Contents 16

Sequences and Series

Learning outcomes

In this Workbook you will learn about sequences and series. You will learn about arithmetic and geometric series and also about infinite series. You will learn how to test the for the convergence of an infinite series. You will then learn about power series, in particular you will study the binomial series. Finally you will apply your knowledge of power series to the process of finding series expansions of functions of a single variable. You will be able to find the Maclaurin and Taylor series expansions of simple functions about a point of interest.

Sequences and Series 16.1

Introduction

In this Section we develop the ground work for later Sections on infinite series and on power series. We begin with simple sequences of numbers and with finite series of numbers. We introduce the summation notation for the description of series. Finally, we consider arithmetic and geometric series and obtain expressions for the sum of n terms of both types of series.

Prerequisites

Before starting this Section you should ...

- understand and be able to use the basic rules of algebra
- be able to find limits of algebraic expressions

Learning Outcomes

On completion you should be able to ...

- check if a sequence of numbers is convergent
- use the summation notation to specify series
- recognise arithmetic and geometric series and find their sums

1. Introduction

A **sequence** is any succession of numbers. For example the sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots$$

which is known as the Fibonacci sequence, is formed by adding two consecutive terms together to obtain the next term. The numbers in this sequence continually increase without bound and we say this sequence **diverges**. An example of a **convergent** sequence is the **harmonic sequence**

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots$$

Here we see the magnitude of these numbers continually decrease and it is obvious that the sequence converges to the number zero. The related **alternating harmonic sequence**

$$1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \ldots$$

is also convergent to the number zero. Whether or not a sequence is convergent is often easy to deduce by graphing the individual terms. The diagrams in Figure 1 show how the individual terms of the harmonic and alternating harmonic series behave as the number of terms increase.

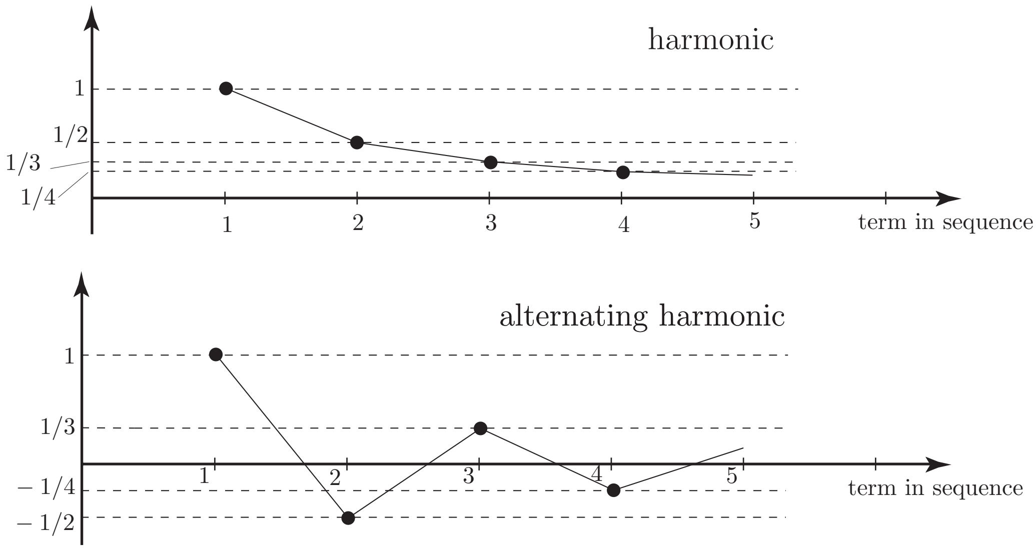


Figure 1

Task

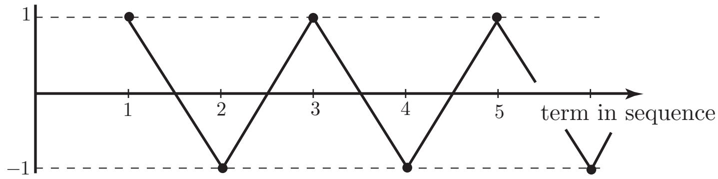
Graph the sequence:

$$1,\ -1,\ 1,\ -1,\ \ldots$$

Is this convergent?

Your solution

Answer



Not convergent.

The terms in the sequence do **not** converge to a particular value. The value **oscillates**.

A general sequence is denoted by

$$a_1,\ a_2,\ \ldots,\ a_n, \ldots$$

in which a_1 is the first term, a_2 is the second term and a_n is the n^{th} term is the sequence. For example, in the harmonic sequence

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \ldots, a_n = \frac{1}{n}$$

whilst for the alternating harmonic sequence the n^{th} term is:

$$a_n = \frac{(-1)^{n+1}}{n}$$

in which $(-1)^n = +1$ if n is an even number and $(-1)^n = -1$ if n is an odd number.

Key Point 1

The sequence $a_1,\ a_2,\ \ldots, a_n,\ \ldots$ is said to be **convergent** if the limit of a_n as n increases can be found. (Mathematically we say that $\lim\limits_{n\to\infty} a_n$ exists.)

If the sequence is not convergent it is said to be **divergent**.

Task

Verify that the following sequence is convergent

$$\frac{3}{1\times 2},\ \frac{4}{2\times 3},\ \frac{5}{3\times 4}, \ \ldots$$

First find the expression for the n^{th} term:

Your solution

Answer

$$a_n = \frac{n+2}{n(n+1)}$$

Now find the limit of a_n as n increases:

Your solution

Answer

$$\frac{n+2}{n(n+1)} = \left[\frac{1+\frac{2}{n}}{n+1}\right] \to \frac{1}{n+1} \to 0 \qquad \text{as } n \text{ increases}$$

Hence the sequence is convergent.

2. Arithmetic and geometric progressions

Consider the sequences:

$$1,\ 4,\ 7,\ 10,\ \ldots \quad \text{and} \quad 3,\ 1,\ -1,\ -3,\ \ldots$$

In both, any particular term is obtained from the previous term by the **addition** of a constant value (3 and -2 respectively). Each of these sequences are said to be an **arithmetic sequence** or **arithmetic progression** and has general form:

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,\ a+(n-1)d,\ \ldots$$

in which a, d are given numbers. In the first example above $a = 1,\ d = 3$ whereas, in the second example, $a = 3,\ d = -2$. The difference between any two successive terms of a given arithmetic sequence gives the value of d which is called the **common difference**.

Two sequences which are **not** arithmetic sequences are:

$$1,\ 2,\ 4,\ 8,\ \ldots$$

$$-1,\ -\frac{1}{3},\ -\frac{1}{9},\ -\frac{1}{27},\ \ldots$$

In each case a particular term is obtained from the previous term by **multiplying** by a constant factor (2 and $\frac{1}{3}$ respectively). Each is an example of a **geometric sequence** or **geometric progression** with the general form:

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

where 'a' is the first term and r is called the **common ratio**, being the ratio of two successive terms. In the first geometric sequence above $a = 1$, $r = 2$ and in the second geometric sequence $a = -1$, $r = \frac{1}{3}$.

Task

Find a, d for the arithmetic sequence $3,\ 9,\ 15, \ldots$

Your solution

$a =$ $d =$

Answer

$a = 3, \quad d = 6$

Task

Find a, r for the geometric sequence $8,\ \dfrac{8}{7},\ \dfrac{8}{49},\ \ldots$

Your solution

$a =$ $r =$

Answer

$a = 8, \quad r = \dfrac{1}{7}$

Task

Write out the first four terms of the geometric series with $a = 4$, $r = -2$.

Your solution

Answer

$4, -8, 16, -32, \ldots$

The reader should note that many sequences (for example the harmonic sequences) are neither arithmetic nor geometric.

3. Series

A **series** is the sum of the terms of a sequence. For example, the **harmonic series** is

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

and the **alternating harmonic series** is

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

The summation notation

If we consider a general sequence

$$a_1,\ a_2,\ \ldots,\ a_n,\ \ldots$$

then the sum of the first k terms $a_1+a_2+a_3+\cdots+a_k$ is concisely denoted by $\displaystyle\sum_{p=1}^{k} a_p$.

That is,

$$a_1+a_2+a_3+\cdots+a_k=\sum_{p=1}^{k} a_p$$

When we encounter the expression $\displaystyle\sum_{p=1}^{k} a_p$ we let the index 'p' in the term a_p take, in turn, the values $1, 2, \ldots, k$ and then add all these terms together. So, for example

$$\sum_{p=1}^{3} a_p=a_1+a_2+a_3 \qquad\qquad \sum_{p=2}^{7} a_p=a_2+a_3+a_4+a_5+a_6+a_7$$

Note that p is a **dummy** index; any letter could be used as the index. For example $\displaystyle\sum_{i=1}^{6} a_i$, and $\displaystyle\sum_{m=1}^{6} a_m$ each represent the same collection of terms: $a_1+a_2+a_3+a_4+a_5+a_6$.

In order to be able to use this 'summation notation' we need to obtain a suitable expression for the 'typical term' in the series. For example, the finite series

$$1^2+2^2+\cdots+k^2$$

may be written as $\displaystyle\sum_{p=1}^{k} p^2$ since the typical term is clearly p^2 in which $p=1, 2, 3, \ldots, k$ in turn.

In the same way

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{16}=\sum_{p=1}^{16}\frac{(-1)^{p+1}}{p}$$

since an expression for the typical term in this alternating harmonic series is $a_p=\dfrac{(-1)^{p+1}}{p}$.

Task

Write in summation form the series

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{21\times 22}$$

First find an expression for the typical term, "the p^{th} term":

Your solution

$a_p =$

Answer

$a_p = \dfrac{1}{p(p+1)}$

Now write the series in summation form:

Your solution

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{21\times 22} =$$

Answer

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots \qquad +\frac{1}{21\times 22} = \sum_{p=1}^{21}\frac{1}{p(p+1)}$$

Task

Write out all the terms of the series $\displaystyle\sum_{p=1}^{5}\frac{(-1)^p}{(p+1)^2}$.

Give p the values $1, 2, 3, 4, 5$ in the typical term $\dfrac{(-1)^p}{(p+1)^2}$:

Your solution

$$\sum_{p=1}^{5}\frac{(-1)^p}{(p+1)^2} =$$

Answer

$-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\dfrac{1}{5^2}-\dfrac{1}{6^2}$.

4. Summing series

The arithmetic series

Consider the finite **arithmetic series** with 14 terms

$$1 + 3 + 5 + \cdots + 23 + 25 + 27$$

A simple way of working out the value of the sum is to create a second series which is the first written in reverse order. Thus we have two series, each with the same value A:

$$A = 1 + 3 + 5 + \cdots + 23 + 25 + 27$$

and

$$A = 27 + 25 + 23 + \cdots + 5 + 3 + 1$$

Now, adding the terms of these series in pairs

$$2A = 28 + 28 + 28 + \cdots + 28 + 28 + 28 = 28 \times 14 = 392 \qquad \text{so} \quad A = 196.$$

We can use this approach to find the sum of n terms of a general arithmetic series.
If

$$A = [a] + [a + d] + [a + 2d] + \cdots + [a + (n-2)d] + [a + (n-1)d]$$

then again simply writing the terms in reverse order:

$$A = [a + (n-1)d] + [a + (n-2)d] + \cdots + [a + 2d] + [a + d] + [a]$$

Adding these two identical equations together we have

$$2A = [2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d]$$

That is, every one of the n terms on the right-hand side has the same value: $[2a + (n-1)d]$. Hence

$$2A = n[2a + (n-1)d] \qquad \text{so} \qquad A = \frac{1}{2}n[2a + (n-1)d].$$

Key Point 2

The **arithmetic series**

$$[a] + [a + d] + [a + 2d] + \cdots + [a + (n-1)d]$$

having n terms has sum A where:

$$A = \frac{1}{2}n[2a + (n-1)d]$$

As an example

$$1 + 3 + 5 + \cdots + 27 \quad \text{has} \quad a = 1,\ d = 2,\ n = 14$$

So $A = 1 + 3 + \cdots + 27 = \dfrac{14}{2}[2 + (13)2] = 196.$

The geometric series

We can also sum a general **geometric series**.
Let

$$G = a + ar + ar^2 + \cdots + ar^{n-1}$$

be a geometric series having exactly n terms. To obtain the value of G in a more convenient form we first multiply through by the common ratio r:

$$rG = ar + ar^2 + ar^3 + \cdots + ar^n$$

Now, writing the two series together:

$$G = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rG = ar + ar^2 + ar^3 + \cdots ar^{n-1} + ar^n$$

Subtracting the second expression from the first we see that all terms on the right-hand side cancel out, except for the first term of the first expression and the last term of the second expression so that

$$G - rG = (1 - r)G = a - ar^n$$

Hence (assuming $r \neq 1$)

$$G = \frac{a(1 - r^n)}{1 - r}$$

(Of course, if $r = 1$ the geometric series simplifies to a simple arithmetic series with $d = 0$ and has sum $G = na$.)

Key Point 3

The **geometric series**

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

having n terms has sum G where

$$G = \frac{a(1 - r^n)}{1 - r}, \quad \text{if } r \neq 1 \qquad \text{and} \qquad G = na, \quad \text{if } r = 1$$

Task

Find the sum of each of the following series:

(a) $1 + 2 + 3 + 4 + \cdots + 100$

(b) $\dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{18} + \dfrac{1}{54} + \dfrac{1}{162} + \dfrac{1}{486}$

(a) In this arithmetic series state the values of a, d, n:

Your solution

$a =$ $\qquad d =$ $\qquad n =$

Answer

$a = 1, \quad d = 1, \quad n = 100.$

Now find the sum:

Your solution

$1 + 2 + 3 + \cdots + 100 =$

Answer

$1 + 2 + 3 + \cdots + 100 = 50(2 + 99) = 50(101) = 5050.$

(b) In this geometric series state the values of a, r, n:

Your solution

$a =$ $\qquad r =$ $\qquad n =$

Answer

$a = \dfrac{1}{2}, \quad r = \dfrac{1}{3}, \quad n = 6$

Now find the sum:

Your solution

$\dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{18} + \dfrac{1}{54} + \dfrac{1}{162} + \dfrac{1}{486} =$

Answer

$$\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{486} = \frac{1}{2}\frac{\left(1 - \left(\frac{1}{3}\right)^6\right)}{1 - \frac{1}{3}} = \frac{3}{4}\left(1 - \left(\frac{1}{3}\right)^6\right) = 0.74897$$

Exercises

1. Which of the following sequences is convergent?

(a) $\sin\frac{\pi}{2},\ \sin\frac{2\pi}{2},\ \sin\frac{3\pi}{2},\ \sin\frac{4\pi}{2},\ \ldots$

(b) $\dfrac{\sin\frac{\pi}{2}}{\frac{\pi}{2}},\ \dfrac{\sin\frac{2\pi}{2}}{\frac{2\pi}{2}},\ \dfrac{\sin\frac{3\pi}{2}}{\frac{3\pi}{2}},\ \dfrac{\sin\frac{4\pi}{2}}{\frac{4\pi}{2}},\ \ldots$

2. Write the following series in summation form:

(a) $\dfrac{\ln 1}{2\times 1}+\dfrac{\ln 3}{3\times 2}+\dfrac{\ln 5}{4\times 3}+\cdots+\dfrac{\ln 27}{15\times 14}$

(b) $-\dfrac{1}{2\times(1+(100)^2)}+\dfrac{1}{3\times(1-(200)^2)}-\dfrac{1}{4\times(1+(300)^2)}+\cdots+\dfrac{1}{9\times(1-(800)^2)}$

3. Write out the first three terms and the last term of the following series:

(a) $\displaystyle\sum_{p=1}^{17}\frac{3^{p-1}}{p!(18-p)}$ (b) $\displaystyle\sum_{p=4}^{17}\frac{(-p)^{p+1}}{p(2+p)}$

4. Sum the series:

(a) $-5\ \ -1\ \ +3\ \ +7\ \ \ldots\ \ \ +27$

(b) $-5\ \ -9\ \ -13\ \ -17\ \ \ldots\ \ \ -37$

(c) $\dfrac{1}{2}-\dfrac{1}{6}+\dfrac{1}{18}-\dfrac{1}{54}+\dfrac{1}{162}-\dfrac{1}{486}$

Answers

1. (a) no; this sequence is $1,\ 0,\ -1,\ 0,\ 1,\ldots$ which does not converge.

(b) yes; this sequence is $\dfrac{1}{\pi/2},\ 0,\ -\dfrac{1}{3\pi/2},\ 0,\ \dfrac{1}{5\pi/2},\ldots$ which converges to zero.

2. (a) $\displaystyle\sum_{p=1}^{14}\frac{\ln(2p-1)}{(p+1)(p)}$ (b) $\displaystyle\sum_{p=1}^{8}\frac{(-1)^p}{(p+1)(1+(-1)^{p+1}p^2 10^4)}$

3. (a) $\dfrac{1}{17},\ \dfrac{3}{2!(16)},\ \dfrac{3^2}{3!(15)},\ \ldots\ ,\dfrac{3^{16}}{17!}$ (b) $-\dfrac{4^5}{(4)(6)},\ \dfrac{5^6}{(5)(7)},\ -\dfrac{6^7}{(6)(8)},\ \ldots\ ,\dfrac{17^{18}}{(17)(19)}$

4. (a) This is an arithmetic series with $a=-5,\ d=4,\ n=9.\quad A=99$

(b) This is an arithmetic series with $a=-5,\ d=-4,\ n=9.\quad A=-189$

(c) This is a geometric series with $a=\dfrac{1}{2},\ r=-\dfrac{1}{3},\ n=6.\quad G\approx 0.3745$