Solution Using Fourier Series

25.4

Introduction

In this Section we continue to use the separation of variables method for solving PDEs but you will find that, to be able to fit certain boundary conditions, Fourier series methods have to be used leading to the final solution being in the (rather complicated) form of an infinite series. The techniques will be illustrated using the two-dimensional Laplace equation but similar situations often arise in connection with other important PDEs.

Prerequisites

Before starting this Section you should ...

- be familiar with the separation of variables method
- be familiar with trigonometric Fourier series

Learning Outcomes

On completion you should be able to ...

- solve the 2-D Laplace equation for given boundary conditions and utilize Fourier series in the solution when necessary

1. Solutions involving infinite Fourier series

We shall illustrate this situation using Laplace's equation but infinite Fourier series can also be necessary for the heat conduction and wave equations.

We recall from the previous Section that using a product solution

$$u(x,t) = X(x)Y(y)$$

in Laplace's equation gives rise to the ODEs:

$$\frac{X''}{X} = K \qquad \frac{Y''}{Y} = -K$$

To determine the sign of K and hence the appropriate solutions for $X(x)$ and $Y(y)$ we must impose appropriate boundary conditions. We will investigate solving Laplace's equation in the square

$$0 \leq x \leq \ell \qquad 0 \leq y \leq \ell$$

for the boundary conditions $u(x,0) = 0 \quad u(0,y) = 0 \quad u(\ell,y) = 0 \quad u(x,\ell) = U_0$, a constant.
See Figure 7.

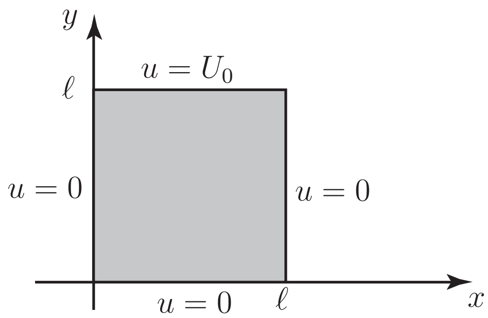


Figure 7

(a) We must first deduce the sign of the separation constant K:
if K is chosen to be positive say $K = \lambda^2$, then the X equation is

$$X'' = \lambda^2 X$$

with general solution

$$X = Ae^{\lambda x} + Be^{-\lambda x}$$

while the Y equation becomes

$$Y'' = -\lambda^2 Y$$

with general solution

$$Y = C\cos\lambda y + D\sin\lambda y$$

If the sign of K is negative $K = -\lambda^2$ the solutions will change to trigonometric in x and exponential in y.

These are the only two possibilities when we solve Laplace's equation using separation of variables and we must look at the boundary conditions of the problem to decide which is appropriate.
Here the boundary conditions are periodic in x (since $u(0,y) = u(\ell,y)$) and non-periodic in y which suggests we need a solution that is periodic in x and non-periodic in y.

Thus we choose $K = -\lambda^2$ to give

$$X(x) = (A\cos\lambda x + B\sin\lambda x)$$
$$Y(y) = (Ce^{\lambda y} + De^{-\lambda y})$$

(Note that had we chosen the incorrect sign for K at this stage we would later have found it impossible to satisfy all the given boundary conditions. You might like to verify this statement.)

The appropriate general solution of Laplace's equation for the given problem is

$$u(x, y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}).$$

(b) Inserting the boundary conditions produces the following consequences:

$$u(0, y) = 0 \qquad \text{gives} \qquad A = 0$$
$$u(\ell, y) = 0 \qquad \text{gives} \qquad \sin\lambda\ell = 0 \qquad \text{i.e. } \lambda = \frac{n\pi}{\ell}$$

where n is a positive integer $1, 2, 3, \ldots$. While $n = 0$ also satisfies the equation it leads to the trivial solution $u = 0$ only.)

$$u(x, 0) = 0 \ \text{ gives } \ C + D = 0 \ \text{ i.e. } \ D = -C$$

At this point the solution can be written

$$u(x, y) = BC\sin\left(\frac{n\pi x}{\ell}\right)\left(e^{\frac{n\pi y}{\ell}} - e^{-\frac{n\pi y}{\ell}}\right)$$

This can be conveniently written as

$$u(x, y) = E\sin\left(\frac{n\pi x}{\ell}\right)\sinh\left(\frac{n\pi y}{\ell}\right) \qquad (1)$$

where $E = 2BC$.

At this stage we have just one final boundary condition to insert to obtain information about the constant E and the integer n. Our solution (1) gives

$$u(x, \ell) = E\sin\left(\frac{n\pi x}{\ell}\right)\sinh(n\pi)$$

and clearly this is not compatible, as it stands, with the given boundary condition

$$u(x, \ell) = U_0 = \text{constant}.$$

The way to proceed is again to **superpose** solutions of the form (1) for all positive integer values of n to give

$$u(x, y) = \sum_{n=1}^{\infty} E_n\sin\left(\frac{n\pi x}{\ell}\right)\sinh\left(\frac{n\pi y}{\ell}\right) \qquad (2)$$

from which the final boundary condition gives

$$\begin{aligned} U_0 &= \sum_{n=1}^{\infty} E_n\sin\left(\frac{n\pi x}{\ell}\right)\sinh(n\pi) \qquad 0 < x < \ell. \qquad (3) \\ &= \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{\ell}\right) \end{aligned}$$

What we have here is a Fourier (sine) series for the function

$$f(x) = U_0 \qquad 0 < x < \ell.$$

Recalling the work on half-range Fourier series (HELM 23.5) we must extend this definition to produce an odd function with period 2ℓ. Hence we define

$$f(x) = \begin{cases} U_0 & 0 < x < \ell \\ -U_0 & -\ell < x < 0 \end{cases}$$

$$f(x + \ell) = f(x)$$

illustrated in Figure 8.

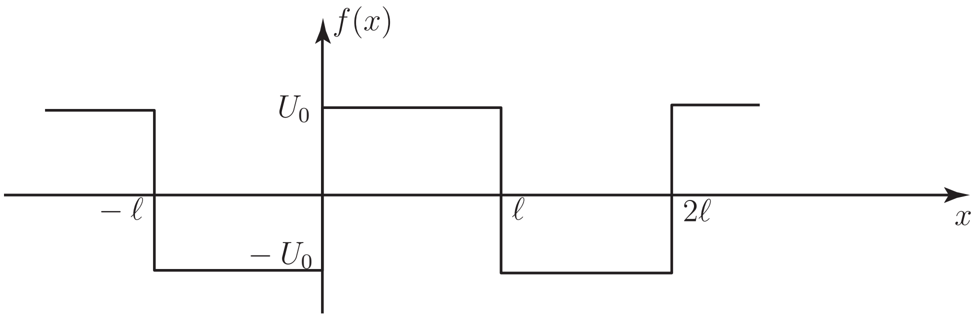


Figure 8

(c) We can now apply standard Fourier series theory to evaluate the Fourier coefficients b_n in (3).

We obtain

$$b_n = E_n \sinh n\pi = \frac{4U_0}{2\ell} \int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right)\, dx$$

(Recall that, in general, $b_n = 2 \times$ the mean value of $f(x) \sin\left(\frac{n\pi x}{\ell}\right)$ over a period. Here, because $f(x)$ is odd, and hence $f(x) \sin\left(\frac{n\pi x}{\ell}\right)$ is even, we may take half the period for our averaging process.)

Carrying out the integration

$$E_n \sinh n\pi = \frac{2U_0}{n\pi}(1 - \cos n\pi) \qquad \text{i.e.} \qquad E_n = \begin{cases} \dfrac{4U_0}{n\pi \sinh n\pi} & n = 1, 3, 5, \ldots \\ 0 & n = 2, 4, 6, \ldots \end{cases}$$

(Since $f(x)$ is a square wave with half-period symmetry we are not surprised that only odd harmonics arise in the Fourier series.)

Finally substituting these results for E_n into (2) we obtain the solution to the given problem as the infinite series:

$$u(x, y) = \frac{4U_0}{\pi} \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{\sin\left(\frac{n\pi x}{\ell}\right) \sinh\left(\frac{n\pi y}{\ell}\right)}{n \sinh n\pi}$$

Task

Solve Laplace's equation to determine the steady state temperature $u(x, y)$ in the semi-infinite plate $0 \leq x \leq 1, \quad y \geq 0$. Assume that the left and right sides are insulated and assume that the solution is bounded. The temperature along the bottom side is a known function $f(x)$.

First write this problem as a mathematical boundary value problem paying particular attention to the mathematical representation of the boundary conditions:

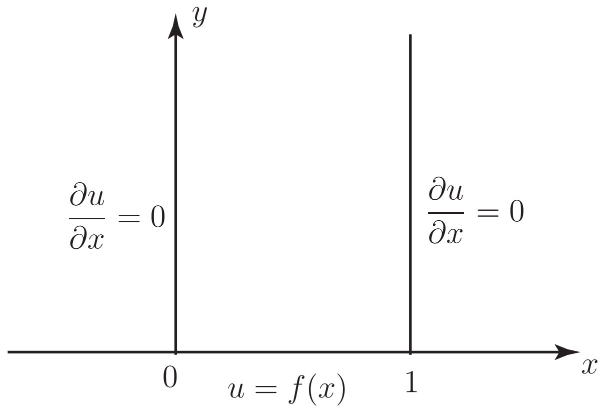
Your solution

Answer

Since the sides $x = 0$ and $x = 1$ are insulated, the temperature gradient across these sides is zero i.e. $\dfrac{\partial u}{\partial x} = 0$ for $x = 0$, $0 < y < \infty$ and $\dfrac{\partial u}{\partial x} = 0$ for $x = 1$, $0 < y < \infty$.

The third boundary condition is $u(x, 0) = f(x)$.

The fourth boundary condition is less obvious: since the solution should be bounded (ie not grow and grow) we must demand that $u(x, y) \to 0$ as $y \to \infty$. (See figure below.)



Now use the separation of variables method, putting $u(x,y) = X(x)Y(y)$, to find the differential equations satisfied by $X(x)$, $Y(y)$ and decide on the sign of the separation constant K:

Your solution

Answer
We have boundary conditions which, like the worked example above, are periodic in x. Hence the differential equations are, again,

$$X'' = -\lambda^2 X \qquad Y'' = +\lambda^2 Y$$

putting the separation constant K as $-\lambda^2$.

Write down the solutions for X, for Y and hence the product solution $u(x,y) = X(x)Y(y)$:

Your solution

Answer
$X = A\cos\lambda x + B\sin\lambda x \qquad Y = Ce^{\lambda y} + De^{-\lambda y}$

so

$$u = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \qquad (4)$$

Impose the derivative boundary conditions on this solution:

Your solution

Answer

$$\frac{\partial u}{\partial x} = (-\lambda A \sin \lambda x + \lambda B \cos \lambda x)(Ce^{\lambda y} + De^{-\lambda y})$$

Hence $\dfrac{\partial u}{\partial x}(0, y) = 0$ gives $\lambda B(Ce^{\lambda y} + De^{-\lambda y}) = 0$ for all y.

The possibility $\lambda = 0$ can be excluded this would give a trivial constant solution in (4). Hence we must choose $B = 0$.

The condition $\dfrac{\partial u}{\partial x}(1, y) = 0$ gives

$$-\lambda A \sin \lambda(Ce^{\lambda y} + De^{-\lambda y}) = 0$$

Choosing $A = 0$ would make $u \equiv 0$ so we must force $\sin \lambda$ to be zero i.e. choose $\lambda = n\pi$ where n is a positive integer.

Thus, at this stage (4) becomes

$$\begin{aligned} u &= A \cos n\pi x(Ce^{n\pi y} + De^{-n\pi y}) \\ &= \cos n\pi x(Ee^{n\pi y} + Fe^{-n\pi y}) \qquad (5) \end{aligned}$$

Now impose the condition that this solution should be bounded:

Your solution

Answer

The region over which we are solving Laplace's equation is semi-infinite i.e. the y coordinate increases without limit. The solution for $u(x, y)$ in (5) will increase without limit as $y \to \infty$ due to the term $e^{n\pi y}$ (n being a positive integer.) This can be avoided i.e. the solution will be bounded if the constant E is chosen as zero.

Finally, use Fourier series techniques to deal with the final boundary condition $u(x, 0) = f(x)$:

Your solution

Your solution

Answer

Superposing solutions of the form (5) (with $E = 0$) gives

$$u(x, y) = \sum_{n=0}^{\infty} F_n \cos(n\pi x)\ e^{-n\pi y} \qquad (6)$$

so the boundary condition gives

$$f(x) = \sum_{n=0}^{\infty} F_n \cos n\pi x$$

We have here a half-range Fourier cosine series representation of a function $f(x)$ defined over $0 < x < 1$. Extending $f(x)$ as an even periodic function with period 2 and using standard Fourier series theory gives

$$F_n = 2\int_0^1 f(x) \cos n\pi x\ dx \qquad n = 1, 2, \ldots$$

with

$$\frac{F_0}{2} = \int_0^1 f(x)\ dx.$$

Hence (6) is the solution of this given boundary value problem, the integrals giving us in principle the Fourier coefficients F_n for a given function $f(x)$.