

# Volume Integrals

## 27.3



### Introduction

In the previous two Sections, surface integrals (or double integrals) were introduced i.e. functions were integrated with respect to one variable and then with respect to another variable. It is often useful in engineering to extend the process to an integration with respect to three variables i.e. a volume integral or triple integral. Many of the processes and techniques involved in double integration are relevant to triple integration.



### Prerequisites

Before starting this Section you should ...

- have a thorough understanding of the various techniques of integration
- be familiar with the concept of a function of two variables
- have studied Sections 27.1 and 27.2 on double integration
- be able to visualise or sketch a function in three variables.



### Learning Outcomes

On completion you should be able to ...

- evaluate triple integrals

# 1. Example of volume integral: mass of water in a reservoir

Sections 27.1 and 27.2 introduced an example showing how the force on a dam can be represented by a double integral. Suppose, instead of the total force on the dam, an engineer wishes to find the total mass of water in the reservoir behind the dam. The mass of a little element of water (dimensions  $\delta x$  in length,  $\delta y$  in breadth and  $\delta z$  in height) with density  $\rho$  is given by  $\rho\delta z\delta y\delta x$  (i.e. the mass of the element is given by its density multiplied by its volume).

The density may vary at different parts of the reservoir e.g. due to temperature variations and the water expanding at higher temperatures. It is important to realise that the density  $\rho$  may be a function of all three variables,  $x$ ,  $y$  and  $z$ . For example, during the spring months, the depths of the reservoir may be at the cold temperatures of the winter while the parts of the reservoir nearer the surface may be at higher temperatures representing the fact that they have been influenced by the warmer air above; this represents the temperature varying with the vertical coordinate  $z$ . Also, the parts of the reservoir near where streams flow in may be extremely cold as melting snow flows into the reservoir. This represents the density varying with the horizontal coordinates  $x$  and  $y$ .

Thus the mass of a small element of water is given by  $\rho(x, y, z)\delta z\delta y\delta x$ . The mass of water in a column is given by the integral  $\int_{-h(x,y)}^0 \rho(x, y, z) dz\delta y\delta x$  where the level  $z = 0$  represents the surface of the reservoir and the function  $h(x, y)$  represents the depth of the reservoir for the particular values of  $x$  and  $y$  under consideration. [Note that the depth is positive but as it is measured downwards, it represents a negative value of  $z$ .]

The mass of water in a slice (aligned parallel to the  $x$ -axis) is given by integrating once more with respect to  $y$  i.e.  $\int_{y_1(x)}^{y_2(x)} \int_{-h(x,y)}^0 \rho(x, y, z) dzdy\delta x$ . Here the functions  $y_1(x)$  and  $y_2(x)$  represent the extreme values of  $y$  for the value of  $x$  under consideration.

Finally the total mass of water in the reservoir can be found by integrating over all  $x$  i.e.

$$\int_a^b \int_{y_1(x)}^{y_2(x)} \int_{-h(x,y)}^0 \rho(x, y, z) dzdydx.$$

To find the total mass of water, it is necessary to integrate the density three times, firstly with respect to  $z$  (between limits dependent on  $x$  and  $y$ ), then with respect to  $y$  (between limits which are functions of  $x$ ) and finally with respect to  $x$  (between limits which are constant).

This is an example of a triple or volume integral.

## 2. Evaluating triple integrals

A triple integral is an integral of the form

$$\int_a^b \int_{p(x)}^{q(x)} \int_{r(x,y)}^{s(x,y)} f(x, y, z) dzdydx$$

The evaluation can be split into an “inner integral” (the integral with respect to  $z$  between limits which are functions of  $x$  and  $y$ ), an “intermediate integral” (the integration with respect to  $y$  between limits which are functions of  $x$ ) and an “outer integral” (the integration with respect to  $x$  between

limits which are constants. Note that there is nothing special about the variable names  $x$ ,  $y$  and  $z$ : other variable names could have been used instead.

Triple integrals can be represented in different ways.  $\int_V f dV$  represents a triple integral where the  $dV$  is replaced by  $dx dy dz$  (or equivalent) and the limit of  $V$  on the integral is replaced by appropriate limits on the three integrals.

Note that the integral  $\int_V dV$  (i.e. integrating the function  $f(x, y, z) = 1$ ) gives the volume of the relevant shape. Hence the alternative name of volume integral.

One special case is where the limits on **all** the integrals are constants (a constant is, of course, a special case of a function). This represents an integral over a cuboidal region.



### Example 16

Consider a cube  $V$  of side 1.

(a) Express the integral  $\int_V f dV$  (where  $f$  is any function of  $x$ ,  $y$  and  $z$ ) as a triple integral.

(b) Hence evaluate  $\int_V (y^2 + z^2) dV$

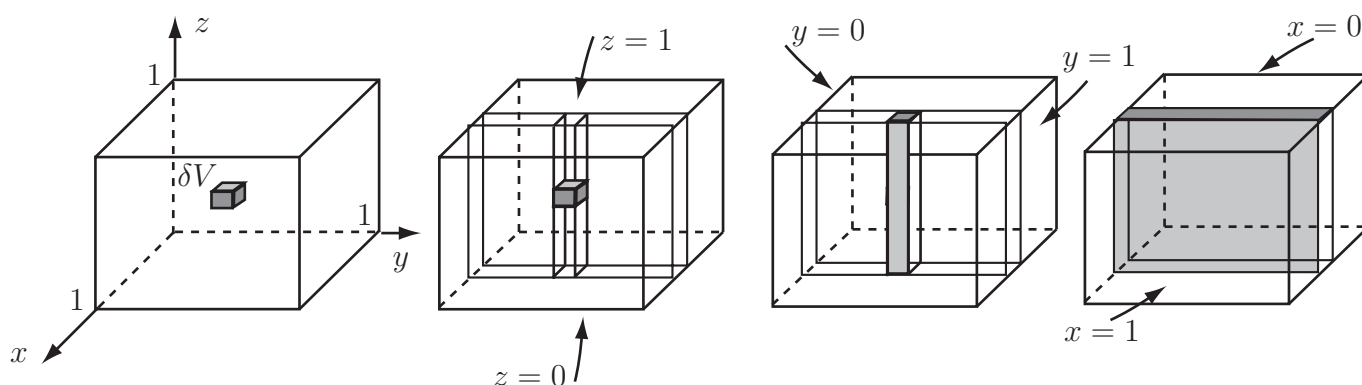


Figure 23

#### Solution

(a) Consider a little element of length  $dx$ , width  $dy$  and height  $dz$ . Then  $\delta V$  (the volume of the small element) is the product of these lengths  $dx dy dz$ . The function is integrated three times. The first integration represents the integral over the vertical strip from  $z = 0$  to  $z = 1$ . The second integration represents this strip sweeping across from  $y = 0$  to  $y = 1$  and is the integration over the slice that is swept out by the strip. Finally the integration with respect to  $x$  represents this slice sweeping from  $x = 0$  to  $x = 1$  and is the integration over the entire cube. The integral therefore becomes

$$\int_0^1 \int_0^1 \int_0^1 f(x, y, z) dz dy dx$$

### Solution (contd.)

(b) In the particular case where the function is  $f(x, y, z) = y^2 + z^2$ , the integral becomes

$$\int_0^1 \int_0^1 \int_0^1 (y^2 + z^2) dz dy dx$$

The inner integral is

$$\int_0^1 (y^2 + z^2) dz = \left[ y^2 z + \frac{1}{3} z^3 \right]_{z=0}^1 = y^2 \times 1 + \frac{1}{3} \times 1 - y^2 \times 0 - \frac{1}{3} \times 0 = y^2 + \frac{1}{3}$$

This inner integral is now placed into the intermediate integral to give

$$\int_0^1 \left( y^2 + \frac{1}{3} \right) dy = \left[ \frac{1}{3} y^3 + \frac{1}{3} y \right]_{y=0}^1 = \frac{1}{3} \times 1^3 + \frac{1}{3} \times 1 - \frac{1}{3} \times 0^3 - \frac{1}{3} \times 0 = \frac{2}{3}$$

Finally, this intermediate integral can be placed into the outer integral to give

$$\int_0^1 \frac{2}{3} dx = \left[ \frac{2}{3} x \right]_0^1 = \frac{2}{3} \times 1 - \frac{2}{3} \times 0 = \frac{2}{3}$$



### Example 17

Evaluate  $\int_0^1 \int_0^2 \int_0^3 8xyz dz dy dx$ . This represents an integral over the cuboid given by  $0 \leq x \leq 1$ ,  $0 \leq y \leq 2$ ,  $0 \leq z \leq 3$ .

### Solution

The inner integral is given by integrating the function with respect to  $z$  while keeping  $x$  and  $y$  constant.

$$\int_0^3 8xyz dz = \left[ 4xyz^2 \right]_0^3 = 4xy \times 9 - 0 = 36xy$$

This result is now integrated with respect to  $y$  while keeping  $x$  constant:

$$\int_0^2 36xy dy = \left[ 18xy^2 \right]_0^2 = 18x \times 4 - 0 = 72x$$

Finally, this result is integrated with respect to  $x$ :

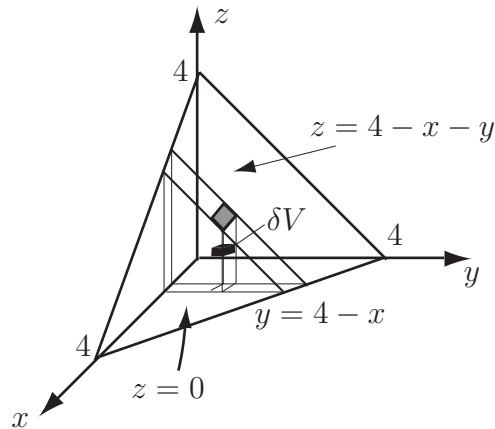
$$\int_0^1 72x dx = \left[ 36x^2 \right]_0^1 = 36 \times 1 - 0 = 36$$

Hence,  $\int_0^1 \int_0^2 \int_0^3 8xyz dz dy dx = 36$

More generally, the limits on the inner integral may be functions of the “intermediate” and “outer” variables and the limits on the intermediate integral may be functions of the “outer” variable.

**Example 18**

$V$  is the tetrahedron bounded by the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$  and  $x + y + z = 4$ . (see Figure 24).

**Figure 24**

- (a) Express  $\int_V f(x, y, z) dV$  (where  $f$  is a function of  $x$ ,  $y$  and  $z$ ) as a triple integral.
- (b) Hence find  $\int_V x dV$ .

**Solution**

The tetrahedron is divided into a series of slices parallel to the  $yz$ -plane and each slice is divided into a series of vertical strips. For each strip, the bottom is at  $z = 0$  and the top is on the plane  $x + y + z = 4$  i.e.  $z = 4 - x - y$ . So the integral up each strip is given by  $\int_{z=0}^{4-x-y} f(x, y, z) dz$  and this (inner) integral will be a function of  $x$  and  $y$ .

This, in turn, is integrated over all strips which form the slice. For each value of  $x$ , one end of the slice will be at  $y = 0$  and the other end at  $y = 4 - x$ . So the integral over the slice is  $\int_{y=0}^{4-x} \int_{z=0}^{4-x-y} f(x, y, z) dz dy$  and this (intermediate) integral will be a function of  $x$ .

Finally, integration is carried out over  $x$ . The limits on  $x$  are  $x = 0$  and  $x = 4$ . Thus the triple integral is  $\int_{x=0}^4 \int_{y=0}^{4-x} \int_{z=0}^{4-x-y} f(x, y, z) dz dy dx$  and this (outer) integral will be a constant.

$$\text{Hence } \int_V f(x, y, z) dV = \int_{x=0}^4 \int_{y=0}^{4-x} \int_{z=0}^{4-x-y} f(x, y, z) dz dy dx.$$

### Solution (contd.)

In the case where  $f(x, y, z) = x$ , the integral becomes

$$\begin{aligned}\int_V f(x, y, z) dV &= \int_{x=0}^4 \int_{y=0}^{4-x} \int_{z=0}^{4-x-y} x dz dy dx \\ &= \int_{x=0}^4 \int_{y=0}^{4-x} \left[ xz \right]_{z=0}^{4-x-y} dy dx \\ &= \int_{x=0}^4 \int_{y=0}^{4-x} [(4-x-y)x - 0] dy dx \\ &= \int_{x=0}^4 \int_{y=0}^{4-x} [4x - x^2 - xy] dy dx \\ &= \int_{x=0}^4 \left[ 4xy - x^2y - \frac{1}{2}xy^2 \right]_{y=0}^{4-x} dx \\ &= \int_{x=0}^4 \left[ 4x(4-x) - x^2(4-x) - \frac{1}{2}x(4-x)^2 - 0 \right]_{y=0}^{4-x} dx \\ &= \int_{x=0}^4 \left[ 16x - 4x^2 - 4x^2 + x^3 - 8x + 4x^2 - \frac{1}{2}x^3 \right] dx \\ &= \int_{x=0}^4 \left[ 8x - 4x^2 + \frac{1}{2}x^3 \right] dx \\ &= \left[ 4x^2 - \frac{4}{3}x^3 + \frac{1}{8}x^4 \right]_0^4 = 4 \times 4^2 - \frac{4}{3} \times 4^3 + \frac{1}{8} \times 4^4 - 0 \\ &= 64 - \frac{256}{3} + 32 \\ &= \frac{192 - 256 + 96}{3} \\ &= \frac{32}{3}\end{aligned}$$



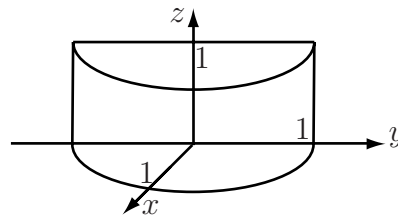
### Key Point 7

#### Triple Integration

The procedure for carrying out a triple integral is very similar to that for a double integral except that the procedure requires three stages rather than two.

**Example 19**

Find the integral of  $x$  over the shape shown in Figure 25. It represents half (positive  $x$ ) of a cylinder centered at  $x = y = 0$  with radius 1 and vertical extent from  $z = 0$  to  $z = 1$ .

**Figure 25****Solution**

In terms of  $x$ , the shape goes from  $x = 0$  to  $x = 1$ . For each value of  $x$ ,  $y$  goes from  $-\sqrt{1-x^2}$  to  $\sqrt{1-x^2}$ . The variable  $z$  varies from  $z = 0$  to  $z = 1$ . Hence the triple integral is

$$\begin{aligned} I &= \int_{x=0}^1 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^1 x \, dz dy dx \\ &= \int_{x=0}^1 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [xz]_{z=0}^1 dy dx = \int_{x=0}^1 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [x - 0] dy dx = \int_{x=0}^1 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x dy dx \\ &= \int_{x=0}^1 [xy]_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{x=0}^1 2x\sqrt{1-x^2} dx \end{aligned}$$

This outer integral can be evaluated by means of the substitution  $U = 1 - x^2$  i.e.  $dU = -2x dx$  and noting that  $U = 1$  when  $x = 0$  and  $U = 0$  when  $x = 1$  i.e.

$$I = \int_{x=0}^1 2x\sqrt{1-x^2} dx = - \int_1^0 U^{1/2} dU = \int_0^1 U^{1/2} dU = \left[ \frac{2}{3} U^{3/2} \right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$$

It is important to note that the three integrations can be carried out in whatever order is most convenient. The result does not depend on the order in which the integrals are carried out. However, when the order of the integrations is changed, it is necessary to consider carefully what the limits should be on each integration. Simply moving the limits from one integration to another will only work in the case of integration over a cuboid (i.e. where all limits are constants).



## Key Point 8

### Order of Integration for Triple Integrals

1. The three integrations can be carried out in whichever order is most convenient.
2. When changing the order of the integrations, it is important to reconsider the limits on each integration; a diagram can often help.



### Example 20

For the triangular prism in Figure 26, with ends given by the planes  $y = 0$  and  $y = 2$  and remaining faces given by the planes  $x = 0$ ,  $z = 0$  and  $x + 4z = 4$ , find the integral of  $x$  over the prism, by

- (a) Integrating first with respect to  $z$ , then  $y$  and finally  $x$ , and
- (b) Changing the order of the integrations to  $x$  first, then  $y$ , then  $z$ .

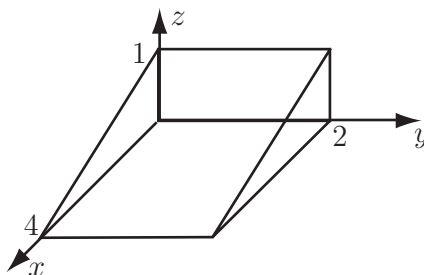


Figure 26

### Solution

For every value of  $x$  and  $y$ , the vertical coordinate  $z$  varies from  $z = 0$  to  $z = 1 - x/4$ . Hence the limits on  $z$  are  $z = 0$  and  $z = 1 - x/4$ . For every value of  $x$ , the limits on  $y$  are  $y = 0$  to  $y = 2$ . The limits on  $x$  are  $x = 0$  and  $x = 4$  (the limits on the figure). Hence the triple integral is

$\int_0^4 \int_0^2 \int_0^{1-x/4} x \, dz \, dy \, dx$  which can be evaluated as follows

$$\begin{aligned}
 I &= \int_0^4 \int_0^2 \int_0^{1-x/4} x \, dz \, dy \, dx = \int_0^4 \int_0^2 \left[ xz \right]_0^{1-x/4} \, dy \, dx \\
 &= \int_0^4 \int_0^2 \left[ x \left( 1 - \frac{x}{4} \right) - 0 \right] \, dy \, dx = \int_0^4 \int_0^2 \left( x - \frac{1}{4}x^2 \right) \, dy \, dx \\
 &= \int_0^4 \left[ \left( x - \frac{1}{4}x^2 \right) y \right]_0^2 \, dx = \int_0^4 \left[ \left( x - \frac{1}{4}x^2 \right) \times 2 - \left( x - \frac{1}{4}x^2 \right) \times 0 \right] \, dx \\
 &= \int_0^4 \left( 2x - \frac{1}{2}x^2 \right) \, dx = \left[ x^2 - \frac{1}{6}x^3 \right]_0^4 = 4^2 - \frac{1}{6} \times 4^3 - 0 = 16 - \frac{32}{3} = \frac{16}{3}
 \end{aligned}$$



**Solution (contd.)**

Now, if the order of the integrations is changed, it is necessary to re-derive the limits on the integrals. For every combination of  $y$  and  $z$ ,  $x$  varies between  $x = 0$  (left) and  $x = 4 - 4z$  (right). Hence the limits on  $x$  are  $x = 0$  and  $x = 4 - 4z$ . The limits on  $y$  are  $y = 0$  and  $y = 2$  (for all  $z$ ). The limits of  $z$  are  $z = 0$  (bottom) and  $z = 1$  (top).

So the triple integral becomes  $\int_0^1 \int_0^2 \int_0^{4-4z} x \, dx \, dy \, dz$  which can be evaluated as follows

$$\begin{aligned} I &= \int_0^1 \int_0^2 \int_0^{4-4z} x \, dx \, dy \, dz = \int_0^1 \int_0^2 \left[ \frac{1}{2} x^2 \right]_0^{4-4z} dy \, dz \\ &= \int_0^1 \int_0^2 \left[ \frac{1}{2} (4 - 4z)^2 \right] dy \, dz = \int_0^1 \int_0^2 (8 - 16z + 8z^2) dy \, dz \\ &= \int_0^1 \left[ (8 - 16z + 8z^2) y \right]_0^2 dz = 2 \int_0^1 (8 - 16z + 8z^2) dz \\ &= 2 \left[ 8z - 8z^2 + \frac{8}{3} z^3 \right]_0^1 = 2 \left( 8 - 8 + \frac{8}{3} - 0 \right) = \frac{16}{3} \end{aligned}$$

**Key Point 9****Limits of Integration**

While for different orders of integration the integral will always evaluate to the same value, the limits of integration will in general be different.



Evaluate the triple integral:  $\int_0^2 \int_0^3 \int_0^2 x^3 y^2 z \, dx \, dy \, dz$

**Your solution**

**Answer**

The inner integral is

$$\int_0^2 x^3 y^2 z \, dx = \left[ \frac{1}{4} x^4 y^2 z \right]_0^2 = \frac{1}{4} 2^4 y^2 z - 0 = 4y^2 z$$

This is put into the intermediate integral i.e.

$$\int_0^3 4y^2 z \, dy = \left[ \frac{4}{3} y^3 z \right]_0^3 = \frac{4}{3} 3^3 z - 0 = 36z$$

Finally, this is put in the outer integral to give

$$I = \int_0^2 36z \, dz = \left[ 18z^2 \right]_0^2 = 18 \times 2^2 - 0 = 72$$

**Exercises**

Evaluate the following triple integrals

$$1. \int_0^2 \int_0^x \int_0^{x+z} (x + y + z) \, dy dz dx$$

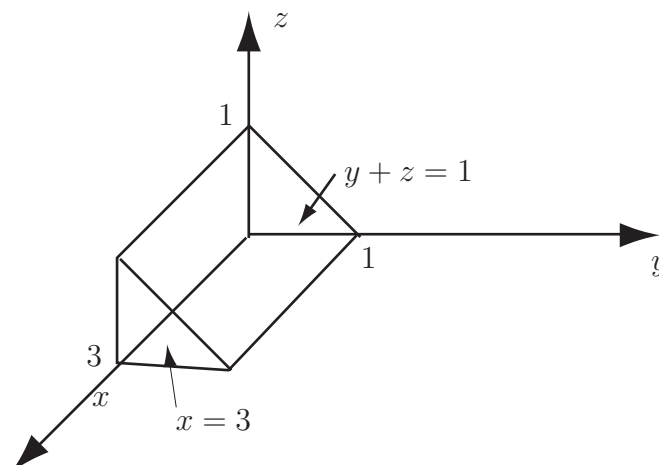
$$2. \int_2^4 \int_{-1}^3 \int_{x/2-2}^{2-x/2} (x + y) \, dz dy dx$$

**Answer**

$$1. 14 \quad 2. \frac{88}{3}$$



Find the volume of the solid prism shown in the diagram below. Check that when the order of integration is changed, the volume remains unaltered.



**Your solution****Answer**

The volume is given by the triple integral  $\iiint dV$ .

Putting  $z$  on the outer integral,  $y$  on the intermediate integral and  $x$  on the inner integral, the limits on  $z$  are  $z = 0$  to  $z = 1$ . For each value of  $z$ ,  $y$  varies from  $y = 0$  (base) to  $y = 1 - z$  on the sloping face. For each combination of  $y$  and  $z$ ,  $x$  varies from  $x = 0$  to  $x = 3$ . Thus, the volume is given by

$$\begin{aligned} V &= \int \int \int dV = \int_{z=0}^1 \int_{y=0}^{1-z} \int_{x=0}^3 dx dy dz \\ &= \int_{z=0}^1 \int_{y=0}^{1-z} \left[ x \right]_0^3 dy dz = \int_{z=0}^1 \int_{y=0}^{1-z} 3 dy dz \\ &= \int_{z=0}^1 \left[ 3y \right]_{y=0}^{1-z} dz = \int_{z=0}^1 (3(1-z) - 0) dz = \int_{z=0}^1 (3 - 3z) dz \\ &= \left[ 3z - \frac{3}{2}z^2 \right]_0^1 = 3 - \frac{3}{2} - (0 - 0) = \frac{3}{2} = 1.5 \end{aligned}$$

### Answers continued

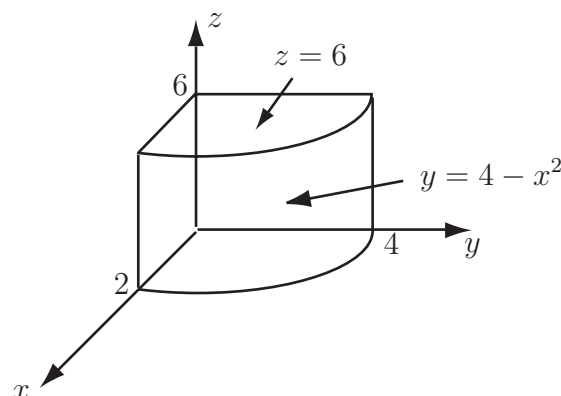
Now, the three integrations can be carried out in a different order. For example, with  $x$  on the outer integral,  $z$  on the intermediate integral and  $y$  on the inner integral, the limits on  $x$  are  $x = 0$  to  $x = 3$ ; for each value of  $x$ ,  $z$  varies from  $z = 0$  to  $z = 1$  and for each combination of  $x$  and  $z$ ,  $y$  varies from  $y = 0$  to  $y = 1 - z$ . The volume is therefore given by

$$\begin{aligned} V &= \int \int \int dV = \int_{x=0}^3 \int_{z=0}^1 \int_{y=0}^{1-z} dydzdx \\ &= \int_{x=0}^3 \int_{z=0}^1 \left[ y \right]_{y=0}^{1-z} dzdx \\ &= \int_{x=0}^3 \int_{z=0}^1 [1 - z] dzdx \\ &= \int_{x=0}^3 \left[ z - \frac{z^2}{2} \right]_{z=0}^1 dx \\ &= \int_{x=0}^3 \left[ 1 - \frac{1^2}{2} - 0 \right] dx \\ &= \int_{x=0}^3 \frac{1}{2} dx \\ &= \left[ \frac{x}{2} \right]_{x=0}^3 = \frac{3}{2} - 0 = 1.5 \end{aligned}$$

There are in all six ways (3!) to order the three integrations; each order gives the same answer of 1.5.

### Exercise

Find the volume of the solid shown in the diagram below. Check that when the order of integration is changed, the volume remains unaltered.



**Answer**

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### 3. Higher order integrals

A function may be integrated over four or more variables. For example, the integral

$$\int_{w=0}^1 \int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^{1-x} (w+y) dz dy dx dw$$

represents the function  $w + y$  being integrated over the variables  $w$ ,  $x$ ,  $y$  and  $z$ . This is an example of a quadruple integral.

The methods of evaluating quadruple integrals are very similar to those for double and triple integrals. Start the integration from the inside and gradually work outwards. Quintuple (five variable) and higher-order integrals also exist and the techniques are similar.



#### Example 21

Evaluate the quadruple integral  $\int_{w=0}^1 \int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^{1-x} (w+y) dz dy dx dw$ .

#### Solution

The first integral, with respect to  $z$  gives

$$\int_0^{1-x} (w+y) dz = \left[ (w+y)z \right]_0^{1-x} = (w+y)(1-x) - 0 = (w+y)(1-x).$$

The second integral, with respect to  $y$  gives

$$\int_0^1 (w+y)(1-x) dy = \left[ \left( wy + \frac{1}{2}y^2 \right) (1-x) \right]_0^1 = \left( w + \frac{1}{2} \right) (1-x) - 0 = \left( w + \frac{1}{2} \right) (1-x).$$

The third integral, with respect to  $x$  gives

$$\int_0^1 \left( w + \frac{1}{2} \right) (1-x) dx = \left[ \left( w + \frac{1}{2} \right) \left( x - \frac{x^2}{2} \right) \right]_0^1 = \left( w + \frac{1}{2} \right) \frac{1}{2} - 0 = \frac{1}{2} \left( w + \frac{1}{2} \right) = \frac{1}{2}w + \frac{1}{4}.$$

Finally, integrating with respect to  $w$  gives

$$\int_0^1 \left( \frac{1}{2}w + \frac{1}{4} \right) dw = \left[ \frac{1}{4}w^2 + \frac{1}{4}w \right]_0^1 = \frac{1}{4} + \frac{1}{4} - 0 = \frac{1}{2}$$

#### Exercise

Evaluate the quadruple integral  $\int_0^1 \int_{-1}^1 \int_{-1}^1 \int_0^{1-y^2} (x+y^2) dz dy dx dw$ .

#### Answer

$$\frac{8}{15}$$

## 4. Applications of triple and higher integrals

The integral  $\int \int \int f(x, y, z) dzdydx$  (or  $\int_V f(x, y, z) dV$ ) may represent many physical quantities depending on the function  $f(x, y, z)$  and the limits used.

### Volume

The integral  $\int_V 1 dV$  (i.e. the integral of the function  $f(x, y, z) = 1$ ) with appropriate limits gives the volume of the solid described by  $V$ . This is sometimes more convenient than finding the volume by means of a double integral.

### Mass

The integral  $\int \int \int \rho(x, y, z) dzdydx$  (or  $\int_V \rho(x, y, z) dV$ ), with appropriate limits, gives the mass of the solid bounded by  $V$ .

### Mass of water in a reservoir

The introduction to this Section concerned the mass of water in a reservoir. Imagine that the reservoir is rectangular in profile and that the width along the dam (i.e. measured in the  $x$  direction) is 100 m. Imagine also that the length of the reservoir (measured away from the dam i.e. in the  $y$  direction) is 400 m. The depth of the reservoir is given by  $40 - y/10$  m i.e. the reservoir is 40 m deep along the dam and the depth reduces to zero at the end away from the dam.

The density of the water can be approximated by  $\rho(z) = a - b \times z$  where  $a = 998 \text{ kg m}^{-3}$  and  $b = 0.05 \text{ kg m}^{-4}$ . I.e. at the surface ( $z = 0$ ) the water has density  $998 \text{ kg m}^{-3}$  (corresponding to a temperature of  $20^\circ\text{C}$ ) while 40 m down i.e.  $z = -40$ , the water has a density of  $1000 \text{ kg m}^{-3}$  (corresponding to the lower temperature of  $4^\circ\text{C}$ ).

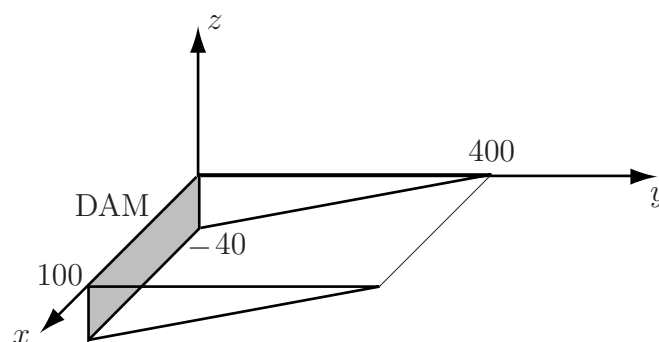


Figure 27

The mass of water in the reservoir is given by the integral of the function  $\rho(z) = a - b \times z$ . For each value of  $x$  and  $y$ , the limits on  $z$  will be from  $y/10 - 40$  (bottom) to 0 (top). Limits on  $y$  will be 0 to 400 m while the limits of  $x$  will be 0 to 100 m. The mass of water is therefore given by the integral

$$M = \int_0^{100} \int_0^{400} \int_{y/10-40}^0 (a - bz) dzdydx$$

which can be evaluated as follows

$$\begin{aligned}
 M &= \int_0^{100} \int_0^{400} \int_{y/10-40}^0 (a - bz) \, dz \, dy \, dx \\
 &= \int_0^{100} \int_0^{400} \left[ az - \frac{b}{2} z^2 \right]_{y/10-40}^0 \, dy \, dx \\
 &= \int_0^{100} \int_0^{400} \left[ 0 - a(y/10 - 40) + \frac{b}{2} (y/10 - 40)^2 \right] \, dy \, dx \\
 &= \int_0^{100} \int_0^{400} \left[ 40a - \frac{a}{10} y + \frac{b}{200} y^2 - 4by + 800b \right] \, dy \, dx \\
 &= \int_0^{100} \left[ 40ay - \frac{a}{20} y^2 + \frac{b}{600} y^3 - 2by^2 + 800by \right]_0^{400} \, dx \\
 &= \int_0^{100} \left[ 16000a - 8000a + \frac{320000}{3} b - 320000b + 320000b \right] \, dx \\
 &= \int_0^{100} \left[ 8000a + \frac{320000}{3} b \right] \, dx \\
 &= 8 \times 10^5 a + \frac{3.2}{3} \times 10^7 b = 7.984 \times 10^8 + \frac{0.16}{3} \times 10^7 = 7.989 \times 10^8 \text{ kg}
 \end{aligned}$$

So the mass of water in the reservoir is  $7.989 \times 10^8$  kg.

Notes :

1. In practice, the profile of the reservoir would not be rectangular and the depth would not vary so smoothly.
2. The variation of the density of water with height is only a minor factor so it would only be taken into account when a very exact answer was required. Assuming that the water had a uniform density of  $\rho = 998 \text{ kg m}^{-3}$  would give a total mass of  $7.984 \times 10^8$  kg while assuming a uniform density of  $\rho = 1000 \text{ kg m}^{-3}$  gives a total mass of  $8 \times 10^8$  kg.

## Centre of mass

The expressions for the centre of mass  $(\bar{x}, \bar{y}, \bar{z})$  of a solid of density  $\rho(x, y, z)$  are given below

$$\bar{x} = \frac{\int \rho(x, y, z) x \, dV}{\int \rho(x, y, z) \, dV} \quad \bar{y} = \frac{\int \rho(x, y, z) y \, dV}{\int \rho(x, y, z) \, dV} \quad \bar{z} = \frac{\int \rho(x, y, z) z \, dV}{\int \rho(x, y, z) \, dV}$$

In the (fairly common) case where the density  $\rho$  does not vary with position and is constant, these results simplify to

$$\bar{x} = \frac{\int x \, dV}{\int dV} \quad \bar{y} = \frac{\int y \, dV}{\int dV} \quad \bar{z} = \frac{\int z \, dV}{\int dV}$$



### Example 22

A tetrahedron is enclosed by the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$  and  $x + y + z = 4$ . Find (a) the volume of this tetrahedron, (b) the position of the centre of mass.

#### Solution

(a) Note that this tetrahedron was considered in Example 18, see Figure 24. It was shown that in this case the volume integral  $\int_V f(x, y, z) dV$  becomes  $\int_{x=0}^4 \int_{y=0}^{4-x} \int_{z=0}^{4-x-y} f(x, y, z) dz dy dx$ . The volume is given by

$$\begin{aligned} V &= \int_V dV = \int_{x=0}^4 \int_{y=0}^{4-x} \int_{z=0}^{4-x-y} dz dy dx \\ &= \int_{x=0}^4 \int_{y=0}^{4-x} \left[ z \right]_{z=0}^{4-x-y} dy dx \\ &= \int_{x=0}^4 \int_{y=0}^{4-x} (4 - x - y) dy dx \\ &= \int_{x=0}^4 \left[ 4y - xy - \frac{1}{2}y^2 \right]_{y=0}^{4-x} dx \\ &= \int_{x=0}^4 \left[ 8 - 4x + \frac{1}{2}x^2 \right] dx \\ &= \left[ 8x - 2x^2 + \frac{1}{6}x^3 \right]_0^4 = 32 - 32 + \frac{64}{6} = \frac{32}{3} \end{aligned}$$

Thus the volume of the tetrahedron is  $\frac{32}{3} \approx 10.3$

(b) The  $x$  coordinate of the centre of mass i.e.  $\bar{x}$  is given by  $\bar{x} = \frac{\int x dV}{\int dV}$ .

The denominator  $\int dV$  is the formula for the volume i.e.  $\frac{32}{3}$  while the numerator  $\int x dV$  was calculated in an earlier Example to be  $\frac{32}{3}$ .

$$\text{Thus } \bar{x} = \frac{\int x dV}{\int dV} = \frac{32/3}{32/3} = 1.$$

By symmetry (or by evaluating relevant integrals), it can be shown that  $\bar{y} = \bar{z} = 1$  i.e. the centre of mass is at  $(1, 1, 1)$ .



## Moment of inertia

The moment of inertia  $I$  of a particle of mass  $M$  about an axis  $PQ$  is defined as

$$I = \text{Mass} \times \text{Distance}^2 \quad \text{or} \quad I = Md^2$$

where  $d$  is the perpendicular distance from the particle to the axis.

To find the moment of inertia of a larger object, it is necessary to carry out a volume integration over all such particles. The distance of a particle at  $(x, y, z)$  from the  $z$ -axis is given by  $\sqrt{x^2 + y^2}$  so the moment of inertia of an object about the  $z$ -axis is given by

$$I_z = \int_V \rho(x, y, z)(x^2 + y^2) dz$$

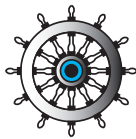
Similarly, the moments of inertia about the  $x$ -axis and  $y$ -axis are given by

$$I_x = \int_V \rho(x, y, z)(y^2 + z^2) dx \quad \text{and} \quad I_y = \int_V \rho(x, y, z)(x^2 + z^2) dy$$

In the case where the density is constant over the object, so  $\rho(x, y, z) = \rho$ , these formulae reduce to

$$I_x = \rho \int_V (y^2 + z^2) dx, \quad I_y = \rho \int_V (x^2 + z^2) dy \quad \text{and} \quad I_z = \rho \int_V (x^2 + y^2) dz$$

When possible, the moment of inertia is expressed in terms of  $M$ , the mass of the object.



### Example 23

Find the moment of inertia (about the  $x$ -axis) of the cube of side 1, mass  $M$  and density  $\rho$  shown in Example 16, page 43.

#### Solution

For the cube,

$$\text{Mass} = \text{Volume} \times \text{Density i.e.} \quad M = 1^3 \times \rho = \rho$$

The moment of inertia (about the  $x$ -axis) is given by

$$I_x = \rho \int_V (y^2 + z^2) dx = \rho \int_0^1 \int_0^1 \int_0^1 (y^2 + z^2) dz dy dx$$

This integral was shown to equal  $\frac{2}{3}$  in Example 16. Thus

$$I_x = \frac{2}{3}\rho = \frac{2}{3}M$$

By applying symmetry, it can also be shown that the moments of inertia about the  $y$ - and  $z$ -axes are also equal to  $\frac{2}{3}M$ .



## Engineering Example 2

### Radioactive decay

#### Introduction

A cube of an impure radioactive ore is of side 10 cm. The number of radioactive decays taking place per cubic metre per second is given by  $R = 10^{23}(0.1 - z)e^{-t/1000}$ . The dependence on time represents a half-life of 693 seconds while the dependence on the vertical coordinate  $z$  represents some gravitational stratification. The value  $z = 0$  represents the bottom of the cube and  $z = 0.1$  represents the top of the cube. (Note that the dimensions are in metres so 10 cm becomes 0.1 m.)

What is the total number of decays taking place over the cube in the 100 seconds between  $t = 0$  and  $t = 100$ ?

#### Solution

The total number of decays is given by the quadruple integral

$$N = \int_{x=0}^{0.1} \int_{y=0}^{0.1} \int_{z=0}^{0.1} \int_{t=0}^{100} 10^{23}(0.1 - z)e^{-t/1000} dt dz dy dx$$

which may be evaluated as follows

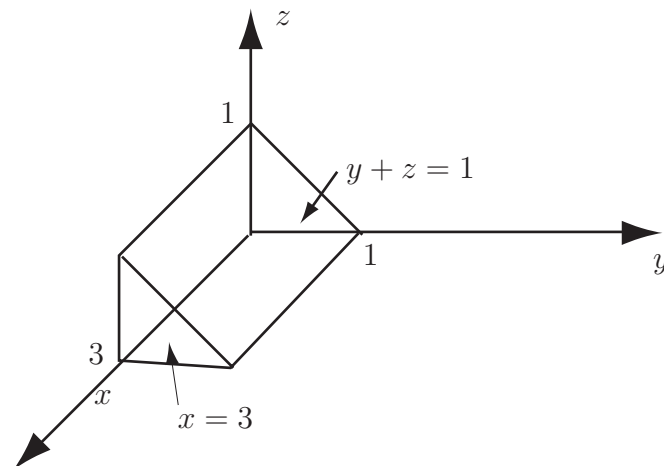
$$\begin{aligned} N &= \int_{x=0}^{0.1} \int_{y=0}^{0.1} \int_{z=0}^{0.1} \int_{t=0}^{100} 10^{23}(0.1 - z)e^{-t/1000} dt dz dy dx \\ &= \int_{x=0}^{0.1} \int_{y=0}^{0.1} \int_{z=0}^{0.1} \left[ -1000 \times 10^{23}(0.1 - z)e^{-t/1000} \right]_{t=0}^{100} dz dy dx \\ &= \int_{x=0}^{0.1} \int_{y=0}^{0.1} \int_{z=0}^{0.1} [10^{26}(0.1 - z)(1 - e^{-0.1})] dz dy dx \\ &= \int_{x=0}^{0.1} \int_{y=0}^{0.1} \int_{z=0}^{0.1} [9.5 \times 10^{24}(0.1 - z)] dz dy dx \\ &= 9.5 \times 10^{24} \int_{x=0}^{0.1} \int_{y=0}^{0.1} \left[ (0.1z - 0.5z^2) \right]_{z=0}^{0.1} dy dx \\ &= 9.5 \times 10^{24} \int_{x=0}^{0.1} \int_{y=0}^{0.1} [0.005] dy dx \\ &= 0.005 \times 9.5 \times 10^{24} \int_{x=0}^{0.1} \int_{y=0}^{0.1} dy dx \\ &= 0.005 \times 9.5 \times 10^{24} \times 0.1 \times 0.1 = 4.75 \times 10^{20} \end{aligned}$$

Thus the number of decays is approximately equal to  $4.75 \times 10^{20}$



For the solid prism shown below (the subject of the Task on page 50) find

- the coordinates of the centre of mass
- the moment of inertia about the  $x$ -,  $y$ - and  $z$ -axes.



**Your solution**

(a)

**Answer**

The  $x$ ,  $y$  and  $z$  coordinates of the centre of mass of a solid of constant density are given on page 55 by

$$\bar{x} = \frac{\int x \, dV}{\int dV} \quad \bar{y} = \frac{\int y \, dV}{\int dV} \quad \bar{z} = \frac{\int z \, dV}{\int dV}$$

For the triangular prism, the task on page 50 showed that the denominator  $\int dV$  has value 1.5. The numerator of the expression for  $\bar{x}$  is given by

$$\begin{aligned} \int x \, dV &= \int_{z=0}^1 \int_{y=0}^{1-z} \int_{x=0}^3 x \, dx \, dy \, dz = \int_{z=0}^1 \int_{y=0}^{1-z} \left[ \frac{x^2}{2} \right]_0^3 \, dy \, dz = \int_{z=0}^1 \int_{y=0}^{1-z} \frac{9}{2} \, dy \, dz \\ &= \int_{z=0}^1 \left[ \frac{9}{2} y \right]_{y=0}^{1-z} \, dz = \int_{z=0}^1 \left( \frac{9}{2}(1-z) - 0 \right) \, dz = \int_{z=0}^1 \left( \frac{9}{2} - \frac{9}{2}z \right) \, dz \\ &= \left[ \frac{9}{2}z - \frac{9}{4}z^2 \right]_0^1 = \frac{9}{2} - \frac{9}{4} - (0 - 0) = \frac{9}{4} = 2.25 \end{aligned}$$

So,  $\bar{x} = \frac{2.25}{1.5} = 1.5$ . By similar integration it can be shown that  $\bar{y} = \frac{1}{3}$ ,  $\bar{z} = \frac{1}{3}$ .

**Your solution**

(b)

**Answer**

The moment of inertia about the  $x$ -axis,  $I_x$  is given by  $I_x = \rho \int_V (y^2 + z^2) dV$  which for the solid under consideration is given by

$$\begin{aligned} I_x &= \rho \int_{x=0}^3 \int_{y=0}^1 \int_{z=0}^{1-y} (y^2 + z^2) dz dy dx = \rho \int_{x=0}^3 \int_{y=0}^1 \left( y^2 - y^3 + \frac{(1-y)^3}{3} \right) dy dx \\ &= \rho \int_{x=0}^3 \frac{1}{6} dx = \frac{1}{2} \rho \end{aligned}$$

Now, the mass  $M$  of the solid is given by  $M = \rho \times \text{Volume} = \frac{3}{2} \rho$  (where the volume had been calculated in a previous example) so

$$I_x = \frac{1}{2} \rho = \frac{1}{2} \rho \times \frac{M}{\frac{3}{2} \rho} = \frac{1}{3} M$$

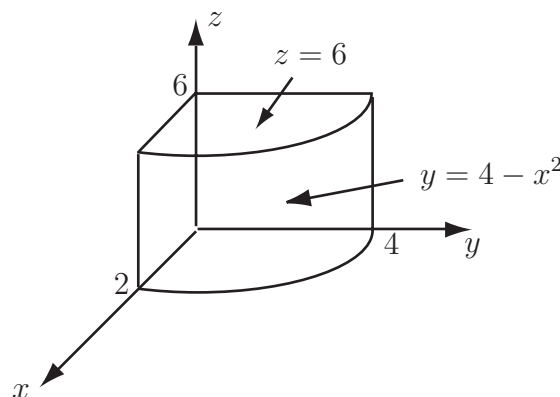
Similarly, the moment of inertia about the  $y$ -axis,  $I_y$  is given by  $I_y = \rho \int_V (x^2 + z^2) dV$  which for the solid under consideration is given by

$$\begin{aligned} I_x &= \rho \int_{x=0}^3 \int_{y=0}^1 \int_{z=0}^{1-y} (x^2 + z^2) dz dy dx = \rho \int_{x=0}^3 \int_{y=0}^1 \left( x^2(1-y) + \frac{(1-y)^3}{3} \right) dy dx \\ &= \rho \int_{x=0}^3 \left( \frac{1}{2} x^2 + \frac{1}{12} \right) dx = \frac{19}{4} \rho \end{aligned}$$

and so  $I_y = \frac{19}{4} \rho = \frac{19}{4} \rho \times \frac{M}{\frac{3}{2} \rho} = \frac{19}{6} M$ . Finally, by symmetry,  $I_z = I_y = \frac{19}{6} M$ .

**Exercise**

For the solid shown below (the subject of the Task on page 47) find the centre of mass and the moment of inertia about the  $x$ -,  $y$ - and  $z$ -axes.


**Answer**

$$(\bar{x}, \bar{y}, \bar{z}) = (0.75, 1.6, 3) \quad I_x = 15.66M \quad I_y = 12.8M \quad I_z = 4.46M$$



A cube of side 2 is made of laminated material so that, with the origin at one corner, the density of the material is  $kx$ .

(a) First find the mass  $M$  of the cube:

**Your solution**

**Answer**

The integrations over the cube are of the form  $\int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 dV$ .

The mass  $M$  is given by

$$\begin{aligned} M &= \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 \rho \, dzdydx \\ &= \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 kx \, dzdydx \\ &= \int_{x=0}^2 \int_{y=0}^2 2kx \, dydx = \int_{x=0}^2 4kx \, dx = \left[ 2kx^2 \right]_0^2 = 8k \end{aligned}$$

(b) Now find the position of the centre of mass of the cube:

### Your solution

### Answer

The  $x$ -coordinate of the centre of mass will be given by  $\frac{\int \rho x dV}{M}$  where the numerator is given by

$$\begin{aligned} \int \rho x dV &= \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 \rho x dz dy dx = \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 kx^2 dz dy dx \\ &= \int_{x=0}^2 \int_{y=0}^2 2kx^2 dy dx = \int_{x=0}^2 4kx^2 dx = \left[ \frac{4}{3} kx^3 \right]_0^2 = \frac{32}{3} k \end{aligned}$$

$$\text{So } \bar{x} = \frac{\frac{32}{3} k}{8k} = \frac{4}{3}.$$

The  $y$ -coordinate of the centre of mass is given by  $\frac{\int \rho y dV}{M}$  where the numerator is given by

$$\begin{aligned} \int \rho y dV &= \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 \rho y dz dy dx = \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 kxy dz dy dx \\ &= \int_{x=0}^2 \int_{y=0}^2 2kxy dy dx = \int_{x=0}^2 4kx dx = \left[ 2kx^2 \right]_0^2 = 8k \end{aligned}$$

$$\text{So } \bar{y} = \frac{8k}{8k} = 1.$$

By symmetry (the density depends only on  $x$ ),  $\bar{z} = \bar{y} = 1$ .

The coordinates of the centre of mass are  $\left(\frac{4}{3}, 1, 1\right)$ .

(c) Finally find the moments of inertia about the  $x$ -,  $y$ - and  $z$ -axes:

**Your solution**



**Answer**

The moment of inertia about the  $x$ -axis is given by  $I_x = \int_V \rho (y^2 + z^2) dV$  (page 58). In this case,

$$\begin{aligned}
 I_x &= \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 kx(y^2 + z^2) dz dy dx \\
 &= \int_{x=0}^2 \int_{y=0}^2 \left[ kx(y^2 z + \frac{1}{3} z^3) \right]_{z=0}^2 dy dx = \int_{x=0}^2 \int_{y=0}^2 kx(2y^2 + \frac{8}{3}) dy dx \\
 &= \int_{x=0}^2 \left[ kx(\frac{2}{3} y^3 + \frac{8}{3} y) \right]_{y=0}^2 dx = \int_{x=0}^2 kx(\frac{32}{3}) dx \\
 &= \left[ \frac{16}{3} kx^2 \right]_0^2 = \frac{64}{3} k = \frac{8}{3} M
 \end{aligned}$$

where the last step involves substituting that the mass  $M = 8k$ .

Similarly, the moment of inertia about the  $y$ -axis is given by  $I_y = \int_V \rho (x^2 + z^2) dV$  i.e.

$$\begin{aligned}
 I_y &= \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 kx(x^2 + z^2) dz dy dx \\
 &= \int_{x=0}^2 \int_{y=0}^2 \int_{z=0}^2 k(x^3 + xz^2) dz dy dx = \int_{x=0}^2 \int_{y=0}^2 \left[ k(x^3 z + \frac{1}{3} xz^3) \right]_{z=0}^2 dy dx \\
 &= \int_{x=0}^2 \int_{y=0}^2 \left( k(2x^3 + \frac{8}{3} x) \right) dy dx = \int_{x=0}^2 \left( k(4x^3 + \frac{16}{3} x) \right) dx \\
 &= \left[ k(x^4 + \frac{8}{3} x^2) \right]_{x=0}^2 = k(16 + \frac{32}{3}) = \frac{80}{3} k = \frac{10}{3} M
 \end{aligned}$$

By symmetry,  $I_z = I_y = \frac{10}{3} M$ .