Contents 4

Trigonometry

Learning outcomes

In this Workbook you will learn about the basic building blocks of trigonometry. You will learn about the sine, cosine, tangent, cosecant, secant, cotangent functions and their many important relationships. You will learn about their graphs and their periodic nature. You will learn how to apply Pythagoras' theorem and the Sine and Cosine rules to find lengths and angles of triangles.

Right-angled Triangles

4.1

Introduction

Right-angled triangles (that is triangles where one of the angles is 90°) are the easiest topic for introducing trigonometry. Since the sum of the three angles in a triangle is 180° it follows that in a right-angled triangle there are no obtuse angles (i.e. angles greater than 90°). In this Section we study many of the properties associated with right-angled triangles.

Prerequisites

Before starting this Section you should ...

- have a basic knowledge of the geometry of triangles

Learning Outcomes

On completion you should be able to ...

- define trigonometric functions both in right-angled triangles and more generally
- express angles in degrees
- calculate all the angles and sides in any right-angled triangle given certain information

1. Right-angled triangles

Look at Figure 1 which could, for example, be a profile of a hill with a constant gradient.

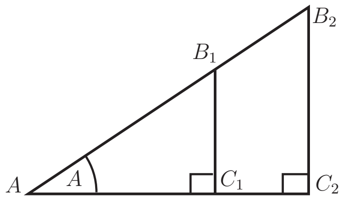


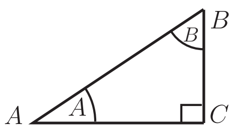
Figure 1

The two right-angled triangles AB_1C_1 and AB_2C_2 are **similar** (because the three angles of triangle AB_1C_1 are equal to the equivalent 3 angles of triangle AB_2C_2). From the basic properties of similar triangles corresponding sides have the same ratio. Thus, for example,

$$\frac{B_1C_1}{AB_1} = \frac{B_2C_2}{AB_2} \quad \text{and} \quad \frac{AC_1}{AB_1} = \frac{AC_2}{AB_2} \qquad (1)$$

The values of the two ratios (1) will clearly depend on the angle A of inclination. These ratios are called the **sine** and **cosine** of the angle A, these being abbreviated to $\sin A$ and $\cos A$.

Key Point 1



$$\sin A = \frac{BC}{AB} \qquad \cos A = \frac{AC}{AB}$$

Figure 2

AC is the side **adjacent** to angle A.

BC is the side **opposite** to angle A.

AB is the **hypotenuse** of the triangle (the longest side).

Task

Referring again to Figure 2 in Key Point 1, write down the ratios which give $\sin B$ and $\cos B$.

Your solution

Answer

$\sin B = \dfrac{AC}{AB} \quad \cos B = \dfrac{BC}{AB}$.

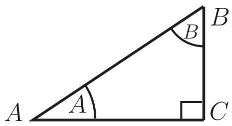
Note that $\sin B = \cos A = \cos(90° - B)$ and $\cos B = \sin A = \sin(90° - B)$

A third result of importance from Figure 1 is

$$\frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2} \qquad (2)$$

These ratios is referred to as the **tangent** of the angle at A, written $\tan A$.

Key Point 2



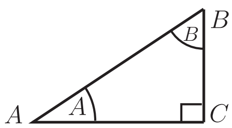
$$\sin A = \frac{BC}{AB} \qquad \cos A = \frac{AC}{AB}$$

Figure 3

$$\tan A = \frac{BC}{AC} = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$

For any right-angled triangle the values of sine, cosine and tangent are given in Key Point 3.

Key Point 3



$$\sin A = \frac{BC}{AB} \qquad \cos A = \frac{AC}{AB}$$

Figure 4

We can write, therefore, for any right-angled triangle containing an angle θ (not the right-angle)

$$\sin\theta = \frac{\text{length of side opposite angle } \theta}{\text{length of hypotenuse}} = \frac{\text{Opp}}{\text{Hyp}}$$

$$\cos\theta = \frac{\text{length of side adjacent to angle } \theta}{\text{length of hypotenuse}} = \frac{\text{Adj}}{\text{Hyp}}$$

$$\tan\theta = \frac{\text{length of side opposite angle } \theta}{\text{length of side adjacent to angle } \theta} = \frac{\text{Opp}}{\text{Adj}}$$

These are sometimes memorised as SOH, CAH and TOA respectively.

These three ratios are called **trigonometric ratios**.

Task

Write $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Your solution

Answer

$$\tan\theta = \frac{\text{Opp}}{\text{Adj}} = \frac{\text{Opp}}{\text{Adj}}.\frac{\text{Hyp}}{\text{Hyp}} = \frac{\text{Opp}}{\text{Hyp}}.\frac{\text{Hyp}}{\text{Adj}} = \frac{\text{Opp}}{\text{Hyp}}\Big/\frac{\text{Adj}}{\text{Hyp}} \qquad \text{i.e. } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Key Point 4

Pythagoras' Theorem

$$a^2 + b^2 = c^2$$

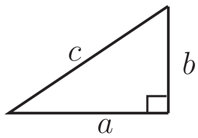


Figure 5

Example 1

Use the isosceles triangle in Figure 6 to obtain the sine, cosine and tangent of $45°$.

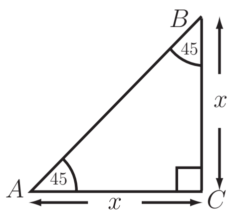


Figure 6

Solution

By Pythagoras' theorem $\quad (AB)^2 = x^2 + x^2 = 2x^2 \quad$ so $\quad AB = x\sqrt{2}$

Hence $\quad \sin 45° = \dfrac{BC}{AB} = \dfrac{x}{x\sqrt{2}} = \dfrac{1}{\sqrt{2}} \qquad \cos 45° = \dfrac{AC}{AB} = \dfrac{1}{\sqrt{2}} \qquad \tan 45° = \dfrac{BC}{AC} = \dfrac{x}{x} = 1$

Engineering Example 1

Noise reduction by sound barriers

Introduction

Audible sound has much longer wavelengths than light. Consequently, sound travelling in the atmosphere is able to bend around obstacles even when these obstacles cause sharp shadows for light. This is the result of the wave phenomenon known as **diffraction**. It can be observed also with water waves at the ends of breakwaters. The extent to which waves bend around obstacles depends upon the wavelength and the source-receiver geometry. So the efficacy of purpose built noise barriers, such as to be found alongside motorways in urban and suburban areas, depends on the frequencies in the sound and the locations of the source and receiver (nearest noise-affected person or dwelling) relative to the barrier. Specifically, the barrier performance depends on the difference in the lengths of the hypothetical ray paths passing from source to receiver either directly or via the top of the barrier (see Figure 7).

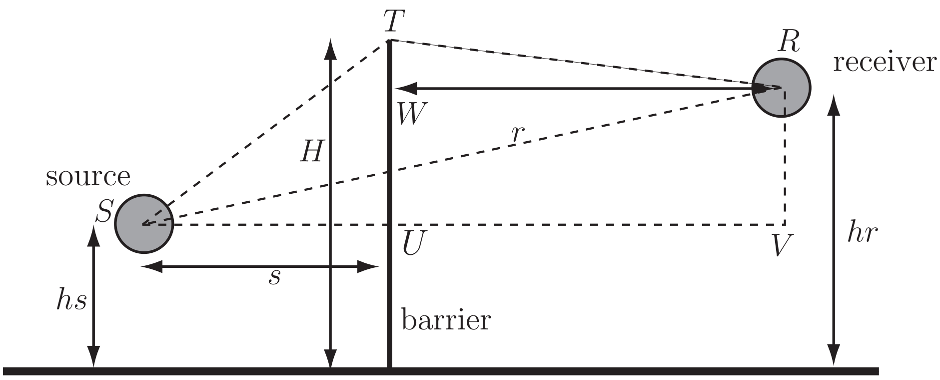


Figure 7

Problem in words

Find the difference in the path lengths from source to receiver either directly or via the top of the barrier in terms of

(i) the source and receiver heights,
(ii) the horizontal distances from source and receiver to the barrier and
(iii) the height of the barrier.

Calculate the path length difference for a 1 m high source, 3 m from a 3 m high barrier when the receiver is 30 m on the other side of the barrier and at a height of 1 m.

Mathematical statement of the problem

Find $ST + TR - SR$ in terms of hs, hr, s, r and H.

Calculate this quantity for $hs = 1,\ s = 3,\ H = 3,\ r = 30$ and $hr = 1$.

Mathematical analysis

Note the labels V, U, W on points that are useful for the analysis. Note that the length of $RV = hr - hs$ and that the horizontal separation between S and R is $r + s$. In the right-angled triangle SRV, Pythagoras' theorem gives

$$(SR)^2 = (r+s)^2 + (hr - hs)^2$$

So

$$SR = \sqrt{(r+s)^2 + (hr-hs)^2} \qquad (3)$$

Note that the length of $TU = H - hs$ and the length of $TW = H - hr$. In the right-angled triangle STU,

$$(ST)^2 = s^2 + (H - hs)^2$$

In the right-angled triangle TWR,

$$(TR)^2 = r^2 + (H - hr)^2$$

So

$$ST + TR = \sqrt{s^2 + (H-hs)^2} + \sqrt{r^2 + (H-hr)^2} \qquad (4)$$

So using (3) and (4)

$$ST + TR - SR = \sqrt{s^2 + (H-hs)^2} + \sqrt{r^2 + (H-hr)^2} - \sqrt{(r+s)^2 + (hr-hs)^2}.$$

For $hs = 1,\ s = 3,\ H = 3,\ r = 30$ and $hr = 1$,

$$\begin{aligned} ST + TR - SR &= \sqrt{3^2 + (3-1)^2} + \sqrt{30^2 + (3-1)^2} - \sqrt{(30+3)^2 + (1-1)^2} \\ &= \sqrt{13} + \sqrt{904} - 33 \\ &= 0.672 \end{aligned}$$

So the path length difference is 0.672 m.

Interpretation

Note that, for equal source and receiver heights, the further either receiver or source is from the barrier, the smaller the path length difference. Moreover if source and receiver are at the same height as the barrier, the path length difference is zero. In fact diffraction by the barrier still gives some sound reduction for this case. The smaller the path length difference, the more accurately it has to be calculated as part of predicting the barriers noise reduction.

Engineering Example 2

Horizon distance

Problem in words

Looking from a height of 2 m above sea level, how far away is the horizon? State any assumptions made.

Mathematical statement of the problem

Assume that the Earth is a sphere. Find the length D of the tangent to the Earth's sphere from the observation point O.

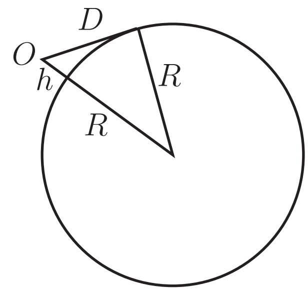


Figure 8: The Earth's sphere and the tangent from the observation point O

Mathematical analysis

Using Pythagoras' theorem in the triangle shown in Figure 8,

$$(R+h)^2 = D^2 + R^2$$

Hence

$$R^2 + 2Rh + h^2 = D^2 + R^2 \quad \rightarrow \quad h(2R+h) = D^2 \quad \rightarrow \quad D = \sqrt{h(2R+h)}$$

If $R = 6.373 \times 10^6$ m, then the variation of D with h is shown in Figure 9.

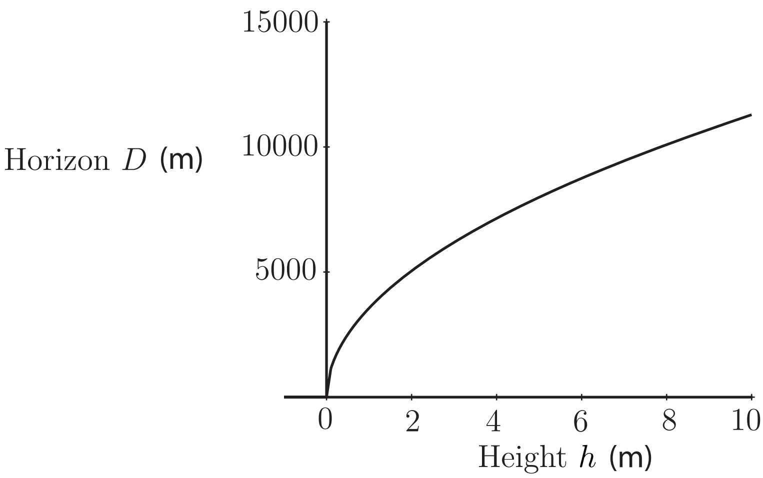


Figure 9

At an observation height of 2 m, the formula predicts that the horizon is just over 5 km away. In fact the variation of optical refractive index with height in the atmosphere means that the horizon is approximately 9% greater than this.

Task

Using the triangle ABC in Figure 10 which can be regarded as one half of the equilateral triangle ABD, calculate sin, cos, tan for the angles $30°$ and $60°$.

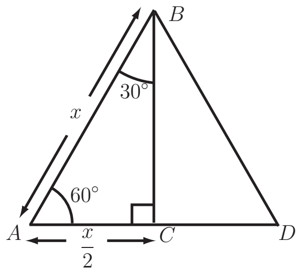


Figure 10

Your solution

Answer

By Pythagoras' theorem: $(BC)^2 = (AB)^2 - (AC)^2 = x^2 - \dfrac{x^2}{4} = \dfrac{3x^2}{4}$ so $BC = x\dfrac{\sqrt{3}}{2}$

Hence $\sin 60° = \dfrac{BC}{AB} = \dfrac{x\frac{\sqrt{3}}{2}}{x} = \dfrac{\sqrt{3}}{2}$ $\quad \sin 30° = \dfrac{AC}{AB} = \dfrac{\frac{x}{2}}{x} = \dfrac{1}{2}$ $\quad \cos 60° = \dfrac{AC}{AB} = \dfrac{1}{2}$

$\cos 30° = \dfrac{BC}{AB} = \dfrac{x\frac{\sqrt{3}}{2}}{x} = \dfrac{\sqrt{3}}{2}$ $\quad \tan 60° = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$ $\quad \tan 30° = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}}$

Values of $\sin\theta,\ \cos\theta$ and $\tan\theta$ can of course be obtained by calculator. When entering the angle in degrees (e.g. $30°$) the calculator must be in degree mode. (Typically this is ensured by pressing the DRG button until 'DEG' is shown on the display). The keystrokes for $\sin 30°$ are usually simply [sin] [30] or, on some calculators, [30] [sin] perhaps followed by [=].

Task

(a) Use your calculator to check the values of $\sin 45°,\ \cos 30°$ and $\tan 60°$ obtained in the previous Task.

(b) Also obtain $\sin 3.2°,\ \cos 86.8°,\ \tan 28°15'$. ($'$ denotes a minute $= \dfrac{1}{60}^{\circ}$)

Your solution

(a)

(b)

Answer

(a) $0.7071,\ 0.8660,\ \ 1.7321$ to 4 d.p.

(b) $\sin 3.2^\circ = \cos 86.8^\circ = 0.0558$ to 4 d.p., $\tan 28^\circ 15' = \tan 28.25^\circ = 0.5373$ to 4 d.p.

Inverse trigonometric functions (a first look)

Consider, by way of example, a right-angled triangle with sides 3, 4 and 5, see Figure 11.

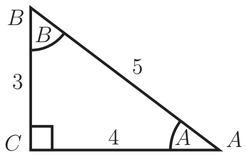


Figure 11

Suppose we wish to find the angles at A and B. Clearly $\sin A = \dfrac{3}{5},\ \cos A = \dfrac{4}{5},\ \tan A = \dfrac{3}{4}$ so we need to solve one of the above three equations to find A.

Using $\sin A = \dfrac{3}{5}$ we write $A = \sin^{-1}\left(\dfrac{3}{5}\right)$ (read as 'A is the inverse sine of $\dfrac{3}{5}$')

The value of A can be obtained by calculator using the '$\sin^{-1}$' button (often a second function to the $\sin$ function and accessed using a [SHIFT] or [INV] or [SECOND FUNCTION] key).

Thus to obtain $\sin^{-1}\left(\dfrac{3}{5}\right)$ we might use the following keystrokes:

[INV] [SIN] [0.6] [=] or [3] ÷ [5] [INV] [SIN] [=]

We find $\sin^{-1}\dfrac{3}{5} = 36.87^\circ$ (to 4 significant figures).

Key Point 5

Inverse Trigonometric Functions

$\sin\theta = x$ implies $\theta = \sin^{-1} x$
$\cos\theta = y$ implies $\theta = \cos^{-1} y$
$\tan\theta = z$ implies $\theta = \tan^{-1} z$

(The alternative notations arcsin, arccos, arctan are sometimes used for these inverse functions.)

Task

Check the values of the angles at A and B in Figure 11 above using the $\cos^{-1}$ functions on your calculator. Give your answers in degrees to 2 d.p.

Your solution

Answer

$A = \cos^{-1}\frac{4}{5} = 36.87^\circ \qquad B = \cos^{-1}\frac{3}{5} = 53.13^\circ$

Task

Check the values of the angles at A and B in Figure 11 above using the $\tan^{-1}$ functions on your calculator. Give your answers in degrees to 2 d.p.

Your solution

Answer

$A = \tan^{-1}\frac{3}{4} = 36.87^\circ \qquad B = \tan^{-1}\frac{4}{3} = 53.13^\circ$

You should note carefully that $\sin^{-1} x$ **does not mean** $\dfrac{1}{\sin x}$.

Indeed the function $\dfrac{1}{\sin x}$ has a special name – the cosecant of x, written $\text{cosec}\, x$. So

$$\text{cosec}\, x \equiv \frac{1}{\sin x} \quad \text{(the **cosecant** function).}$$

Similarly

$$\sec x \equiv \frac{1}{\cos x} \quad \text{(the **secant** function)}$$

$$\cot x \equiv \frac{1}{\tan x} \quad \text{(the **cotangent** function).}$$

Use your calculator to obtain to 3 d.p. $\text{cosec}\ 38.5°,\ \sec 22.6°,\ \cot 88.32°$ (Use the sin, cos or tan buttons unless your calculator has specific buttons.)

Your solution

Answer

$\text{cosec}\ 38.5° = \dfrac{1}{\sin 38.5°} = 1.606 \qquad \sec 22.6° = \dfrac{1}{\cos 22.6°} = 1.083$

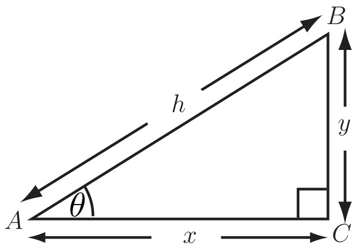
$\cot 88.32° = \dfrac{1}{\tan 88.32°} = 0.029$

2. Solving right-angled triangles

Solving right-angled triangles means obtaining the values of all the angles and all the sides of a given right-angled triangle using the trigonometric functions (and, if necessary, the inverse trigonometric functions) and perhaps Pythagoras' theorem.
There are three cases to be considered:

Case 1 Given the hypotenuse and an angle

We use $\sin$ or cos as appropriate:



(a)

Figure 12

Assuming h and θ in Figure 12 are given then

$$\cos\theta = \frac{x}{h} \text{ which gives } x = h\cos\theta$$

from which x can be calculated.

Also

$$\sin\theta = \frac{y}{h} \text{ so } y = h\sin\theta \text{ which enables us to calculate } y.$$

Clearly the third angle of this triangle (at B) is $90° - \theta$.

Case 2 Given a side other than the hypotenuse and an angle.

We use $\tan$:

(a) If x and θ are known then, in Figure 12, $\tan\theta = \dfrac{y}{x}$ so $y = x\tan\theta$

which enables us to calculate y.

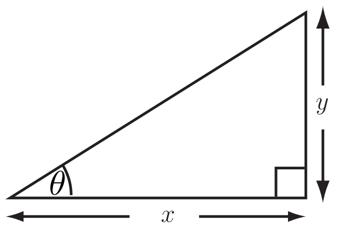
(b) If y and θ are known then $\tan\theta = \dfrac{y}{x}$ gives $x = \dfrac{y}{\tan\theta}$ from which x can be calculated.

Then the hypotenuse can be calculated using Pythagoras' theorem: $h = \sqrt{x^2 + y^2}$

Case 3 Given two of the sides

We use $\tan^{-1}$ or $\sin^{-1}$ or $\cos^{-1}$:

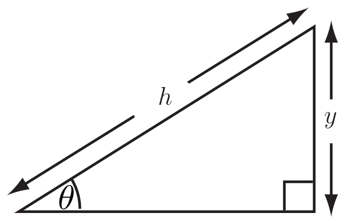
(a)



$$\tan\theta = \frac{y}{x} \qquad \text{so} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Figure 13

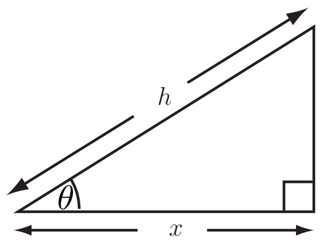
(b)



$$\sin\theta = \frac{y}{h} \qquad \text{so} \qquad \theta = \sin^{-1}\left(\frac{y}{h}\right)$$

Figure 14

(c)



$$\cos\theta = \frac{x}{h} \qquad \text{so} \qquad \theta = \cos^{-1}\left(\frac{x}{h}\right)$$

Figure 15

Note: since two sides are given we can use Pythagoras' theorem to obtain the length of the third side at the outset.

Engineering Example 3

Vintage car brake pedal mechanism

Introduction

Figure 16 shows the structure and some dimensions of a vintage car brake pedal arrangement as far as the brake cable. The **moment** of a force about a point is the product of the force and the perpendicular distance from the point to the line of action of the force. The pedal is pivoted about the point A. The moments about A must be equal as the pedal is stationary.

Problem in words

If the driver supplies a force of 900 N, to act at point B, calculate the force (F) in the cable.

Mathematical statement of problem

The perpendicular distance from the line of action of the force provided by the driver to the pivot point A is denoted by x_1 and the perpendicular distance from the line of action of force in the cable to the pivot point A is denoted by x_2. Use trigonometry to relate x_1 and x_2 to the given dimensions. Calculate clockwise and anticlockwise moments about the pivot and set them equal.

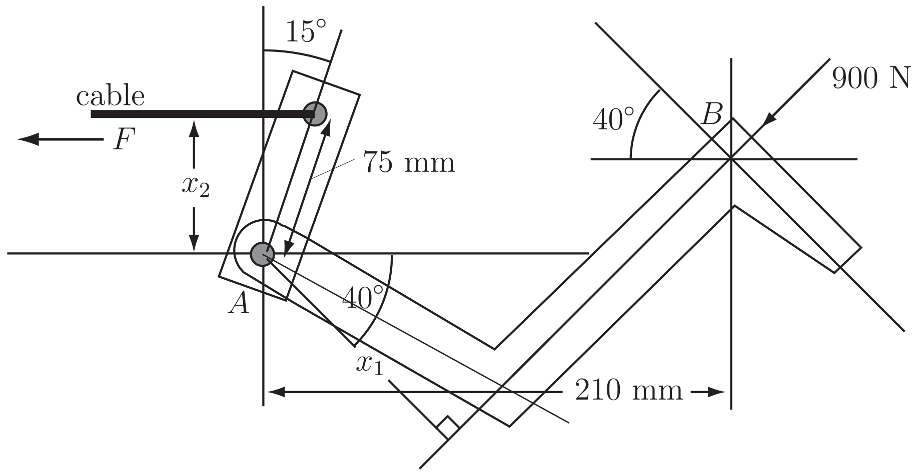
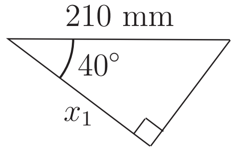


Figure 16: Structure and dimensions of vintage car brake pedal arrangement

Mathematical Analysis

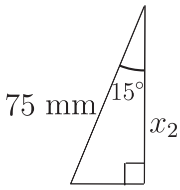
The distance x_1 is found by considering the right-angled triangle shown in Figure 17 and using the definition of cosine.



$$\cos(40°) = \frac{x_1}{0.210} \qquad \text{hence} \qquad x_1 = 161 \text{ mm.}$$

Figure 17

The distance x_2 is found by considering the right-angled triangle shown in Figure 18.



$$\cos(15°) = \frac{x_2}{0.075} \qquad \text{hence} \qquad x_2 = 72 \text{ mm.}$$

Figure 18

Equating moments about A:

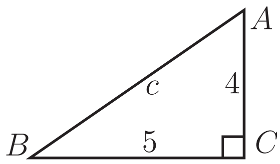
$$900x_1 = Fx_2 \qquad \text{so} \qquad F = 2013 \text{ N.}$$

Interpretation

This means that the force exerted by the cable is 2013 N in the direction of the cable. This force is more than twice that applied by the driver. In fact, whatever the force applied at the pedal the force in the cable will be more than twice that force. The pedal structure is an example of a lever system that offers a mechanical gain.

Task

Obtain all the angles and the remaining side for the triangle shown:



Your solution

Answer

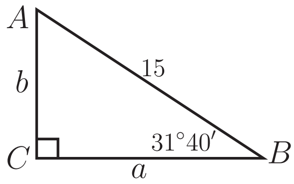
This is Case 3. To obtain the angle at B we use $\tan B = \dfrac{4}{5}$ so $B = \tan^{-1}(0.8) = 38.66°$.

Then the angle at A is $180° - (90° - 38.66°) = 51.34°$.

By Pythagoras' theorem $c = \sqrt{4^2 + 5^2} = \sqrt{41} \approx 6.40$.

Task

Obtain the remaining sides and angles for the triangle shown.



Your solution

Answer

This is Case 1. Since $31°40' = 31.67°$ then $\cos 31.67° = \dfrac{a}{15}$ so $a = 15\cos 31.67° = 12.77$.

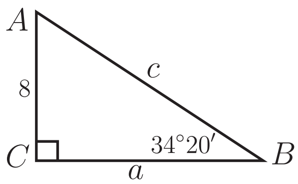
The angle at A is $180° - (90 + 31.67°) = 58.33°$.

Finally $\sin 31.67° = \dfrac{b}{15}$ $\therefore$ $b = 15 \sin 31.67° = 7.85$.

(Alternatively, of course, Pythagoras' theorem could be used to calculate the length b.)

Task

Obtain the remaining sides and angles of the following triangle.



Your solution

Answer

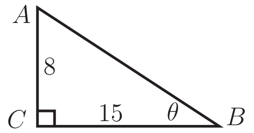
This is Case 2.

Here $\tan 34.33° = \dfrac{8}{a}$ so $a = \dfrac{8}{\tan 34.33°} = 11.7$

Also $c = \sqrt{8^2 + 11.7^2} = 14.18$ and the angle at A is $180° - (90° + 34.33°) = 55.67°$.

Exercises

1. Obtain $\operatorname{cosec}\theta,\ \sec\theta,\ \cot\theta,\ \theta$ in the following right-angled triangle.



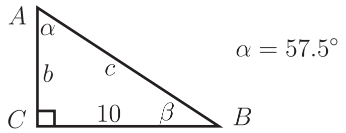
2. Write down $\sin\theta,\ \cos\theta,\ \tan\theta,\ \operatorname{cosec}\theta$ for each of the following triangles:

3. If θ is an acute angle such that $\sin\theta = 2/7$ obtain, without use of a calculator, $\cos\theta$ and $\tan\theta$.

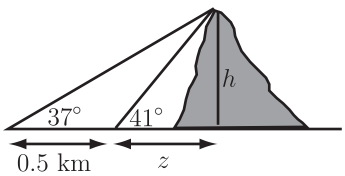
4. Use your calculator to obtain the acute angles θ satisfying

   (a) $\sin\theta = 0.5260,$ (b) $\tan\theta = 2.4,$ (c) $\cos\theta = 0.2$

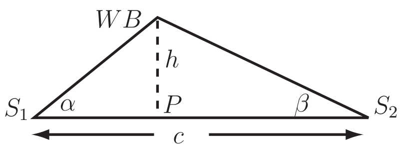
5. Solve the right-angled triangle shown:



6. A surveyor measures the **angle of elevation** between the top of a mountain and ground level at two different points. The results are shown in the following figure. Use trigonometry to obtain the distance z (which cannot be measured) and then obtain the height h of the mountain.

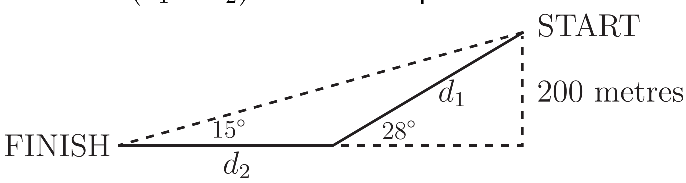


7. As shown below two tracking stations S_1 and S_2 sight a weather balloon (WB) between them at elevation angles α and β respectively.



   Show that the height h of the balloon is given by $h = \dfrac{c}{\cot\alpha + \cot\beta}$

8. A vehicle entered in a 'soap box derby' rolls down a hill as shown in the figure. Find the total distance $(d_1 + d_2)$ that the soap box travels.



Answers

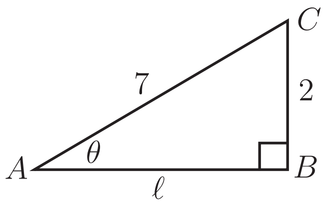
1. $h = \sqrt{15^2 + 8^2} = 17, \quad \text{cosec}\,\theta = \dfrac{1}{\sin\theta} = \dfrac{17}{8} \quad \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{17}{15} \qquad \cot\theta = \dfrac{1}{\tan\theta} = \dfrac{15}{8}$

   $\theta = \sin^{-1}\dfrac{8}{17}$ (for example) $\quad \therefore \quad \theta = 28.07°$

2. (a) $\sin\theta = \dfrac{2}{5} \quad \cos\theta = \dfrac{\sqrt{21}}{5} \quad \tan\theta = \dfrac{2\sqrt{21}}{21} \quad \text{cosec}\,\theta = \dfrac{5}{2}$

   (b) $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}} \quad \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}} \quad \tan\theta = \dfrac{y}{x} \quad \text{cosec}\,\theta = \dfrac{\sqrt{x^2+y^2}}{y}$

3. Referring to the following diagram



   $\ell = \sqrt{7^2 - 2^2} = \sqrt{45} = 3\sqrt{5}$

   Hence $\cos\theta = \dfrac{3\sqrt{5}}{7} \quad \tan\theta = \dfrac{2}{3\sqrt{5}} = \dfrac{2\sqrt{5}}{15}$

4. (a) $\theta = \sin^{-1} 0.5260 = 31.73°$ (b) $\theta = \tan^{-1} 2.4 = 67.38°$ (c) $\theta = \cos^{-1} 0.2 = 78.46°$

5. $\beta = 90 - \alpha = 32.5°, \qquad b = \dfrac{10}{\tan 57.5°} \simeq 6.37 \qquad c = \dfrac{10}{\sin 57.5°} \simeq 11.86$

6. $\tan 37° = \dfrac{h}{z + 0.5} \qquad \tan 41° = \dfrac{h}{z}$ from which

   $h = (z + 0.5)\tan 37° = z\tan 41°$, so $z\tan 37° - z\tan 41° = -0.5\tan 37°$

   $\therefore \quad z = \dfrac{-0.5\tan 37°}{\tan 37° - \tan 41°} \simeq 3.2556$ km, so $h = z\tan 41° = 3.2556\tan 41° \simeq 2.83$ km

7. Since the required answer is in terms of $\cot\alpha$ and $\cot\beta$ we proceed as follows:

   Using x to denote the distance S_1P $\quad \cot\alpha = \dfrac{1}{\tan\alpha} = \dfrac{x}{h} \qquad \cot\beta = \dfrac{1}{\tan\beta} = \dfrac{c - x}{h}$

   Adding: $\cot\alpha + \cot\beta = \dfrac{x}{h} + \dfrac{c-x}{h} = \dfrac{c}{h} \qquad \therefore \quad h = \dfrac{c}{\cot\alpha + \cot\beta}$ as required.

8. From the smaller right-angled triangle $\quad d_1 = \dfrac{200}{\sin 28°} = 426.0$ m. The base of this triangle then has length $\ell = 426\cos 28° = 376.1$ m

   From the larger right-angled triangle the straight-line distance from START to FINISH is $\dfrac{200}{\sin 15°} = 772.7$ m. Then, using Pythagoras' theorem $(d_2 + \ell) = \sqrt{772.7^2 - 200^2} = 746.4$ m from which $d_2 = 370.3$ m $\therefore \quad d_1 + d_2 = 796.3$ m