Implicit Differentiation

11.7

Introduction

This Section introduces implicit differentiation which is used to differentiate functions expressed in implicit form (where the variables are found together). Examples are $x^3 + xy + y^2 = 1$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which represents an ellipse.

Prerequisites

Before starting this Section you should . . .

- be able to differentiate standard functions
- be competent in using the chain rule

Learning Outcomes

On completion you should be able to . . .

- differentiate functions expressed implicitly

1. Implicit and explicit functions

Equations such as $y = x^2$, $y = \dfrac{1}{x}$, $y = \sin x$ are said to define y **explicitly** as a function of x because the variable y appears alone on one side of the equation.

The equation

$$yx + y + 1 = x$$

is not of the form $y = f(x)$ but can be put into this form by simple algebra.

Task

Write y as the subject of

$$yx + y + 1 = x$$

Your solution

Answer

We have $y(x+1) = x - 1$ so

$$y = \frac{x-1}{x+1}$$

We say that y is defined **implicitly** as a function of x by means of $yx+y+1 = x$, the actual function being given **explicitly** as

$$y = \frac{x-1}{x+1}$$

We note than an equation relating x and y can implicitly define **more than one** function of x.

For example, if we solve

$$x^2 + y^2 = 1$$

we obtain $y = \pm\sqrt{1-x^2}$ so $x^2 + y^2 = 1$ defines implicitly two functions

$$f_1(x) = \sqrt{1-x^2} \qquad f_2(x) = -\sqrt{1-x^2}$$

Task

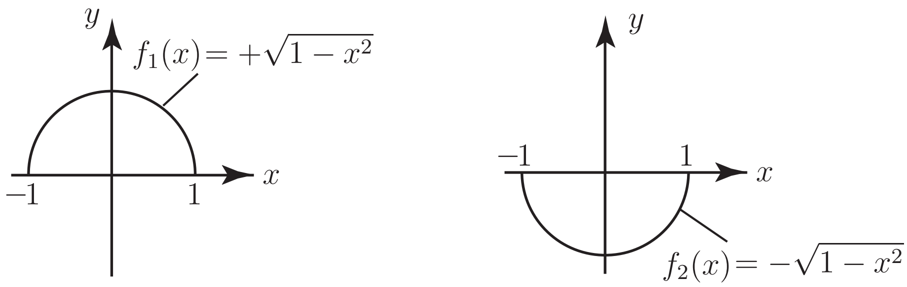
Sketch the graphs of $f_1(x) = \sqrt{1-x^2}$ $\quad f_2(x) = -\sqrt{1-x^2}$

(The equation $x^2 + y^2 = 1$ should give you the clue.)

Your solution

Answer

Since $x^2 + y^2 = 1$ is the well-known equation of the circle with centre at the origin and radius 1, it follows that the graphs of $f_1(x)$ and $f_2(x)$ are the upper and lower halves of this circle.



Sometimes it is difficult or even impossible to solve an equation in x and y to obtain y explicitly in terms of x.

Examples where explicit expressions for y cannot be obtained are

$$\sin(xy) = y \qquad x^2 + \sin y = 2y$$

2. Differentiation of implicit functions

Fortunately it is not necessary to obtain y in terms of x in order to **differentiate** a function defined implicitly.

Consider the simple equation

$$xy = 1$$

Here it **is** clearly possible to obtain y as the subject of this equation and hence obtain $\dfrac{dy}{dx}$.

Task Express y explicitly in terms of x and find $\frac{dy}{dx}$ for the case $xy = 1$.

Your solution

Answer

We have immediately

$$y = \frac{1}{x} \quad \text{so} \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

We now show an alternative way of obtaining $\frac{dy}{dx}$ which does **not** involve writing y explicitly in terms of x at the outset. We simply treat y as an (unspecified) function of x.

Hence if $xy = 1$ we obtain

$$\frac{d}{dx}(xy) = \frac{d}{dx}(1).$$

The right-hand side differentiates to zero as 1 is a constant. On the left-hand side we must use the **product** rule of differentiation:

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y\frac{dx}{dx} = x\frac{dy}{dx} + y$$

Hence $xy = 1$ becomes, after differentiation,

$$x\frac{dy}{dx} + y = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{y}{x}$$

In this case we can of course substitute $y = \frac{1}{x}$ to obtain

$$y = -\frac{1}{x^2}$$

as before.

The method used here is called **implicit differentiation** and, apart from the final step, it can be applied even if y cannot be expressed explicitly in terms of x. Indeed, on occasions, it is **easier** to differentiate implicitly even if an explicit expression is possible.

Example 15

Obtain the derivative $\frac{dy}{dx}$ where

$$x^2 + y = 1 + y^3$$

Solution

We begin by differentiating the left-hand side of the equation with respect to x to get:

$$\frac{d}{dx}(x^2 + y) = 2x + \frac{dy}{dx}.$$

We now differentiate the right-hand side of with respect to x. Using the chain (or function of a function) rule to deal with the y^3 term:

$$\frac{d}{dx}(1 + y^3) = \frac{d}{dx}(1) + \frac{d}{dx}(y^3) = 0 + 3y^2\frac{dy}{dx}$$

Now by equating the left-hand side and right-hand side derivatives, we have:

$$2x + \frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

We can make $\frac{dy}{dx}$ the subject of this equation:

$$\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -2x \quad \text{which gives} \quad \frac{dy}{dx} = \frac{2x}{3y^2 - 1}$$

We note that $\frac{dy}{dx}$ has to be expressed in terms of both x and y. This is quite usual if y cannot be obtained explicitly in terms of x. Now try this Task requiring implicit differentiation.

Task

Find $\frac{dy}{dx}$ if $2y = x^2 + \sin y$

Note that your answer will be in terms of both y and x.

Your solution

Answer

We have, on differentiating both sides of the equation with respect to x and using the chain rule on the $\sin y$ term:

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin y) \quad \text{i.e.} \quad 2\frac{dy}{dx} = 2x + \cos y\frac{dy}{dx} \qquad \text{leading to} \qquad \frac{dy}{dx} = \frac{2x}{2 - \cos y}.$$

We sometimes need to obtain the second derivative $\dfrac{d^2y}{dx^2}$ for a function defined implicitly.

Example 16

Obtain $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(4, 2)$ on the curve defined by the equation

$$x^2 - xy - y^2 - 2y = 0$$

Solution

Firstly we obtain $\dfrac{dy}{dx}$ by differentiating the equation implicitly and then evaluate it at $(4, 2)$.

We have $$2x - x\frac{dy}{dx} - y - 2y\frac{dy}{dx} - 2\frac{dy}{dx} = 0 \qquad (1)$$

from which $$\frac{dy}{dx} = \frac{2x - y}{x + 2y + 2} \qquad (2)$$

so at $(4, 2)$ $\dfrac{dy}{dx} = \dfrac{6}{10} = \dfrac{3}{5}$.

To obtain the second derivative $\dfrac{d^2y}{dx^2}$ it is easier to use (1) than (2) because the latter is a quotient. We simplify (1) first:

$$2x - y - (x + 2y + 2)\frac{dy}{dx} = 0 \qquad (3)$$

We will have to use the product rule to differentiate the third term here.

Hence differentiating (3) with respect to x:

$$2 - \frac{dy}{dx} - (x + 2y + 2)\frac{d^2y}{dx^2} - (1 + 2\frac{dy}{dx})\frac{dy}{dx} = 0$$

or

$$2 - 2\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2 - (x + 2y + 2)\frac{d^2y}{dx^2} = 0 \qquad (4)$$

Note carefully that the third term here, $\left(\dfrac{dy}{dx}\right)^2$, is the square of the first derivative. It should not be confused with the second derivative denoted by $\dfrac{d^2y}{dx^2}$.

Finally, at $(4, 2)$ where $\dfrac{dy}{dx} = \dfrac{3}{5}$ we obtain from (4): $\quad 2 - 2(\dfrac{3}{5}) - 2(\dfrac{9}{25}) - (4 + 4 + 2)\dfrac{d^2y}{dx^2} = 0$

from which $\quad \dfrac{d^2y}{dx^2} = \dfrac{1}{125}$ at $(4, 2)$.

Task

This Task involves finding a formula for the curvature of a bent beam. When a horizontal beam is acted on by forces which bend it, then each small segment of the beam will be slightly curved and can be regarded as an arc of a circle. The radius R of that circle is called the **radius of curvature** of the beam at the point concerned. If the shape of the beam is described by an equation of the form $y = f(x)$ then there is a formula for the radius of curvature R which involves only the first and second derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Find that equation as follows.

Start with the equation of a circle in the simple implicit form

$$x^2 + y^2 = R^2$$

and perform implicit differentiation twice. Now use the result of the first implicit differentiation to find a simple expression for the quantity $1 + (dy/dx)^2$ in terms of R and y; this can then be used to simplify the result of the second differentiation, and will lead to a formula for $\frac{1}{R}$ (called the **curvature**) in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Your solution

Answer

Differentiating: $x^2 + y^2 = R^2$ gives:

$$2x + 2y\frac{dy}{dx} = 0 \qquad (1)$$

Differentiating again: $$2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0 \qquad (2)$$

From (1)

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{y^2} = \frac{y^2 + x^2}{y^2} = \left(\frac{R}{y}\right)^2 \qquad (3)$$

So $1 + \left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{R}{y}\right)^2$.

Thus (2) becomes $$2\left(\frac{R}{y}\right)^2 + 2y\left(\frac{d^2y}{dx^2}\right) = 0 \qquad \therefore \qquad \frac{d^2y}{dx^2} = -\frac{R^2}{y^3} = -\left(\frac{1}{R}\right)\left(\frac{R}{y}\right)^3$$

so $$\frac{d^2y}{dx^2} = -\frac{1}{R}\left(\frac{R}{y}\right)^3 \qquad (4)$$

Rearranging (4) to make $\dfrac{1}{R}$ the subject and substituting for $\left(\dfrac{R}{y}\right)$ from (3) gives the result:

$$\frac{1}{R} = -\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

The equation usually found in textbooks omits the minus sign but the sign indicates whether the circle is above or below the curve, as you will see by sketching a few examples. When the gradient is small (as for a slightly deflected horizontal beam), i.e. $\dfrac{dy}{dx}$ is small, the denominator in the equation for $(1/R)$ is close to 1, and so the second derivative alone is often used to estimate the radius of curvature in the theory of bending beams.