

# Oscillating Functions and Modelling

## 5.3

### Introduction

This Section describes ways in which trigonometric functions can be used to model situations involving periodic motion, which occur in a wide variety of scientific and engineering situations, and in nature.



### Prerequisites

Before starting this Section you should ...

- be competent at algebraic manipulation
- be familiar with trigonometric functions



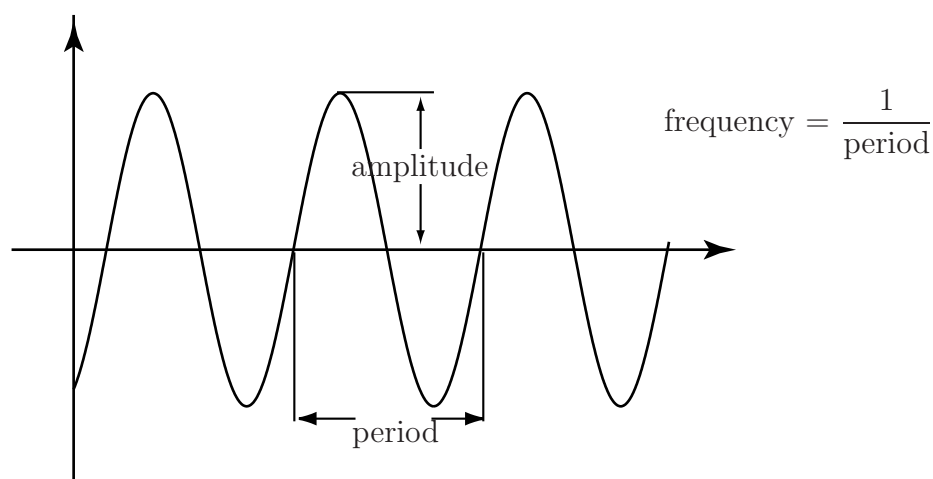
### Learning Outcomes

On completion you should be able to ...

- use trigonometric functions to model periodic motion
- define terms associated with the description of periodic motion

# 1. Oscillating functions: amplitude, period and frequency

Particular types of periodic functions (HELM 2.2) that are especially important in engineering are the sine and cosine functions. These are possible choices when modelling behaviour that involves oscillation or motion in a circle. The usefulness of these functions is rather limited if we confine our attention only to  $\sin(x)$  and  $\cos(x)$ . Use of functions such as  $3\sin(2x)$ ,  $5\cos(3x)$  and so on, and other functions made up of sums of functions of this type, enables the modelling of a great variety of situations where the quantity being modelled is known to change in a periodic way. Here we will examine the behaviour of sine and cosine functions and consider a modelling context where choice of a sine function is appropriate. Figure 6 shows how the terms **amplitude**, **period** and **frequency** are defined with respect to a general **sinusoid** (the name for any general sine or cosine function).



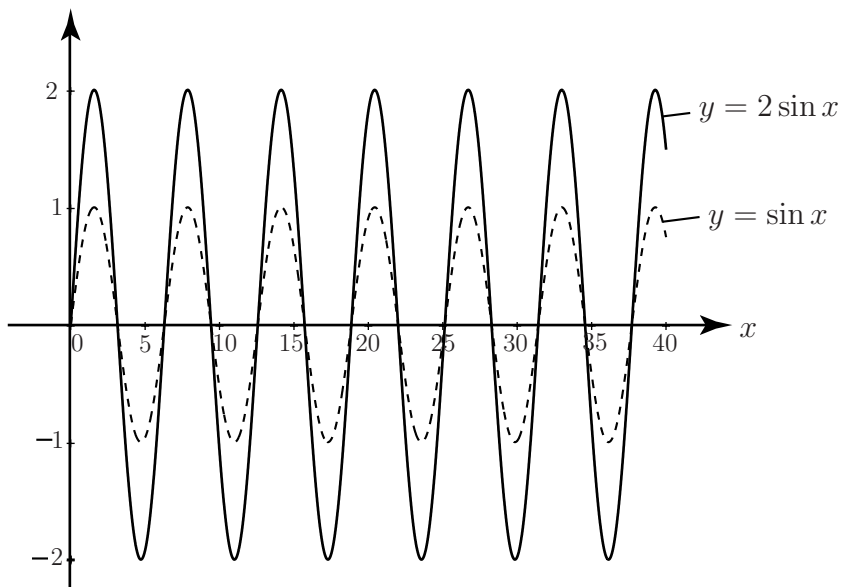
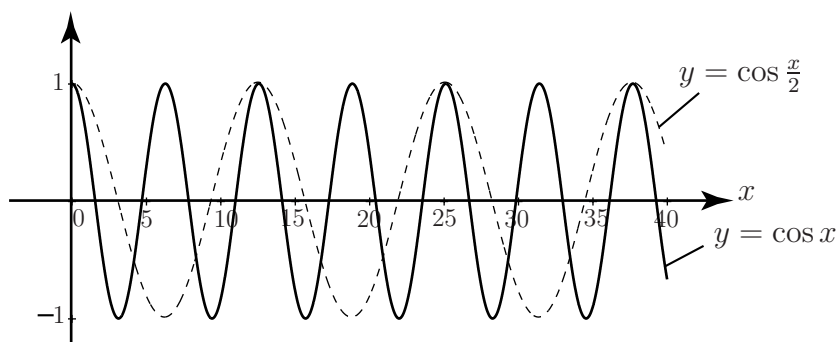
**Figure 6:** Defining amplitude and period for a sinusoid

The amplitude represents the difference between the maximum (or minimum) value of a sinusoidal function and its mean value (which is zero in Figure 6). The frequency represents the number of complete cycles of the function in each unit change in  $x$ . The period is such that  $f(x + T) = f(x)$  for all  $x$ , e.g. for  $\sin x$ ,  $T = 2\pi$ .

**Example 2**

Sketch the sinusoids:

(a)  $y = \sin x$    (b)  $y = 2 \sin x$    (c)  $y = \cos x$    (d)  $y = \cos \frac{x}{2}$

**Solution****Figure 7****Figure 8**



Using the graphs in Figures 7 and 8 on page 37, state the amplitude, frequency and period of

- (a)  $\sin x$  (b)  $2\sin x$  (c)  $\cos x$  (d)  $\cos \frac{x}{2}$

Give frequency and period in terms of  $\pi$ .

### Your solution

### Answer

- (a) amplitude = 1, frequency =  $1/2\pi$ , period =  $2\pi$ .  
(b) amplitude = 2, frequency =  $1/2\pi$ , period =  $2\pi$ .  
(c) amplitude = 1, frequency =  $1/2\pi$ , period =  $2\pi$ .  
(d) amplitude = 1, frequency =  $1/4\pi$ , period =  $4\pi$ .

See Figure 7 for the sine functions and Figure 8 for the cosine functions.

Note that (b) has twice the amplitude of (a) and (d) has half the frequency and twice the period of (c).

Note that the cosine functions  $\cos nx$  have the same shape as the sine functions  $\sin nx$  but, at  $x = 0$ , the cosine functions have a peak or maximum, whereas the sine functions have the value zero, which is the mean value for both of these functions. Indeed the graph of  $y = \cos x$  is exactly like that for  $y = \sin x$  with all the  $x$  values displaced by  $\pi/2$ .

More general forms of sine and cosine function are given by  $y = a \sin(bx)$ , and  $y = a \cos(bx)$  where  $a$  and  $b$  are arbitrary constants. These are functions with frequency  $\frac{b}{2\pi}$ , period  $\frac{2\pi}{b}$  and amplitude  $a$ . The peak values of the sine functions occur at  $x$  values equal to  $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$  etc. The minimum values occur at  $x$  values equal to  $\frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$  etc.

When the period is measured in seconds, frequency is measured in cycles per second or Hz which has units of 1/time.

## Exercises

- Figure 7 on page 37 shows on the same axes the graphs of  $y = \sin x$  and  $y = 2 \sin x$ .
  - State in words how the graph of  $y = 2 \sin x$  relates to the graph of  $y = \sin x$
  - Sketch the graphs of (i)  $y = \frac{1}{2} \sin x$ , (ii)  $y = \frac{1}{2} \sin x + \frac{1}{2}$
- Figure 8 on page 37 shows on the same axes the graph  $y = \cos x$  and  $y = \cos \frac{x}{2}$ 
  - State in words how the graph of  $y = \cos x$  relates to the graph of  $y = \cos \frac{x}{2}$
  - Sketch graphs of (i)  $y = \cos 2x$ , (ii)  $y = 2 \cos x$

### Answers

- $y = \sin 2x$  has the same form as  $y = \sin x$  but all the  $y$  values are doubled. The graph is 'stretched' vertically.
- $y = \cos \frac{x}{2}$  has the same form as  $y = \cos x$  but all the  $y$  values are halved. The graph is 'shrunk' vertically.

## 2. Oscillating functions: modelling tides

We consider how the function

$$h = 3.2 \sin(2.7t + 8.5)$$

might be used to model the rise and fall of the tide in a harbour.

Figure 9 shows a graph of this function for  $(0 \leq t \leq 5)$ .

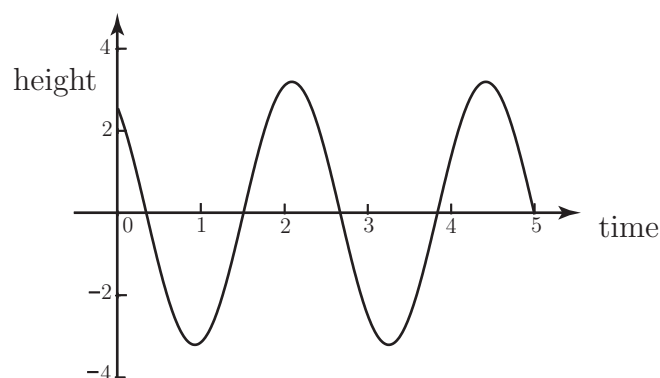


Figure 9

We consider some aspects of this graph and model. It seems reasonable to suppose that the tide creates an oscillation of the water level in the harbour of  $h$  m about some mean value represented on the graph by  $h = 0$ . There seems to be a low tide near  $t = 1$  and another low tide just after  $t = 3$ . Since we expect intervals of 12 to 14 hours between low tides around the U.K., this suggests that time in this graph is specified in 6-hour intervals.



Write down the amplitude, period and frequency of  $h = 3.2 \sin(2.7t + 8.5)$

### Your solution

### Answer

The amplitude of the change in water level in the harbour is 3.2 m. The period of the function is given by  $2\pi/2.7 = 2.3271$  between successive high tides or successive low tides. This corresponds to  $2.3271 \times 6$  hours = 13.96 hours between high tides. The frequency of the function is  $2.7/2\pi = 0.4297$ .

The peak levels of the graph correspond to times when the sine function has the value 1. The lowest points correspond to times when the sine function is  $-1$ . At these times the arguments of the sine function (i.e.  $2.7t + 8.5$ ) are an odd number of  $\pi/2$  starting at  $3\pi/2$  for the first low tide.

So far all of this may be deduced from the general form  $y = a\sin(bx)$  and from the modelling context. However there is an additional term in the function being considered here. This is a constant 8.5 within the sine function. When  $t = 0$  the presence of this constant means that the intercept on the height axis is  $3.2\sin(8.5) = 2.56$ , implying that the water level is 2.56 m above the mean value at the start of timing. The constant 8.5 has displaced the sine curve sideways. This constant is known as the **phase** of the function. Phase is measured in radians as it is an angle.

As remarked earlier, at  $t = 0$ , this function has the value  $3.2\sin(8.5)$ . Since  $\sin(8.5) = \sin(8.5 - 2\pi) \approx \sin(2.2168)$ , we can replace the constant 8.5 by 2.2168 without altering the values on the graph. This means that the function

$$h = 3.2 \sin(2.74t + 2.2168)$$

does just as well as the original function in representing the tidal variation in the harbour. We now rewrite this latest form of the function, representing the variation of water level in the harbour, so that time is measured in hours rather than in six-hourly intervals. The effect of changing the units of time to hours from 6 hours is to decrease the coefficient of  $t$  in the sine function by a factor of 6, so that the new function is

$$h = 3.2 \sin(0.45t + 2.168). \quad \text{See Figure 10.}$$

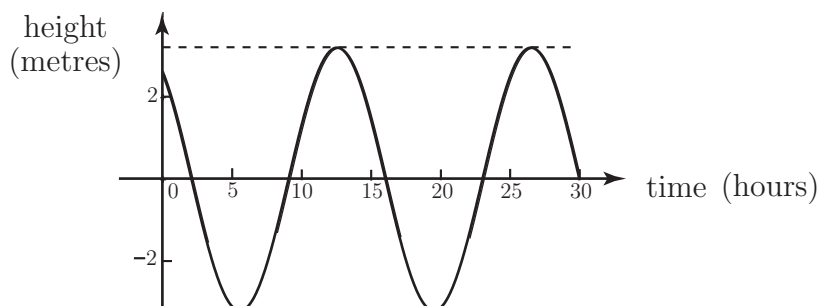


Figure 10

We can use the latest form of the function to calculate the time of the first low tide assuming that  $t = 0$  corresponds to midnight.

At the first low tide,  $h = -3.2$  and  $\sin(0.45t + 2.2168) = -1$ ,

Using the fact that  $\sin\left(\frac{3\pi}{2}\right) = -1$ , we have

$$0.45t + 2.2168 = 3\pi/2, \text{ giving } t = 5.5458 = 5.55 \text{ to 2 d.p.}$$

so the first low tide is at 5:30 a.m.



For the above tide modelling situation, assume that  $t = 0$  corresponds to midnight. Calculate

- the time of the first high tide after midnight
- the times either side of midnight at which the water is at its mean level.

### Your solution

### Answer

(a) At the first high tide,  $h = 3.2$  and  $\sin(0.45t + 2.2168) = 1$ , so  $0.45t + 2.2168 = 5\pi/2$  giving  $t = 12.5271$  so the first high tide is at half past midday

(b) When the water level is at the mean value,

$$\sin(0.45t + 2.2168) = 0.$$

At the mean level before midnight, using the fact that  $\sin(0) = 0$  we have

$$0.45t = -2.2168 \text{ so } t = -4.9262 = -4.93 \text{ to 2 d.p.}$$

So this mean level occurs nearly 5 hours before midnight, i.e. about 7 p.m. the previous day. The next mean level will occur one period, or 13.963 hours, later, at approximately 9 a.m.

There are various rules connected with sine and cosine functions that can be summarised at this point.

- (1) Placing a multiplier before  $\sin x$  or  $\cos x$  (e.g.  $2 \sin x$ ) changes the amplitude without changing the period.
- (2) Placing a multiplier before  $x$  in  $\sin x$  or  $\cos x$ , (e.g.  $\sin 3x$ ), changes the period or frequency without changing the amplitude.
- (3) As with any function, the addition of a constant (e.g.  $4 + \sin x$ ) raises or lowers the whole graph of the sine or cosine function. It alters the mean value without changing the amplitude.
- (4) Changing the sign within a cosine function has no effect, (e.g.  $\cos(-x) = \cos x$ ).
- (5) Changing the sign within a sine function changes the sign of the function, (e.g.  $\sin(-x) = -\sin x$ ).
- (6) Placing a constant or altering the constant  $b$  in  $\sin(ax + b)$  or  $\cos(ax + b)$  changes the phase and shifts the sine or cosine function along the  $x$ -axis.



- (a) Write down the amplitude and period of  $y = \sin(3x)$
- (b) Write down the amplitude and frequency of  $y = 3\sin(2x)$
- (c) Write down the amplitude, period and frequency of  $y = a \sin(bx)$
- (d) Write down the amplitude, period, frequency and phase of

$$y = 4 \sin(2x + 7).$$

- (e) Write down an equivalent expression to that in (d) but with the phase less than  $2\pi$ .

### Your solution

### Answer

- (a) amplitude = 1      period =  $2\pi/3$
- (b) amplitude = 3      frequency =  $\frac{2}{2\pi} = \frac{1}{\pi}$
- (c) amplitude =  $a$       period =  $2\pi/b$       frequency =  $b/2\pi$
- (d) amplitude = 4      period =  $\frac{2\pi}{2} = \pi$       frequency =  $\frac{1}{\pi}$       phase = 7
- (e)  $y = 4\sin(2x + 7 - 2\pi) = 4\sin(2x + 0.7168)$





Write down a function relating water level ( $L$  m) in a harbour to time ( $T$  hours), starting when the level is equal to the mean level of 5 m, that has an amplitude of 2 m and has a period of twelve hours.

### Your solution

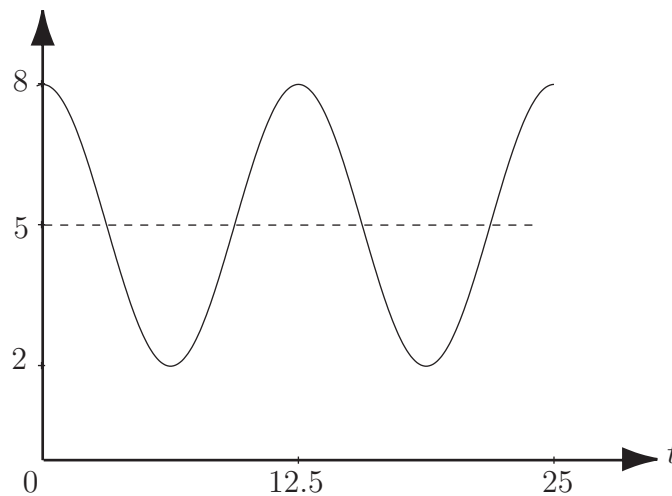
### Answer

In the general form  $y = a \sin(bx + c) + d$ , the phase  $c = 0$ , the period  $\frac{2\pi}{b} = 12$ , so  $b = \frac{\pi}{6}$  the amplitude  $a = 2$ , the mean value  $d = 5$ .

$$L = 2 \sin\left(\frac{\pi}{6}T\right) + 5 \quad (T \geq 0)$$



The diagram shows a graph of a typical variation of the depth ( $d$  metres) of water in a particular harbour with time ( $t$  hours) as the depth changes with the tide.



(a) Find a suitable equation for the curve in the diagram:

### Your solution

**Answer**

Equation is of the form

$$h = a + b \cos(\omega t) \quad (\text{or } h = a + b \sin(\omega t + \frac{\pi}{2}))$$

By inspection,  $a = 5$  and  $b = 3$ .

The period  $T = 12.5 = \frac{2\pi}{\omega}$  so  $\omega = \frac{4\pi}{25}$  ( $= 0.502655$ )

so the equation of the curve is  $h = 5 + 3 \cos(\frac{4\pi}{25}t)$

(b) A boat enters the harbour in late morning on a day when the high tide is at 2 p.m. The boat needs a water depth of 4 m to sail safely. What advice would you give to its pilot about when to leave the harbour if the boat is not to be forced to wait in the harbour through the evening low tide?

**Your solution****Answer**

Put  $h = 4$  into the equation:

$$4 = 5 + 3 \cos(\frac{4\pi}{25}t) \quad \text{implying} \quad -\frac{1}{3} = \cos(\frac{4\pi}{25}t)$$

Now, inverting the cosine:

$$\frac{4\pi}{25}t = \cos^{-1}(-\frac{1}{3}) = 1.91063 \quad \text{giving} \quad t = 3.80108 \text{ hours.}$$

So the advice to the pilot should be that he needs to be clear of the harbour by 5:45 pm at the very latest - and that he should allow a safety margin.

(c) State two modelling assumptions you have made:

**Your solution****Answer**

Assumptions likely are:

The tide on the day in question is *typical*.

No waves.

A sinusoidal function accurately models the effect of the tide on sea level.