  $2\rho$ , an area of 3 units must have mass  $3\rho$ , and so on. Any portion of the lamina which has area  $A$  has mass  $\rho A$ .



### Key Point 4

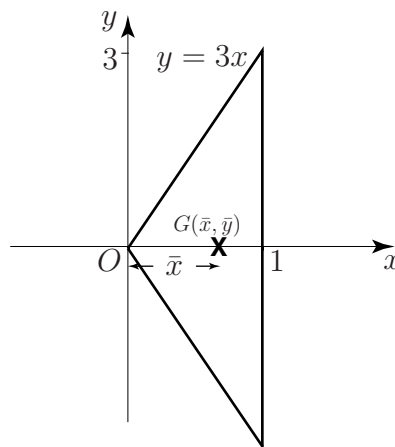
If a lamina has mass per unit area,  $\rho$ , then the mass of part of the lamina having area  $A$  is  $A\rho$ .

We will investigate the calculation of centre of mass through the following Tasks.





Consider the plane sheet, or lamina, shown below. Find the location of its centre of mass  $(\bar{x}, \bar{y})$ . (By symmetry the centre of mass of this lamina lies on the  $x$ -axis.)



(a) First inspect the figure and note the symmetry of the lamina. Purely from the symmetry, what must be the  $y$  coordinate,  $\bar{y}$ , of the centre of mass ?

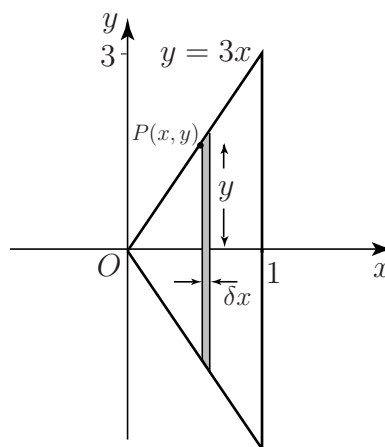
**Your solution**

**Answer**

$\bar{y} = 0$  since the centre of mass must lie on the  $x$ -axis

(b) Let  $\rho$  stand for the mass per unit area of the lamina. The total area is 3 units. The total mass is therefore  $3\rho$ . Its moment about the  $y$ -axis is  $3\rho\bar{x}$ .

To find  $\bar{x}$  first divide the lamina into a large number of thin vertical slices. In the figure below a typical slice has been highlighted. Note that the slice has been drawn from the point  $P$  on the line  $y = 3x$ . The point  $P$  has coordinates  $(x, y)$ . The thickness of the slice is  $\delta x$ .



A typical slice of this sheet has been shaded.

Assuming that the slice is rectangular in shape, write down its area:

**Your solution**

**Answer**

$$2y\delta x$$

(c) Writing  $\rho$  as the mass per unit area, write down the mass of the slice:

**Your solution**

**Answer**

$$(2y\delta x)\rho$$

(d) The centre of mass of this slice lies on the  $x$ -axis. So the slice can be assumed to be a point mass,  $2y\rho\delta x$ , located a distance  $x$  from  $O$ .

Write down the moment of the mass of the slice about the  $y$ -axis:

**Your solution**

**Answer**

$$(2y\delta x)\rho x$$

(e) By adding up contributions from all such slices in the lamina we obtain the sum of the moments of the individual masses:

$$\sum_{x=0}^{x=1} 2\rho xy\delta x$$

The limits on the sum are chosen so that all slices are included.

Write down the integral defined by letting  $\delta x \rightarrow 0$ :

**Your solution**

**Answer**

$$\int_{x=0}^{x=1} 2\rho xy \, dx$$

(f) Noting that  $y = 3x$ , express the integrand in terms of  $x$  and evaluate it:

**Your solution**

**Answer**

$$\int_0^1 6\rho x^2 \, dx = \left[ 2\rho x^3 \right]_0^1 = 2\rho$$

(g) Calculate  $\bar{x}$  and hence find the centre of mass of the lamina:

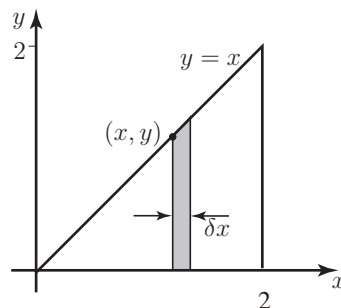
**Your solution**

**Answer**

This must equal the moment of the total mass acting at the centre of mass so  $3\rho\bar{x} = 2\rho$  giving  $\bar{x} = \frac{2}{3}$ . Now the coordinates of the centre of mass are thus  $(\frac{2}{3}, 0)$ .



Find the centre of mass of the plane lamina shown below.



The coordinates of  $\bar{x}$  and  $\bar{y}$  must be calculated separately.

**Stage 1: To calculate  $\bar{x}$**

(a) Let  $\rho$  equal the mass per unit area. Write down the total area, the total mass, and its moment about the  $y$ -axis:

**Your solution**

**Answer**

$2, 2\rho, 2\rho\bar{x}$

(b) To calculate  $\bar{x}$  the lamina is divided into thin slices; a typical slice is shown in the figure above. We assume that the shaded slice is rectangular, which is a reasonable approximation.

Write down the height of the typical strip shown in the figure, its area, and its mass:

**Your solution**

**Answer**

$y, y\delta x, (y\delta x)\rho$

(c) Write down the moment about the  $y$ -axis of the typical strip:

**Your solution**

**Answer**

$$(y\delta x)\rho x$$

(d) The sum of the moments of all strips is

$$\sum_{x=0}^{x=2} \rho xy \delta x$$

Write down the integral which follows as  $\delta x \rightarrow 0$ :

**Your solution**

**Answer**

$$\int_0^2 \rho xy \, dx$$

(e) In this example,  $y = x$  because the line  $y = x$  defines the upper limit of each strip (and hence its height). Substitute this value for  $y$  in the integral, and evaluate it:

**Your solution**

**Answer**

$$\int_0^2 \rho x^2 \, dx = \frac{8}{3}\rho$$

(f) Equating the sum of individual moments and the total moment gives  $2\rho\bar{x} = \frac{8}{3}\rho$ . Deduce  $\bar{x}$ :

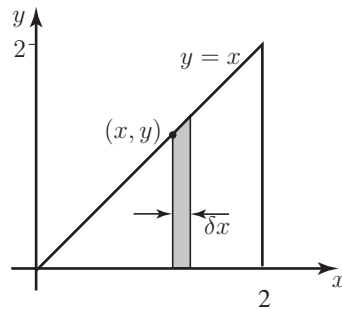
**Your solution**

**Answer**

$$\bar{x} = \frac{4}{3}$$

We will illustrate two alternative ways of calculating  $\bar{y}$ .

Stage 2: To calculate  $\bar{y}$  using vertical strips



(a) Referring to the figure again, which we repeat here, the centre of mass of the slice must lie half way along its length, that is its  $y$  coordinate is  $\frac{y}{2}$ . Assume that all the mass of the slice,  $y\rho\delta x$ , acts at this point. Then its moment about the  $x$ -axis is  $y\rho\delta x \frac{y}{2}$ . Adding contributions from all slices gives the sum

$$\sum_{x=0}^{x=2} \frac{y^2\rho}{2} \delta x$$

(b) Write down the integral which is defined as  $\delta x \rightarrow 0$ :

**Your solution**

**Answer**

$$\int_{x=0}^2 \frac{\rho y^2}{2} dx$$

(c) We can write the above as

$$\rho \int_{x=0}^2 \frac{y^2}{2} dx \quad \text{and in this example } y = x, \text{ so the integral becomes}$$

$$\rho \int_{x=0}^2 \frac{x^2}{2} dx$$

Evaluate this.

**Your solution**

**Answer**

$$\frac{4\rho}{3}$$

(d) This is the sum of the individual moments about the  $x$ -axis and must equal the moment of the total mass about the  $x$ -axis which has already been found as  $2\rho\bar{y}$ . Therefore

$$2\rho\bar{y} = \frac{4\rho}{3} \quad \text{from which} \quad \bar{y} = \frac{2}{3}$$

(e) Finally deduce  $\bar{y}$  and state the coordinates of the centre of mass:

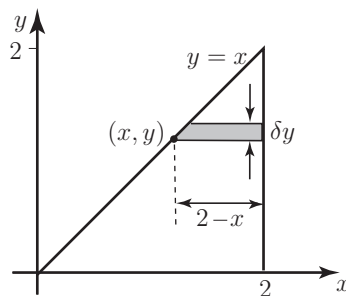
**Your solution**

**Answer**

$$\bar{y} = \frac{2}{3} \text{ and the coordinates of the centre of mass are } \left(\frac{4}{3}, \frac{2}{3}\right)$$

**Stage 3: To calculate  $\bar{y}$  using horizontal strips**

(a) This time the lamina is divided into a number of horizontal slices; a typical slice is shown below.



A typical horizontal slice is shaded.

The length of the typical slice shown is  $2 - x$ .

Write down its area, its mass and its moment **about the  $x$ -axis**:

**Your solution**

**Answer**

$$(2 - x)\delta y, \quad \rho(2 - x)\delta y, \quad \rho(2 - x)y\delta y$$

(b) Write down the expression for the sum of all such moments and the corresponding integral as  $\delta y \rightarrow 0$ .

**Your solution**

**Answer**

$$\sum_{y=0}^{y=2} \rho(2 - x)y\delta y, \quad \int_0^2 \rho(2 - x)y \, dy$$

(c) Now, since  $y = x$  the integral can be written entirely in terms of  $y$  as

$$\int_0^2 \rho(2 - y)y \, dy$$

Evaluate the integral and hence find  $\bar{y}$ :

**Your solution**

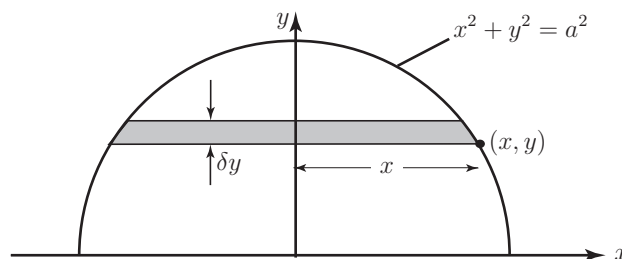
**Answer**

$\frac{4\rho}{3}$ ; As before the total mass is  $2\rho$ , and its moment about the  $x$ -axis is  $2\rho\bar{y}$ . Hence

$$2\rho\bar{y} = \frac{4\rho}{3} \quad \text{from which} \quad \bar{y} = \frac{2}{3} \quad \text{which was the result obtained before in Stage 2.}$$



Find the position of the centre of mass of a uniform semi-circular lamina of radius  $a$ , shown below.



A typical horizontal strip is shaded.

The equation of a circle centre the origin, and of radius  $a$  is  $x^2 + y^2 = a^2$ .

By symmetry  $\bar{x} = 0$ . However it is necessary to calculate  $\bar{y}$ .

(a) The lamina is divided into a number of horizontal strips and a typical strip is shown. Assume that each strip is rectangular. Writing the mass per unit area as  $\rho$ , state the area and the mass of the strip:

**Your solution**

**Answer**

$$2x\delta y, \quad 2x\rho\delta y$$

(b) Write down the moment of the mass about the  $x$ -axis:

**Your solution**

**Answer**

$$2x\rho y\delta y$$

(c) Write down the expression representing the sum of the moments of all strips and the corresponding integral obtained as  $\delta y \rightarrow 0$ :

**Your solution**

**Answer**

$$\sum_{y=0}^{y=a} 2x\rho y\delta y, \quad \int_0^a 2x\rho y \, dy$$

(d) Now since  $x^2 + y^2 = a^2$  we have  $x = \sqrt{a^2 - y^2}$  and the integral becomes:

$$\int_0^a 2\rho y\sqrt{a^2 - y^2} \, dy$$

Evaluate this integral by making the substitution  $u^2 = a^2 - y^2$  to obtain the total moment.

**Your solution**

**Answer**

$$\frac{2\rho a^3}{3}$$

(e) The total area is half that of a circle of radius  $a$ , that is  $\frac{1}{2}\pi a^2$ . The total mass is  $\frac{1}{2}\pi a^2\rho$  and its moment is  $\frac{1}{2}\pi a^2\rho\bar{y}$ .

Deduce  $\bar{y}$ :

**Your solution**

**Answer**

$$\frac{1}{2}\pi a^2\rho\bar{y} = \frac{2\rho a^3}{3} \quad \text{from which} \quad \bar{y} = \frac{4a}{3\pi}$$





## Engineering Example 1

### Suspended cable

#### Introduction

A cable of constant line density is suspended between two vertical poles of equal height such that it takes the shape of a curve,  $y = 6 \cosh(x/6)$ . The origin of the curve is a point mid-way between the feet of the poles and  $y$  is the height above the ground. If the cable is 600 metres long show that the distance between the poles is 55 metres to the nearest metre. Find the height of the centre of mass of the cable above the ground to the nearest metre.

#### Mathematical statement of the problem

We can draw a picture of the cable as in Figure 7 where  $A$  and  $B$  denote the end points.

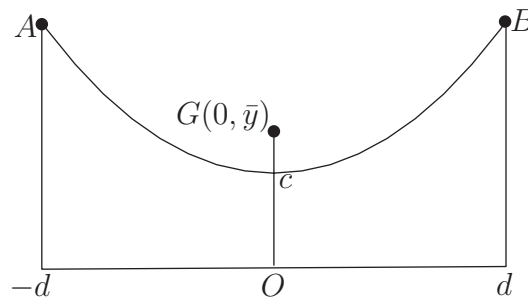


Figure 7

For the first part of this problem we use the result found in HELM 14 that the distance along a curve

$$y = f(x) \text{ from } x = a \text{ to } x = b \text{ is given by } s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

where in this case we are given  $y = 6 \cosh\left(\frac{x}{6}\right)$  and therefore  $\frac{dy}{dx} = \sinh\left(\frac{x}{6}\right)$ .

If we take the distance between the poles to be  $2d$  then the values of  $x$  in this integration go from  $-d$  to  $+d$ . So we need to find  $d$  such that:

$$600 = \int_{-d}^d \sqrt{1 + \left(\sinh\left(\frac{x}{6}\right)\right)^2} dx. \quad (1)$$

For the second part of this problem we need to find the centre of mass of the cable. From the symmetry of the problem we know that the centre of mass must lie on the  $y$ -axis. To find the height of the centre of mass we need to take each section of the cable and consider the moment about the  $x$ -axis through the origin. A section of the cable has mass  $\rho \delta s$  where  $\rho$  is the line density of the cable and  $\delta s$  is the length of a small section of the cable.

so the moment about the  $x$ -axis will be  $\sum_{x=-d}^{x=d} \rho y \delta s$

taking the limit as  $\delta s \rightarrow 0$  and using the fact that  $\delta s = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \delta x$

we get that the moment about the  $x$ -axis to be  $\rho \int_{-d}^d y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

This must equal the moment of a single point mass, equal to the total mass of the cable, placed at its centre of mass. As the length of the cable is 600 metres then the mass of the cable is  $600\rho$  and we have

$$600\rho\bar{y} = \rho \int_{-d}^d y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Dividing both sides of this equation by  $\rho$  we get:

$$600\bar{y} = \int_{-d}^d y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

where we have already established the value of  $d$  from Equation (1) so we can solve this equation to find  $\bar{y}$ .

### Mathematical analysis

We need to find  $d$  so that  $600 = \int_{-d}^d \sqrt{1 + \left(\sinh\left(\frac{x}{6}\right)\right)^2} dx$

Rearranging the hyperbolic identity  $\cosh^2(u) - \sinh^2(u) \equiv 1$  we obtain  $\sqrt{1 + (\sinh(u))^2} = \cosh(u)$

so the integral becomes  $\int_{-d}^d \cosh\left(\frac{x}{6}\right) dx = \left[6 \sinh\left(\frac{x}{6}\right)\right]_{-d}^d = 6 \left(\sinh\left(\frac{d}{6}\right) - \sinh\left(-\frac{d}{6}\right)\right)$

so

$$12 \sinh\left(\frac{d}{6}\right) = 600 \text{ and } d = 6 \sinh^{-1}(50).$$

Using the log identity for the  $\sinh^{-1}$  function:

$$\sinh^{-1}(x) \equiv \ln(x + \sqrt{x^2 + 1})$$

we find that  $d = 27.63$  m so the distance between the poles is 55 m to the nearest metre.

To find the height of the centre of mass above the ground we use

$$600\bar{y} = \int_{-d}^d y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Substituting  $y = 6 \cosh\left(\frac{x}{6}\right)$  and therefore  $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\sinh\left(\frac{x}{6}\right)\right)^2} = \cosh\left(\frac{x}{6}\right)$  we get

$$\int_{-d}^d 6 \cosh\left(\frac{x}{6}\right) \cosh\left(\frac{x}{6}\right) dx = \int_{-d}^d 6 \cosh^2\left(\frac{x}{6}\right) dx$$

From the hyperbolic identities we know that  $\cosh^2(x) \equiv \frac{1}{2}(\cosh(2x) + 1)$

so this integral becomes  $\int_{-d}^d 3 \left(\cosh\left(\frac{x}{3}\right) + 1\right) dx = \left[9 \sinh\left(\frac{x}{3}\right) + 3x\right]_{-d}^d = 18 \sinh\left(\frac{d}{3}\right) + 6d$

So we have that  $600\bar{y} = 18 \sinh\left(\frac{d}{3}\right) + 6d$

From the first part of this problem we found that  $d = 27.63$  so substituting for  $d$  we find  $\bar{y} = 150$  metres to the nearest metre.

**Interpretation**

We have found that the two vertical poles holding the cable have a distance between them of 55 metres and the height of the centre of mass of the cable above the ground is 150 metres.

**Exercise**

Find the centre of mass of a lamina bounded by  $y^2 = 4x$ , for  $0 \leq x \leq 9$ .

**Answer**  $(\frac{27}{5}, 0)$ .