Applications of Trigonometry to Waves

4.5

Introduction

Waves and vibrations occur in many contexts. The water waves on the sea and the vibrations of a stringed musical instrument are just two everyday examples. If the vibrations are simple 'to and fro' oscillations they are referred to as 'sinusoidal' which implies that a knowledge of trigonometry, particularly of the sine and cosine functions, is a necessary pre-requisite for dealing with their analysis. In this Section we give a brief introduction to this topic.

Prerequisites

Before starting this Section you should . . .

- have a knowledge of the basics of trigonometry
- be aware of the standard trigonometric identities

Learning Outcomes

On completion you should be able to . . .

- use simple trigonometric functions to describe waves
- combine two waves of the same frequency as a single wave in amplitude-phase form

1. Applications of trigonometry to waves

Two-dimensional motion

Suppose that a wheel of radius R is rotating anticlockwise as shown in Figure 38.

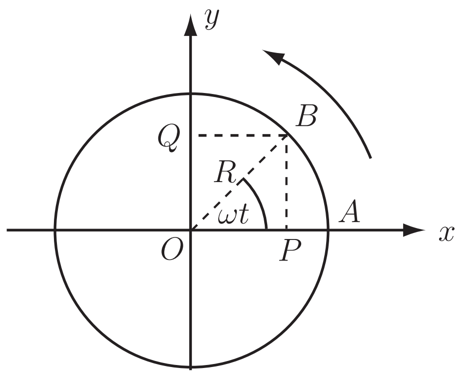


Figure 38

Assume that the wheel is rotating with an angular velocity ω radians per second about O so that, in a time t seconds, a point (x, y) initially at position A on the rim of the wheel moves to a position B such that angle $AOB = \omega t$ radians.
Then the coordinates (x, y) of B are given by

$$x = OP = R\cos\omega t$$

$$y = OQ = PB = R\sin\omega t$$

We know that both the standard sine and cosine functions have period 2π. Since the angular velocity is ω radians per second the wheel will make one complete revolution in $\dfrac{2\pi}{\omega}$ seconds.

The time $\dfrac{2\pi}{\omega}$ (measured in seconds in this case) for one complete revolution is called the **period** of rotation of the wheel. The number of complete revolutions per second is thus $\dfrac{1}{T} = f$ say which is called the **frequency** of revolution. Clearly $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$ relates the three quantities introduced here. The angular velocity $\omega = 2\pi f$ is sometimes called the **angular frequency**.

One-dimensional motion

The situation we have just outlined is two-dimensional motion. More simply we might consider one-dimensional motion.

An example is the motion of the projection onto the x-axis of a point B which moves with uniform angular velocity ω round a circle of radius R (see Figure 39). As B moves round, its projection P moves to and fro across the diameter of the circle.

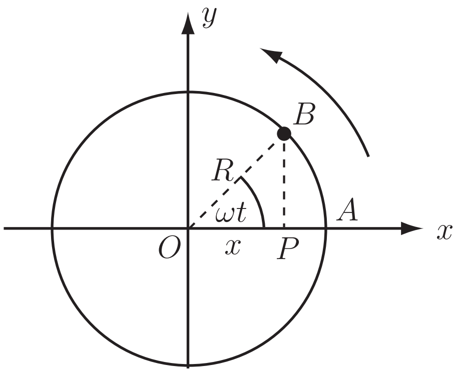


Figure 39

The position of P is given by

$$x = R\cos\omega t \qquad (1)$$

Clearly, from the known properties of the cosine function, we can deduce the following:

1. x varies periodically with t with period $T = \dfrac{2\pi}{\omega}$.
2. x will have maximum value $+R$ and minimum value $-R$.
   (This quantity R is called the **amplitude** of the motion.)

Task

Using (1) write down the values of x at the following times:

$t = 0$, $t = \dfrac{\pi}{2\omega}$, $t = \dfrac{\pi}{\omega}$, $t = \dfrac{3\pi}{2\omega}$, $t = \dfrac{2\pi}{\omega}$.

Your solution

t	0	$\frac{\pi}{2\omega}$	$\frac{\pi}{\omega}$	$\frac{3\pi}{2\omega}$	$\frac{2\pi}{\omega}$
x					

Answer

t	0	$\frac{\pi}{2\omega}$	$\frac{\pi}{\omega}$	$\frac{3\pi}{2\omega}$	$\frac{2\pi}{\omega}$
x	R	0	$-R$	0	R

Using (1) this 'to and fro' or ‘vibrational’ or ‘oscillatory’ motion between R and $-R$ continues indefinitely. The technical name for this motion is **simple harmonic**. To a good approximation it is the motion exhibited (i) by the end of a pendulum pulled through a small angle and then released (ii) by the end of a hanging spring pulled down and then released. See Figure 40 (in these cases damping of the pendulum or spring is ignored).

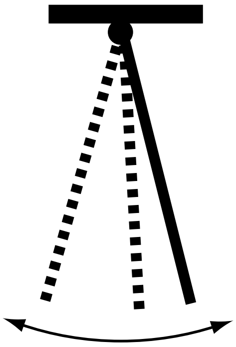


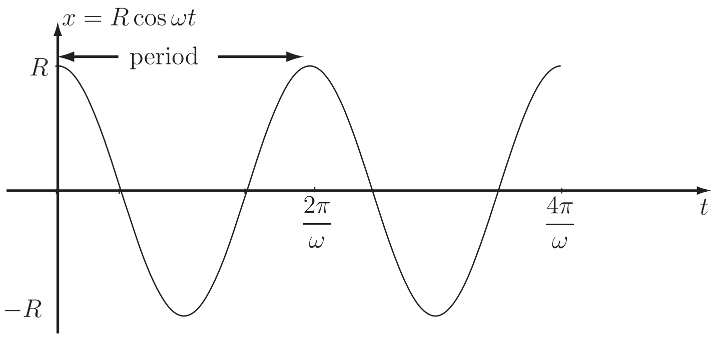
Figure 40

Using your knowledge of the cosine function and the results of the previous Task sketch the graph of x against t where

$$x = R\cos\omega t \quad \text{for } t = 0 \text{ to } t = \frac{4\pi}{\omega}$$

Your solution

Answer

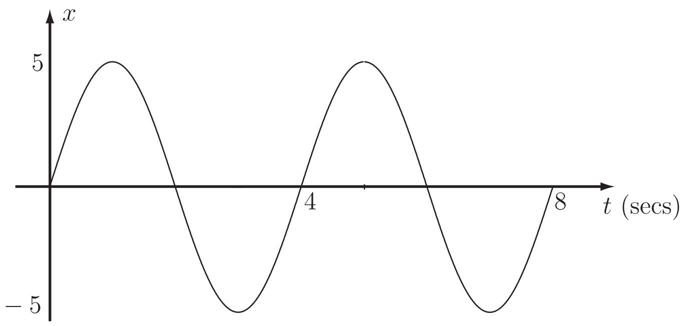


This graph shows part of a cosine **wave**, specifically two **periods** of oscillation. The shape of the graph suggests that the term wave is indeed an appropriate description.

We know that the **shape** of the cosine graph and the sine graph are identical but offset by $\frac{\pi}{2}$ radians horizontally. Bearing this in mind, attempt the following Task.

Task

Write the equation of the wave $x(t)$, part of which is shown in the following graph. You will need to find the period T and angular frequency ω.



Your solution

Answer

From the shape of the graph we have a **sine** wave rather than a cosine wave. The amplitude is 5. The period $T = 4s$ so the angular frequency $\omega = \frac{2\pi}{4} = \frac{\pi}{2}$. Hence $x = 5\sin\left(\frac{\pi t}{2}\right)$.

The quantity x, a function of t, is referred to as the **displacement** of the wave.

Time shifts between waves

We recall that $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$ which means that the graph of $x = \sin\theta$ is the same shape as that of $x = \cos\theta$ but is shifted to the right by $\frac{\pi}{2}$ radians.

Suppose now that we consider the waves

$$x_1 = R\cos 2t \qquad x_2 = R\sin 2t$$

Both have amplitude R, angular frequency $\omega = 2$ rad s^{-1}. Also

$$x_2 = R\cos\left(2t - \frac{\pi}{2}\right) = R\cos\left[2\left(t - \frac{\pi}{4}\right)\right]$$

The graphs of x_1 against t and of x_2 against t are said to have a **time shift** of $\frac{\pi}{4}$. Specifically x_1 is ahead of, or 'leads' x_2 by a time $\frac{\pi}{4}$ s.

More generally, consider the following two sine waves of the same amplitude and frequency:

$$x_1(t) = R\sin\omega t$$

$$x_2(t) = R\sin(\omega t - \alpha)$$

Now $x_1\left(t-\dfrac{\alpha}{\omega}\right)=R\sin\left[\omega\left(t-\dfrac{\alpha}{\omega}\right)\right]=R\sin(\omega t-\alpha)=x_2(t)$
so it is clear that the waves x_1 and x_2 are shifted in time by $\dfrac{\alpha}{\omega}$. Specifically x_1 leads x_2 by $\dfrac{\alpha}{\omega}$ (if $\alpha\ >0$).

Task

Calculate the time shift between the waves

$$\begin{aligned} x_1 &= 3\cos(10\pi t) \\ x_2 &= 3\cos\left(10\pi t+\frac{\pi}{4}\right) \end{aligned}$$

where the time t is in seconds.

Your solution

Answer

Note firstly that the waves have the same amplitude 3 and angular frequency 10π (corresponding to a common period $\dfrac{2\pi}{10\pi}=\dfrac{1}{5}$ s)

Now $\cos\left(10\pi t+\dfrac{\pi}{4}\right)=\cos\left(10\pi\left(t+\dfrac{1}{40}\right)\right)$

so $x_1\left(t+\dfrac{1}{40}\right)=x_2(t)$.

In other words the time shift is $\dfrac{1}{40}$ s, the wave x_2 **leads** the wave x_1 by this amount. Alternatively we could say that x_1 **lags** x_2 by $\dfrac{1}{40}$ s.

Key Point 20

The equations

$$x = R\cos\omega t \qquad\qquad x = R\sin\omega t$$

both represent waves of amplitude R and period $\dfrac{2\pi}{\omega}$.

The **time shift** between these waves is $\dfrac{\pi}{2\omega}$ because $\cos\left\{\omega\left(t - \dfrac{\pi}{2\omega}\right)\right\} = \sin\omega t$.

The **phase difference** between these waves is said to be $\dfrac{\pi}{2}$ because $\cos\left(\omega t - \dfrac{\pi}{2}\right) = \sin\omega t$

Combining two wave equations

A situation that arises in some applications is the need to combine two trigonometric terms such as

$A\cos\theta + B\sin\theta$ where A and B are constants.

For example this sort of situation might arise if we wish to combine two waves of the same frequency but not necessarily the same amplitude or phase. In particular we wish to be able to deal with an expression of the form

$$R_1\cos\omega t + R_2\sin\omega t$$

where the individual waves have, as we have seen, a time shift of $\dfrac{\pi}{2\omega}$ or a phase difference of $\dfrac{\pi}{2}$.

General Theory

Consider an expression $A\cos\theta + B\sin\theta$. We seek to transform this into the single form $C\cos(\theta - \alpha)$ (or $C\sin(\theta - \alpha)$), where C and α have to be determined. The problem is easily solved with the aid of trigonometric identities.
We know that

$$C\cos(\theta - \alpha) \equiv C(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

Hence if $A\cos\theta + B\sin\theta = C\cos(\theta - \alpha)$ then

$$A\cos\theta + B\sin\theta = (C\cos\alpha)\cos\theta + (C\sin\alpha)\sin\theta$$

For this to be an identity (true for all values of θ) we must be able to equate the coefficients of $\cos\theta$ and $\sin\theta$ on each side.
Hence

$$A = C\cos\alpha \qquad \text{and} \qquad B = C\sin\alpha \qquad (2)$$

Task

By squaring and adding the Equations (2), obtain C in terms of A and B.

Your solution

Answer

$A = C\cos\alpha$ and $B = C\sin\alpha$ gives

$$A^2 + B^2 = C^2\cos^2\alpha + C^2\sin^2\alpha = C^2(\cos^2\alpha + \sin^2\alpha) = C^2$$

$\therefore \quad C = \sqrt{A^2 + B^2}$ (We take the positive square root.)

Task

By eliminating C from Equations (2) and using the result of the previous Task, obtain α in terms of A and B.

Your solution

Answer

By division, $\dfrac{B}{A} = \dfrac{C\sin\alpha}{C\cos\alpha} = \tan\alpha$ so α is obtained by solving $\tan\alpha = \dfrac{B}{A}$. However, care must be taken to obtain the correct quadrant for α.

Key Point 21

If $A\cos\theta + B\sin\theta = C\cos(\theta - \alpha)$ then $C = \sqrt{A^2 + B^2}$ and $\tan\alpha = \dfrac{B}{A}$.

Note that the following cases arise for the location of α:

1. $A > 0,\ B > 0:$ 1st quadrant
2. $A < 0,\ B > 0:$ 2nd quadrant
3. $A < 0,\ B < 0:$ 3rd quadrant
4. $A > 0,\ B < 0:$ 4th quadrant

In terms of waves, using Key Point 21 we have

$$R_1 \cos \omega t + R_2 \sin \omega t = R \cos(\omega t - \alpha)$$

where $R = \sqrt{R_1^2 + R_2^2}$ and $\tan \alpha = \dfrac{R_2}{R_1}$.

The form $R\cos(\omega t - \alpha)$ is said to be the **amplitude/phase** form of the wave.

Example 5

Express in the form $C\cos(\theta - \alpha)$ each of the following:

(a) $3\cos\theta + 3\sin\theta$

(b) $-3\cos\theta + 3\sin\theta$

(c) $-3\cos\theta - 3\sin\theta$

(d) $3\cos\theta - 3\sin\theta$

Solution

In each case $C = \sqrt{A^2 + B^2} = \sqrt{9+9} = \sqrt{18}$

(a) $\tan\alpha = \dfrac{B}{A} = \dfrac{3}{3} = 1$ gives $\alpha = 45°$ (A and B are both positive so the first quadrant is the correct one.) Hence $3\cos\theta + 2\sin\theta = \sqrt{18}\cos(\theta - 45°) = \sqrt{18}\cos\left(\theta - \dfrac{\pi}{4}\right)$

(b) The angle α must be in the second quadrant as $A = -3 < 0$, $B = +3 > 0$. By calculator : $\tan\alpha = -1$ gives $\alpha = -45°$ but this is in the 4th quadrant. Remembering that $\tan\alpha$ has period π or $180°$ we must therefore add $180°$ to the calculator value to obtain the correct α value of $135°$. Hence

$$-3\cos\theta + 3\sin\theta = \sqrt{18}\cos(\theta - 135°)$$

(c) Here $A = -3$, $B = -3$ so α must be in the 3rd quadrant. $\tan\alpha = \dfrac{-3}{-3} = 1$ giving $\alpha = 45°$ by calculator. Hence adding $180°$ to this tells us that

$$-3\cos\theta - 3\sin\theta = \sqrt{18}\cos(\theta - 225°)$$

(d) Here $A = 3$ $B = -3$ so α is in the 4th quadrant. $\tan\alpha = -1$ gives us (correctly) $\alpha = -45°$ so

$$3\cos\theta - 3\sin\theta = \sqrt{18}\cos(\theta + 45°).$$

Note that in the amplitude/phase form the angle may be expressed in degrees or radians.

Task

Write the wave form $x = 3\cos\omega t + 4\sin\omega t$ in amplitude/phase form. Express the phase in radians to 3 d.p..

Your solution

Answer

We have $x = R\cos(\omega t - \alpha)$ where $R = \sqrt{3^2 + 4^2} = 5$ and $\tan\alpha = \frac{4}{3}$ from which, using the calculator in radian mode, $\alpha = 0.927$ radians. This is in the first quadrant $\left(0 < \alpha < \dfrac{\pi}{2}\right)$ which is correct since $A = 3$ and $B = 4$ are both positive. Hence $x = 5\cos(\omega t - 0.927)$.

Exercises

1. Write down the amplitude and the period of $y = \frac{5}{2}\sin 2\pi t$.

2. Write down the amplitude, frequency and time shift of

   (a) $y = 3\sin\left(2t - \dfrac{\pi}{3}\right)$ (b) $y = 15\cos\left(5t - \dfrac{3\pi}{2}\right)$

3. The current in an a.c. circuit is $i(t) = 30\sin 120\pi t$ amp where t is measured in seconds. What is the maximum current and at what times does it occur?

4. The depth y of water at the entrance to a small harbour at time t is $y = a\sin b\left(t - \dfrac{\pi}{2}\right) + k$ where k is the average depth. If the tidal period is 12 hours, the depths at high tide and low tide are 18 metres and 6 metres respectively, obtain $a,\ b,\ k$ and sketch two cycles of the graph of y.

5. The Fahrenheit temperature at a certain location over 1 complete day is modelled by

   $$F(t) = 60 + 10\sin\frac{\pi}{12}(t-8) \qquad 0 \leq t \leq 24$$

   where t is in the time in hours after midnight.

   (a) What are the temperatures at 8.00 am and 12.00 noon?

   (b) At what time is the temperature $60°$F?

   (c) Obtain the maximum and minimum temperatures and the times at which they occur.

6. In each of the following write down expressions for time-shifted sine and time-shifted cosine functions that satisfy the given conditions:

   (a) Amplitude 3, Period $\dfrac{2\pi}{3}$, Time shift $\dfrac{\pi}{3}$

   (b) Amplitude 0.7, Period 0.5, Time shift 4.

7. Write the a.c. current $i = 3\cos 5t + 4\sin 5t$ in the form $i = C\cos(5\pi - \alpha)$.

8. Show that if $A\cos\omega t + B\sin\omega t = C\sin(\omega t + \alpha)$ then

   $$C = \sqrt{A^2+B^2}, \qquad \cos\alpha = \frac{B}{C}, \qquad \sin\alpha = \frac{A}{C}.$$

9. Using Exercise 8 express the following in the amplitude/phase form $C\sin(\omega t + \alpha)$

   (a) $y = -\sqrt{3}\sin 2t + \cos 2t$ (b) $y = \cos 2t + \sqrt{3}\sin 2t$

10. The motion of a weight on a spring is given by $y = \dfrac{2}{3}\cos 8t - \dfrac{1}{6}\sin 8t$.

    Obtain C and α such that $y = C\sin(8t + \alpha)$

11. Show that for the two a.c. currents

    $i_1 = \sin\left(\omega t + \dfrac{\pi}{3}\right)$ and $i_2 = 3\cos\left(\omega t - \dfrac{\pi}{6}\right)$ then $i_1 + i_2 = 4\cos\left(\omega t - \dfrac{\pi}{6}\right)$.

12. Show that the power $P = \dfrac{v^2}{R}$ in an electrical circuit where $v = V_0 \cos\left(\omega t + \frac{\pi}{4}\right)$ is

$$P = \frac{V_0^2}{2R}(1 - \sin 2\omega t)$$

13. Show that the product of the two signals

$f_1(t) = A_1 \sin \omega t$ $f_2(t) = A_2 \sin\{\omega(t+\tau) + \phi\}$ is given by

$$f_1(t)f_2(t) = \frac{A_1 A_2}{2}\{\cos(\omega\tau + \phi) - \cos(2\omega t + \omega\tau + \phi)\}.$$

Answers

1. $y = \dfrac{5}{2}\sin 2\pi t$ has amplitude $\dfrac{5}{2}$. The period is $\dfrac{2\pi}{2\pi} = 1.$

   Check: $y(t+1) = \dfrac{5}{2}\sin(2\pi(t+1)) = \dfrac{5}{2}\sin(2\pi t + 2\pi) = \dfrac{5}{2}\sin 2\pi t = y(t)$

2. (a) Amplitude 3, Period $\dfrac{2\pi}{2} = \pi$. Writing $y = 3\sin 2\left(t - \dfrac{\pi}{6}\right)$ we see that there is a time shift of $\dfrac{\pi}{6}$ in this wave compared with $y = 3\sin 2t$.

   (b) Amplitude 15, Period $\dfrac{2\pi}{5}$. Clearly $y = 15\cos 5\left(t - \dfrac{3\pi}{10}\right)$ so there is a time shift of $\dfrac{3\pi}{10}$ compared with $y = 15\cos 5t$.

3. Maximum current $= 30$ amps at a time t such that $120\pi t = \dfrac{\pi}{2}$. i.e. $t = \dfrac{1}{240}\ s$.

   This maximum will occur again at $\left(\dfrac{1}{240} + \dfrac{n}{60}\right) s,\quad n = 1, 2, 3, \ldots$

4. $y = a\sin\left\{b\left(t - \dfrac{\pi}{2}\right)\right\} + h$. The period is $\dfrac{2\pi}{b} = 12$ hr $\quad \therefore\ b = \dfrac{\pi}{6}$ hr^{-1}.

   Also since $y_{\max} = a + k \quad y_{\min} = -a + k$ we have $a + k = 18 \quad -a + k = 6$ so $k = 12$ m, $a = 6$ m. i.e. $y = 6\sin\left\{\dfrac{\pi}{6}\left(t - \dfrac{\pi}{2}\right)\right\} + 12$.

5. $F(t) = 60 + 10\sin\dfrac{\pi}{12}(t - 8) \qquad 0 \leq t < 24$

   (a) At $t = 8:$ temp $= 60°$F. At $t = 12$: temp $= 60 + 10\sin\dfrac{\pi}{3} = 68.7°$F

   (b) $F(t) = 60$ when $\dfrac{\pi}{12}(t - 8) = 0,\ \pi,\ 2\pi, \ldots$ giving $t - 8 = 0, 12, 24, \ldots$ hours so $t = 8, 20, 32, \ldots$ hours i.e. in 1 day at $t = 8$ (8.00 am) and $t = 20$ (8.00 pm)

   (c) Maximum temperature is $70°$ F when $\dfrac{\pi}{12}(t = 8) = \dfrac{\pi}{2}$ i.e. at $t = 14$ (2.00 pm).

   Minimum temperature is $50°$F when $\dfrac{\pi}{12}(t - 8) = \dfrac{3\pi}{2}$ i.e. at $t = 26$ (2.00 am).

Answers

6. (a) $y = 3\sin(3t-\pi)$ $\quad y = 3\cos(3t-\pi)$ $\quad$ (b) $y = 0.7\sin(4\pi t-16\pi)$ $\quad y = 0.7\cos(4\pi t-16\pi)$

7. $C = \sqrt{3^2+4^2} = 5$ $\tan\alpha = \dfrac{4}{3}$ and α must be in the first quadrant (since $A = 3,\ B = 4$ are both positive.) $\quad \therefore\ \ \alpha = \tan^{-1}\dfrac{4}{3} = 0.9273$ rad $\qquad \therefore\ \ i = 5\cos(5t - 0.9273)$

8. Since $\sin(\omega t+\alpha) = \sin\omega t\cos\alpha + \cos\omega t\sin\alpha$ then $A = C\sin\alpha$ (coefficients of $\cos\omega t$)
$B = C\cos\alpha$ (coefficients of $\sin\omega t$) from which $C^2 = A^2+B^2,\ \sin\alpha = \dfrac{A}{C},\ \ \cos\alpha = \dfrac{B}{C}$

9. (a) $C = \sqrt{3+1} = 2;\ \ \cos\alpha = -\dfrac{\sqrt{3}}{2} \qquad \sin\alpha - \dfrac{1}{2}$ so α is in the second quadrant, $\alpha = \dfrac{5\pi}{6}\ \therefore \quad y = 2\sin\left(2t+\dfrac{5\pi}{6}\right)$ $\qquad$ (b) $y = 2\sin\left(2t+\dfrac{\pi}{6}\right)$

10. $C^2 = \dfrac{4}{9}+\dfrac{1}{36} = \dfrac{17}{36}$ so $\quad C = \dfrac{\sqrt{17}}{6} \quad \cos\alpha = \dfrac{-\frac{1}{6}}{\frac{\sqrt{17}}{6}} = -\dfrac{1}{\sqrt{17}} \qquad \sin\alpha = \dfrac{\frac{2}{3}}{\frac{\sqrt{17}}{6}} = \dfrac{4}{\sqrt{17}}$

so α is in the second quadrant. $\alpha = 1.8158$ radians.

11. Since $\sin x = \cos\left(x-\dfrac{\pi}{2}\right)$ $\sin\left(\omega t+\dfrac{\pi}{3}\right) = \cos\left(\omega t+\dfrac{\pi}{3}-\dfrac{\pi}{2}\right) = \cos\left(\omega t-\dfrac{\pi}{6}\right)$

$\therefore\ \ i_1+i_2 = \cos\left(\omega t-\dfrac{\pi}{6}\right)+3\cos\left(\omega t-\dfrac{\pi}{6}\right) = 4\cos\left(\omega t-\dfrac{\pi}{6}\right)$

12. $$v = V_0\cos\left(\omega t+\frac{\pi}{4}\right) = V_0\left(\cos\omega t\cos\tfrac{\pi}{4} - \sin\omega t\sin\tfrac{\pi}{4}\right) = \tfrac{V_0}{\sqrt{2}}\left(\cos\omega t - \sin\omega t\right)$$

$$\therefore \qquad v^2 = \frac{V_0^2}{2}(\cos^2\omega t+\sin^2\omega t - 2\sin\omega t\cos\omega t) = \frac{V_0^2}{2}(1-\sin 2\omega t)$$

and hence $\quad P = \dfrac{v^2}{R} = \dfrac{V_0^2}{2R}(1-\sin 2\omega t.)$

13. Since the required answer involves the difference of two cosine functions we use the identity

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

Hence with $\dfrac{A+B}{2} = \omega t,\quad \dfrac{B-A}{2} - \omega t + \omega\tau + \phi.$

We find, by adding these equations $\ B = 2\omega t+\omega\tau+\phi$ and by subtracting $\ A = -\omega\tau-\phi$.

Hence $\qquad \sin(\omega t)\sin(\omega t+\omega\tau+\phi) = \dfrac{1}{2}\left\{\cos(\omega\tau+\phi) - \cos(2\omega t+\omega\tau+\phi)\right\}.$

(Recall that $\cos(-x) = \cos x$.) The required result then follows immediately.