Applications of Trigonometry to Triangles

4.4

Introduction

We originally introduced trigonometry using right-angled triangles. However, the subject has applications in dealing with **any** triangles such as those that might arise in surveying, navigation or the study of mechanisms.
In this Section we show how, given certain information about a triangle, we can use appropriate rules, called the **Sine rule** and the **Cosine rule**, to fully 'solve the triangle' i.e. obtain the lengths of all the sides and the size of all the angles of that triangle.

Prerequisites

Before starting this Section you should . . .

- have a knowledge of the basics of trigonometry
- be aware of the standard trigonometric identities

Learning Outcomes

On completion you should be able to . . .

- use trigonometry in everyday situations
- fully determine all the sides and angles and the area of any triangle from partial information

1. Applications of trigonometry to triangles

Area of a triangle

The area S of any triangle is given by $S = \frac{1}{2} \times \text{(base)} \times \text{(perpendicular height)}$ where 'perpendicular height' means the perpendicular distance from the side called the 'base' to the opposite vertex. Thus for the right-angled triangle shown in Figure 33(a) $S = \frac{1}{2}\ b\ a$. For the obtuse-angled triangle shown in Figure 33(b) the area is $S = \frac{1}{2}bh$.

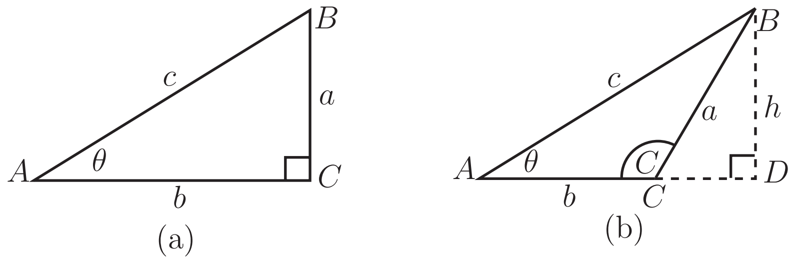


Figure 33

If we use C to denote the angle ACB in Figure 33(b) then

$$\sin(180 - C) = \frac{h}{a} \qquad \text{(triangle } BCD \text{ is right-angled)}$$

$$\therefore \quad h = a\sin(180 - C) = a\sin C \quad \text{(see the graph of the sine wave or expand } \sin(180 - c))$$

$$\therefore \quad \boxed{S = \frac{1}{2}\ b\ a \sin C} \qquad 1(a)$$

By other similar constructions we could demonstrate that

$$\boxed{S = \frac{1}{2}\ a\ c\ \sin B} \qquad 1(b)$$

and

$$\boxed{S = \frac{1}{2}\ b\ c\ \sin A} \qquad 1(c)$$

Note the pattern here: in each formula for the area the angle involved is the one **between** the sides whose lengths occur in that expression.
Clearly if C is a right-angle (so $\sin C = 1$) then

$$S = \frac{1}{2}\ b\ a \quad \text{as for Figure 33(a).}$$

Note: from now on we will not generally write '$\equiv$' but use the more usual '$=$'.

The Sine rule

The Sine rule is a formula which, if we are given certain information about a triangle, enables us to fully 'solve the triangle' i.e. obtain the lengths of all three sides and the value of all three angles. To show the rule we note that from the formulae (1a) and (1b) for the area S of the triangle ABC in Figure 33 we have

$$ba\sin C = ac\sin B \quad \text{or} \quad \frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly using (1b) and (1c)

$$ac\sin B = bc\sin A \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

Key Point 18

The Sine Rule

For **any** triangle ABC where a is the length of the side opposite angle A, b the side length opposite angle B and c the side length opposite angle C states

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use of the Sine rule

To be able to fully determine all the angles and sides of a triangle it follows from the Sine rule that we must know

either two angles and one side : (knowing two angles of a triangle really means that all three are known since the sum of the angles is $180°$)

or two sides and an angle **opposite** one of those two sides.

Example 3

Solve the triangle ABC given that $a = 32$ cm, $b = 46$ cm and angle $B = 63.25°$.

Solution

Using the first pair of equations in the Sine rule (Key Point 18) we have

$$\frac{32}{\sin A} = \frac{46}{\sin 63.25°} \qquad \therefore \qquad \sin A = \frac{32}{46}\sin 63.25° = 0.6212$$

so $\quad A = \sin^{-1}(0.6212) = 38.4° \quad$ (by calculator)

Solution (contd.)

You should, however, note carefully that because of the form of the graph of the sine function there are **two** angles between $0°$ and $180°$ which have the same value for their sine i.e. x and $(180 - x)$. See Figure 34.

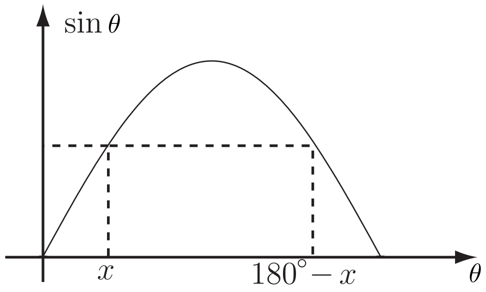


Figure 34

In our example

$$A = \sin^{-1}(0.6212) = 38.4°$$

or

$$A = 180° - 38.4° = 141.6°.$$

However since we are given that angle B is $63.25°$, the value of $141.6°$ for angle A is clearly impossible.

To complete the problem we simply note that

$$C = 180° - (38.4° + 63.25°) = 78.35°$$

The remaining side c is calculated from the Sine rule, using either a and $\sin A$ or b and $\sin B$.

Task

Find the length of side c in Example 3.

Your solution

Answer

Using, for example, $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

we have $c = a\dfrac{\sin C}{\sin A} = 32 \times \dfrac{\sin 78.35°}{0.6212} = \dfrac{32 \times 0.9794}{0.6212} = 50.45$ cm.

The ambiguous case

When, as in Example 3, we are given two sides and the non-included angle of a triangle, particular care is required.
Suppose that sides b and c and the angle B are given. Then the angle C is given by the Sine rule as

$$\sin C = c\frac{\sin B}{b}$$

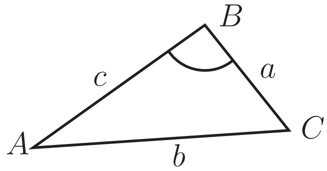


Figure 35

Various cases can arise:

(i) $c \sin B > b$

This implies that $\dfrac{c \sin B}{b} > 1$ in which case no triangle exists since $\sin C$ cannot exceed 1.

(ii) $c \sin B = b$

In this case $\sin C = \dfrac{c \sin B}{b} = 1$ so $C = 90°$.

(iii) $c \sin B < b$

Hence $\sin C = \dfrac{c \sin B}{b} < 1.$

As mentioned earlier there are two possible values of angle C in the range 0 to $180°$, one acute angle $(< 90°)$ and one obtuse (between $90°$ and $180°$.) These angles are $C_1 = x$ and $C_2 = 180 - x$. See Figure 36.
If the given angle B is greater than $90°$ then the obtuse angle C_2 is not a possible solution because, of course, a triangle cannot possess two obtuse angles.

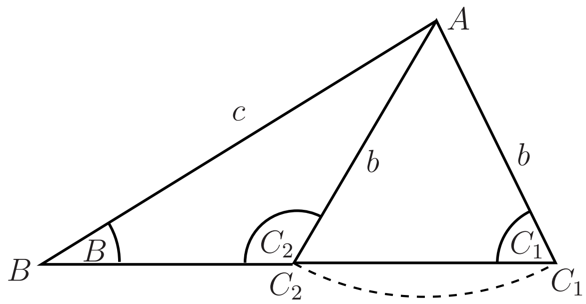


Figure 36

For B less than $90°$ there are still two possibilities.
If the given side b is greater than the given side c, the obtuse angle solution C_2 is not possible because then the larger angle would be opposite the smaller side. (This was the situation in Example 3.)
The final case

$$b < c, \qquad B < 90°$$

does give rise to two possible values $C_1,\ C_2$ of the angle C and is referred to as the **ambiguous case**. In this case there will be two possible values a_1 and a_2 for the third side of the triangle corresponding to the two angle values

$$A_1 = 180° - (B + C_1)$$
$$A_2 = 180° - (B + C_2)$$

Show that two triangles fit the following data for a triangle ABC:

$a = 4.5$ cm $\qquad b = 7$ cm $\qquad A = 35°$

Obtain the sides and angle of both possible triangles.

Your solution

Answer

We have, by the Sine rule, $\sin B = \dfrac{b \sin A}{a} = \dfrac{7 \sin 35°}{4.5} = 0.8922$

So $B = \sin^{-1} 0.8922 - 63.15°$ (by calculator) or $180 - 63.15° = 116.85°$.

In this case, **both** values of B are indeed possible since both values are larger than angle A (side b is longer than side a). This is the ambiguous case with two possible triangles.

$B = B_1 = 63.15°$

$C = C_1 = 81.85°$

$c = c_1$ where $\dfrac{c_1}{\sin 81.85°} = \dfrac{4.5}{\sin 35°}$

$$c_1 = \frac{4.5 \times 0.9899}{0.5736}$$

$= 7.766$ cm

$B = B_2 = 116.85°$

$C = C_2 = 28.15°$

$c = c_2$ where $\dfrac{c_2}{\sin 28.15} = \dfrac{4.5}{\sin 35°}$

$$c_2 = \frac{4.5 \times 0.4718}{0.5736}$$

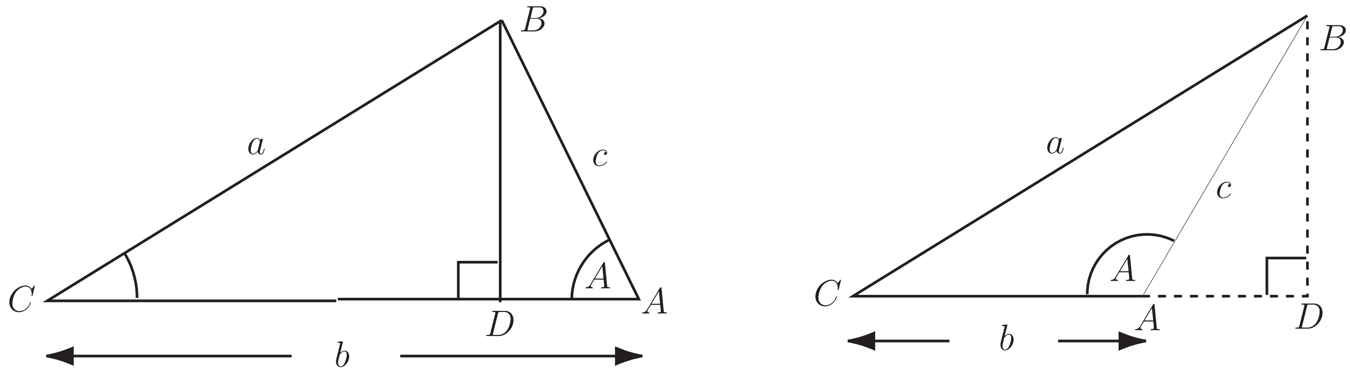
$= 3.701$ cm

You can clearly see that we have one acute angled triangle AB_1C_1 and one obtuse angled AB_2C_2 corresponding to the given data.

The Cosine rule

The Cosine rule is an alternative formula for 'solving a triangle' ABC. It is particularly useful for the case where the Sine rule cannot be used, i.e. when two sides of the triangle are known together with the angle **between** these two sides.

Consider the two triangles ABC shown in Figure 37.



(a) (b)

Figure 37

In Figure 37(a) using the right-angled triangle ABD, $BD = c\sin A$.

In Figure 37(b) using the right-angled triangle ABD, $BD = c\sin(\pi - A) = c\sin A$.

In Figure 37(a) $DA = c\cos A \quad \therefore \quad CD = b - c\cos A$

In Figure 37(b) $DA = c\cos(180 - A) = -c\cos A \quad \therefore \quad CD = b + AD = b - c\cos A$

In both cases, in the right-angled triangle BDC

$$(BC)^2 = (CD)^2 + (BD)^2$$

So, using the above results,

$$a^2 = (b - c\cos A)^2 + c^2(\sin A)^2 = b^2 - 2bc\cos A + c^2(\cos^2 A + \sin^2 A)$$

giving

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A} \qquad (3)$$

Equation (3) is one form of the Cosine rule. Clearly it can be used, as we stated above, to calculate the side a if the sides b and c and the **included** angle A are known.

Note that if $A = 90°$, $\cos A = 0$ and (3) reduces to Pythagoras' theorem.

Two similar formulae to (3) for the Cosine rule can be similarly derived - see following Key Point:

Key Point 19

Cosine Rule

For any triangle with sides a, b, c and corresponding angles A, B, C

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = c^2 + a^2 - 2ca\cos B \qquad \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$c^2 = a^2 + b^2 - 2bc\cos C \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 4

Solve the triangle where $b = 7.00$ cm, $c = 3.59$ cm, $A = 47°$.

Solution

Since two sides and the angle A between these sides is given we must first use the Cosine rule in the form (3a):

$$a^2 = (7.00)^2 + (3.59)^2 - 2(7.00)(3.59)\cos 47° = 49 + 12.888 - 34.277 = 27.610$$

so $$a = \sqrt{27.610} = 5.255 \text{ cm}.$$

We can now most easily use the Sine rule to solve one of the remaining angles:

$$\frac{7.00}{\sin B} = \frac{5.255}{\sin 47°} \quad \text{so} \quad \sin B = \frac{7.00\sin 47°}{5.255} = 0.9742$$

from which $B = B_1 = 76.96°$ or $B = B_2 = 103.04°$.

At this stage it is not obvious which value is correct or whether this is the ambiguous case and both values of B are possible.

The two possible values for the remaining angle C are

$$C_1 = 180° - (47° + 76.96°) = 56.04°$$

$$C_2 = 180° - (47 + 103.04) = 29.96°$$

Since for the sides of this triangle $b > a > c$ then similarly for the angles we must have $B > A > C$ so the value $C_2 = 29.96°$ is the correct one for the third side.

The Cosine rule can also be applied to some triangles where the lengths a, b and c of the three sides are known and the only calculations needed are finding the angles.

Task

A triangle ABC has sides

$a = 7$cm $\quad b = 11$ cm $\quad c = 12$ cm.

Obtain the values of all the angles of the triangle. (Use Key Point 19.)

Your solution

Answer

Suppose we find angle A first using the following formula from Key Point 19

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Here $\quad \cos A = \dfrac{11^2 + 12^2 - 7^2}{2 \times 11 \times 12} = 0.818$ so $A = \cos^{-1}(0.818) = 35.1°$

(There is no other possibility between $0°$ and $180°$ for A. No 'ambiguous case' arises using the Cosine rule!)

Another angle B or C could now be obtained using the Sine rule or the Cosine rule.

Using the following formula from Key Point 19:

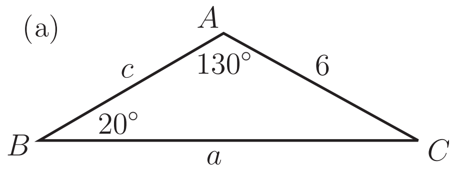
$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{12^2 + 7^2 - 11^2}{2 \times 12 \times 7} = 0.429 \quad \text{so} \quad B = \cos^{-1}(0.429) = 64.6°$$

Since $A + B + C = 180°$ we can deduce $C = 80.3°$

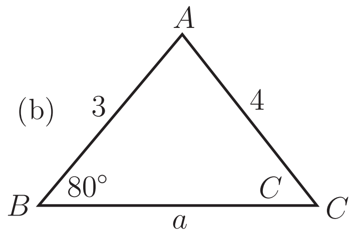
Exercises

1. Determine the remaining angles and sides for the following triangles:

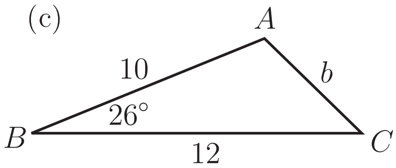
(a)



(b)



(c)

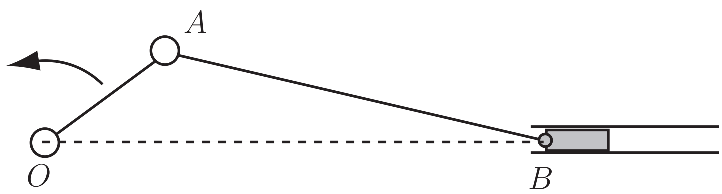


(d) The triangles ABC with $B = 50°,\ b = 5,\ c = 6$. (Take special care here!)

2. Determine all the angles of the triangles ABC where the sides have lengths $a = 7,\ b = 66$ and $c = 9$

3. Two ships leave a port at 8.00 am, one travelling at 12 knots (nautical miles per hour) the other at 10 knots. The faster ship maintains a bearing of $N47°W$, the slower one a bearing $S20°W$. Calculate the separation of the ships at midday. (Hint: Draw an appropriate diagram.)

4. The crank mechanism shown below has an arm OA of length 30 mm rotating anticlockwise about 0 and a connecting rod AB of length 60 mm. B moves along the horizontal line OB. What is the length OB when OA has rotated by $\frac{1}{8}$ of a complete revolution from the horizontal?



Answers

1.

(a) Using the Sine rule $\dfrac{a}{\sin 130^\circ} = \dfrac{6}{\sin 20^\circ} = \dfrac{c}{\sin C}$. From the two left-hand equations $a = 6\dfrac{\sin 130^\circ}{\sin 20^\circ} \simeq 13.44$.

Then, since $C = 30^\circ$, the right hand pair of equations give $c = 6\dfrac{\sin 30^\circ}{\sin 20^\circ} \simeq 8.77$.

(b) Again using the Sine rule $\dfrac{a}{\sin A} = \dfrac{4}{\sin 80^\circ} = \dfrac{3}{\sin C}$ so $\sin C = \dfrac{3}{4}\sin 80^\circ = 0.7386$ there are two possible angles satisfying $\sin C = 0.7386$ or $C = \sin^{-1}(0.7386)$.

These are 47.61° and $180^\circ - 47.614^\circ = 132.39^\circ$. However the obtuse angle value is impossible here because the angle B is 80° and the sum of the angles would then exceed 180° Hence $c = 47.01^\circ$ so $A = 180^\circ - (80^\circ + 47.61^\circ) = 52.39^\circ$.

Then, $\dfrac{a}{\sin 52.39^\circ} = \dfrac{4}{\sin 80^\circ}$ so $a = 4\dfrac{\sin 52.39^\circ}{\sin 80^\circ} \simeq 3.22$

(c) In this case since two sides and the included angle are given we must use the Cosine rule. The appropriate form is

$$b^2 = c^2 + a^2 - 2ca\cos B = 10^2 + 12^2 - (2)(10)(12)\cos 26^\circ = 28.2894$$

so $b = \sqrt{28.2894} = 5.32$

Continuing we use the Cosine rule again to determine say angle C where

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \text{that is} \qquad 10^2 = 12^2 + (5.32)^2 - 2(1.2)(5.32)\cos C$$

from which $\cos C = 0.5663$ and $C = 55.51^\circ$ (There is no other possibility for C between 0° and 180°. Recall that the cosine of an angle between 90° and 180° is negative.) Finally, $A = 180 - (26^\circ + 55.51^\circ) = 98.49^\circ$.

(d) By the Sine rule

$$\frac{a}{\sin A} = \frac{5}{\sin 50^\circ} = \frac{6}{\sin C} \qquad \therefore \qquad \sin C = 6\frac{\sin 50^\circ}{5} = 0.9193$$

Then $C = \sin^{-1}(0.9193) = 66.82^\circ$ (calculator) or $180^\circ - 66.82^\circ = 113.18^\circ$. In this case both values of C say $C_1 = 66.82^\circ$ and $C_2 = 113.18^\circ$ are possible and there are two possible triangles satisfying the given data. Continued use of the Sine rule produces

(i) with $C_1 = 66.82$ (acute angle triangle) $A = A_1 = 180 - (66.82^\circ + 50^\circ) = 63.18^\circ$ $a = a_1 = 5.83$

(ii) with $C_2 = 113.18^\circ$ $\quad A = A_2 = 16.82^\circ \quad a = a_2 = 1.89$

Answers continued

2. We use the Cosine rule firstly to find the angle opposite the longest side. This will tell us whether the triangle contains an obtuse angle. Hence we solve for c using

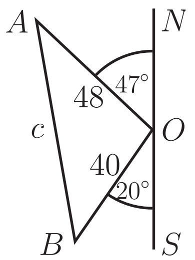
   $$c^2 = a^2 + b^2 - 2ab\cos C \qquad 81 = 49 + 36 - 84\cos C$$

   from which $84\cos C = 4 \qquad \cos C = 4/84$ giving $C = 87.27°$.

   So there is no obtuse angle in this triangle and we can use the Sine rule knowing that there is only one possible triangle fitting the data. (We could continue to use the Cosine rule if we wished of course.) Choosing to find the angle B we have

   $$\frac{6}{\sin B} = \frac{9}{\sin 87.27°}$$

   from which $\sin B = 0.6659$ giving $B = 41.75°$. (The obtuse case for B is not possible, as explained above.) Finally $A = 180° - (41.75° + 87.27°) = 50.98°$.

3. 

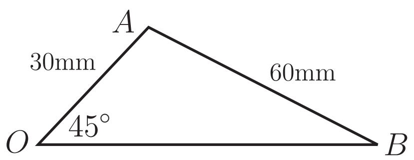
   At midday (4 hours travelling) ships A and B are respectively 48 and 40 nautical miles from the port O. In triangle AOB we have

   $$AOB = 180° - (47° + 20°) = 113°.$$

   We must use the Cosine rule to obtain the required distance apart of the ships. Denoting the distance AB by c, as usual,

   $c^2 = 48^2 + 40^2 - 2(48)(40)\cos 113°$ from which $c^2 = 5404.41$ and $c = 73.5$ nautical miles.

4. By the Sine rule $\dfrac{30}{\sin B} = \dfrac{60}{\sin 45} \qquad \therefore \qquad \sin B = \dfrac{30}{60}\sin 45° = 0.353$ so $B = 20.704°$.

   (Position after $\frac{1}{8}$ revolution)

   The obtuse value of $\sin^{-1}(0.353)$ is impossible. Hence,

   $$A = 180° - (45° + 20.704°) = 114.296°.$$

   Using the sine rule again $\dfrac{30}{0.353} = \dfrac{OB}{\sin 114.296}$ from which $OB = 77.5$ mm.