Differentiating Products and Quotients 11.4

Introduction

We have seen, in the first three Sections, how standard functions like x^n, e^{ax}, $\sin ax$, $\cos ax$, $\ln ax$ may be differentiated.

In this Section we see how more complicated functions may be differentiated. We concentrate, for the moment, on products and quotients of standard functions like $x^n e^{ax}$, $\dfrac{e^{ax}\ln x}{\sin x}$.

We will see that two simple rules may be consistently employed to obtain the derivatives of such functions.

Prerequisites

Before starting this Section you should . . .

- be able to differentiate the standard functions: logarithms, polynomials, exponentials, and trigonometric functions
- be able to manipulate algebraic expressions

Learning Outcomes

On completion you should be able to . . .

- differentiate products and quotients of the standard functions
- differentiate a quotient using the product rule

1. Differentiating a product

In previous Sections we have examined the process of differentiating functions. We found how to obtain the derivative of many commonly occurring functions. These are recorded in the following table (remember, arguments of trigonometric functions are assumed to be in **radians**).

Table 2

y	$\dfrac{dy}{dx}$
x^n	nx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$
$\sec ax$	$a\sec x\tan x$
$\ln ax$	$\dfrac{1}{x}$
e^{ax}	ae^{ax}
$\cosh ax$	$a\sinh ax$
$\sinh ax$	$a\cosh ax$

In this Section we consider how to differentiate non-standard functions - in particular those which can be written as the **product** of standard functions. Being able to differentiate such functions depends upon the following Key Point.

Key Point 9

Product Rule

If $y = f(x)g(x)$ then $\dfrac{dy}{dx} = \dfrac{df}{dx}g(x) + f(x)\dfrac{dg}{dx}$ (or $y' = f'g + fg'$)

If $y = u.v$ then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ (or $y' = uv' + vu'$)

These versions are equivalent, and called the **product rule**.

We shall not prove this result, instead we shall concentrate on its use.

Example 8

Differentiate (a) $y = x^2 \sin x$ (b) $y = x \ln x$

Solution

(a) Here $f(x) = x^2$, $g(x) = \sin x$ $\therefore \dfrac{df}{dx} = 2x$, $\dfrac{dg}{dx} = \cos x$

and so $\dfrac{dy}{dx} = 2x(\sin x) + x^2(\cos x) = x(2\sin x + x\cos x)$

(b) Here $f(x) = x$, $g(x) = \ln x$ $\therefore \dfrac{df}{dx} = 1$, $\dfrac{dg}{dx} = \dfrac{1}{x}$

and so $\dfrac{dy}{dx} = 1.(\ln x) + x.\left(\dfrac{1}{x}\right) = \ln x + 1$

Task

Determine the derivatives of the following functions (a) $y = e^x \ln x$, (b) $y = \dfrac{e^{2x}}{x^2}$

(a) Use the product rule:

Your solution

$f(x) =$ $\dfrac{df}{dx} =$ $g(x) =$ $\dfrac{dg}{dx} =$

$\therefore \dfrac{dy}{dx} =$

Answer

$\dfrac{dy}{dx} = e^x \ln x + \dfrac{e^x}{x}$

(b) Write $y = (x^{-2})e^{2x}$ and then differentiate:

Your solution

$f(x) =$ $\dfrac{df}{dx} =$ $g(x) =$ $\dfrac{dg}{dx} =$

$\therefore \dfrac{dy}{dx} =$

Answer

$\dfrac{dy}{dx} = (-2x^{-3})e^{2x} + x^{-2}(2e^{2x}) = \dfrac{2e^{2x}}{x^3}(-1 + x)$

The rule for differentiating a product can be extended to any number of products. If, for example, $y = f(x)g(x)h(x)$ then

$$\begin{aligned} \frac{dy}{dx} &= \frac{df}{dx}[g(x)h(x)] + f(x)\frac{d}{dx}[g(x)h(x)] \\ &= \frac{df}{dx}g(x)h(x) + f(x)\left\{\frac{dg}{dx}h(x) + g(x)\frac{dh}{dx}\right\} \\ &= \frac{df}{dx}g(x)h(x) + f(x)\frac{dg}{dx}h(x) + f(x)g(x)\frac{dh}{dx} \end{aligned}$$

That is, each function in the product is differentiated in turn and the three results added together.

Example 9

If $y = xe^{2x}\sin x$ then find $\dfrac{dy}{dx}$.

Solution

Here $f(x) = x$, $g(x) = e^{2x}$, $h(x) = \sin x$

$$\frac{df}{dx} = 1, \quad \frac{dg}{dx} = 2e^{2x}, \quad \frac{dh}{dx} = \cos x$$

$$\begin{aligned} \therefore \qquad \frac{dy}{dx} &= 1(e^{2x}\sin x) + x(2e^{2x})\sin x + xe^{2x}(\cos x) \\ &= e^{2x}(\sin x + 2x\sin x + x\cos x) \end{aligned}$$

Task

Obtain the first derivative of $y = x^2(\ln x)\sinh x$.

Firstly identify the three functions:

Your solution

$f(x) =$ $\qquad$ $g(x) =$ $\qquad$ $h(x) =$

Answer

$f(x) = x^2, \quad g(x) = \ln x, \quad h(x) = \sinh x$

Now find the derivative of each of these functions:

Your solution

$$\frac{df}{dx} = \qquad \frac{dg}{dx} = \qquad \frac{dh}{dx} =$$

Answer

$\dfrac{df}{dx} = 2x, \quad \dfrac{dg}{dx} = \dfrac{1}{x}, \quad \dfrac{dh}{dx} = \cosh x$

Finally obtain $\dfrac{dy}{dx}$:

Your solution

Answer

$$\begin{aligned}\frac{dy}{dx} &= 2x(\ln x)\sinh x + x^2\left(\frac{1}{x}\right)\sinh x + x^2 \ln x(\cosh x)\\ &= 2x \ln x \sinh x + x \sinh x + x^2 \ln x \cosh x\end{aligned}$$

Task

Find the second derivative of $y = x^2(\ln x)\sinh x$ by differentiating each of the three terms making up $\dfrac{dy}{dx}$ found in the previous Task ($2x \ln x \sinh x$, $x \sinh x$, $x^2 \ln x \cosh x$), and finally, simplify your answer by collecting like terms together:

Your solution

$\dfrac{d}{dx}(2x \ln x \sinh x) =$

$\dfrac{d}{dx}(x \sinh x) =$

$\dfrac{d}{dx}(x^2 \ln x \cosh x) =$

$\dfrac{d^2y}{dx^2} =$

Answer

$\dfrac{d^2y}{dx^2} = (2 + x^2)\ln x \sinh x + 3\sinh x + 2x\cosh x + 4x \ln x \cosh x$

Exercises

1. In each case find the derivative of the function

   (a) $y = x \tan x$

   (b) $y = x^4 \ln(2x)$

   (c) $y = \sin^2 x$

   (d) $y = e^{2x} \cos 3x$

2. Find the derivatives of:

   (a) $y = \dfrac{x}{\cos x}$

   (b) $y = e^x \sin x$

   (c) Obtain the derivative of $y = xe^x \tan x$ using the results of parts (a) and (b).

Answers

1. (a) $\dfrac{dy}{dx} = \tan x + x \sec^2 x$

   (b) $\dfrac{dy}{dx} = 4x^3 \ln(2x) + \dfrac{x^4}{x} = x^3(4\ln(2x) + 1)$

   (c) $y = \sin x . \sin x$

   $\therefore \quad \dfrac{dy}{dx} = \cos x \sin x + \sin x \cos x = 2 \sin x \cos x = \sin 2x$

   (d) $\dfrac{dy}{dx} = (2e^{2x}) \cos 3x + e^{2x}(-3 \sin 3x) = e^{2x}(2\cos 3x - 3 \sin 3x)$

2. (a) $y = x \sec x \qquad \therefore \quad \dfrac{dy}{dx} = \sec x + x \sec x \tan x$

   (b) $\dfrac{dy}{dx} = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$

   (c) The derivative of $y = xe^x \tan x = (x \sec x)(e^x \sin x)$ is found by applying the product rule to the results of (a) and (b):

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x \sec x).e^x \sin x + (x \sec x)\frac{d}{dx}(e^x \sin x)\\ &= (\sec x + x \sec x \tan x)e^x \sin x + x \sec x(e^x)(\sin x + \cos x)\\ &= e^x(x + \tan x + x \tan x + x \tan^2 x)\end{aligned}$$

2. Differentiating a quotient

In this Section we consider functions of the form $y = \dfrac{f(x)}{g(x)}$. To find the derivative of such a function we make use of the following Key Point:

Key Point 10

Quotient Rule

If $\quad y = \dfrac{f(x)}{g(x)} \quad$ then $\quad \dfrac{dy}{dx} = \dfrac{g(x)\dfrac{df}{dx} - \dfrac{dg}{dx}f(x)}{[g(x)]^2} \qquad$ (or $\; y' = \dfrac{gf' - g'f}{g^2}$)

If $\quad y = \dfrac{u}{v} \quad$ then $\quad \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - \dfrac{dv}{dx}u}{v^2} \qquad$ (or $\; y' = \dfrac{vu' - v'u}{v^2}$)

These are two equivalent versions of the **quotient rule**.

Example 10

Find the derivative of $y = \dfrac{\ln x}{x}$

Solution

Here $f(x) = \ln x$ and $g(x) = x$

$\therefore \quad \dfrac{df}{dx} = \dfrac{1}{x}$ and $\dfrac{dg}{dx} = 1$

Hence $\dfrac{dy}{dx} = \dfrac{x\left(\dfrac{1}{x}\right) - 1(\ln x)}{[x]^2} = \dfrac{1 - \ln x}{x^2}$

Task

Obtain the derivative of $y = \dfrac{\sin x}{x^2}$ (a) using the formula for differentiating a product and (b) using the formula for differentiating a quotient.

(a) Write $y = x^{-2}\sin x$ then use the product rule to find $\dfrac{dy}{dx}$:

Your solution

Answer

$y = x^{-2}\sin x \qquad \therefore \qquad \dfrac{dy}{dx} = (-2x^{-3})\sin x + x^{-2}\cos x \qquad \therefore \qquad \dfrac{dy}{dx} = \dfrac{-2\sin x + x\cos x}{x^3}$

(b) Now use the quotient rule instead to find $\dfrac{dy}{dx}$:

Your solution

Answer

$y = \dfrac{\sin x}{x^2} \qquad \therefore \qquad \dfrac{dy}{dx} = \dfrac{x^2(\cos x) - (2x)\sin x}{(x^2])^2} = \dfrac{x\cos x - 2\sin x}{x^3}$

Exercise

Find the derivatives of the following:

(a) $(2x^3 - 4x^2)(3x^5 + x^2)$

(b) $\dfrac{2x^3 + 4}{x^2 - 4x + 1}$

(c) $\dfrac{x^2 + 2x + 1}{x^2 - 2x + 1}$

(d) $(x^2 + 3)(2x - 5)(3x + 2)$

(e) $\dfrac{(2x + 1)(3x - 1)}{x + 5}$

(f) $(\ln x)\sin x$

(g) $(\ln x)/\sin x$

(h) e^x/x^2

(i) $\dfrac{e^x \sin x}{\cos 2x}$

Answer

(a) $48x^7 - 84x^6 + 10x^4 - 16x^3$

(b) $\dfrac{2x^4 - 16x^3 + 6x^2 - 8x + 16}{(x^2 - 4x + 1)^2}$

(c) $-\dfrac{4(x + 1)}{(x - 1)^3}$

(d) $24x^3 - 33x^2 + 16x - 33$

(e) $\dfrac{6(x^2 + 10x + 1)}{(x + 5)^2}$

(f) $\dfrac{1}{x}\sin x + (\ln x)\cos x$

(g) $\dfrac{\sin x \left(\dfrac{1}{x}\right) - (\ln x)\cos x}{\sin^2 x} = \text{cosec}\,x(\dfrac{1}{x} - \cot x \ln x)$

(h) $\dfrac{x^2 e^x - 2xe^x}{x^4} = (x^{-2} - 2x^{-3})e^x$

(i) $\dfrac{\cos 2x(e^x \sin x + e^x \cos x) + 2\sin 2x e^x \sin x}{\cos^2 2x}$

$$= e^x[(\sin x + \cos x)\sec 2x + 2\sin x \sin 2x \sec^2 2x]$$