Contents 14

Applications of Integration 1

Learning outcomes

In this Workbook you will learn to interpret an integral as the limit of a sum. You will learn how to apply this approach to the meaning of an integral to calculate important attributes of a curve: the area under the curve, the length of a curve segment, the volume and surface area obtained when a segment of a curve is rotated about an axis. Other quantities of interest which can also be calculated using integration is the position of the centre of mass of a plane lamina and the moment of inertia of a lamina about an axis. You will also learn how to determine the mean value of an integal.

Integration as the Limit of a Sum

14.1

Introduction

In HELM 13, integration was introduced as the reverse of differentiation. A more rigorous treatment would show that integration is a process of adding or 'summation'. By viewing integration from this perspective it is possible to apply the techniques of integration to finding areas, volumes, centres of gravity and many other important quantities.

The content of this Section is important because it is here that integration is defined more carefully. A thorough understanding of the process involved is essential if you need to apply integration techniques to practical problems.

Prerequisites

Before starting this Section you should ...

- be able to calculate definite integrals

Learning Outcomes

On completion you should be able to ...

- explain integration as the limit of a sum
- evaluate the limit of a sum in simple cases

1. The limit of a sum

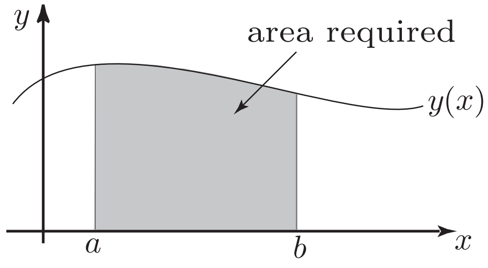
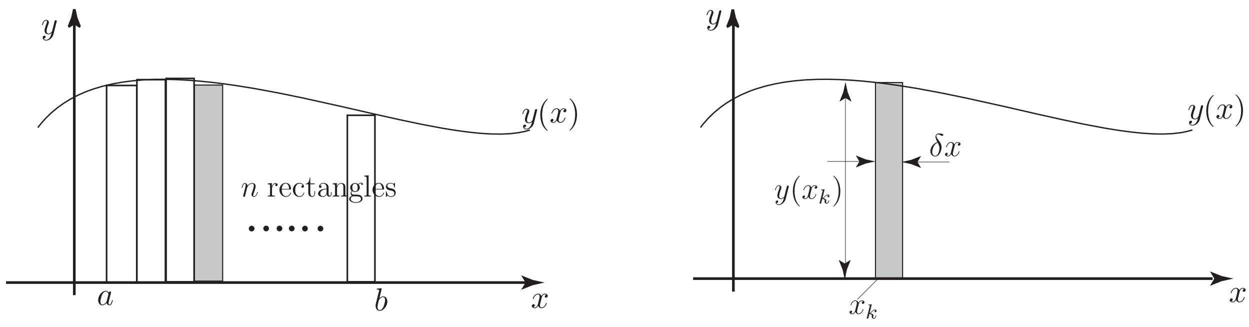


Figure 1: The area under a curve

Consider the graph of the positive function $y(x)$ shown in Figure 1. Suppose we are interested in finding the area under the graph between $x = a$ and $x = b$. One way in which this area can be **approximated** is to divide it into a number of rectangles of equal width, find the area of each rectangle, and then add up all these individual rectangular areas. This is illustrated in Figure 2a, which shows the area divided into n rectangles (with some small discrepancies at the tops), and Figure 2b which shows the dimensions of a typical rectangle which is located at $x = x_k$.



(a) The area approximated by n rectangles

(b) A typical rectangle

Figure 2

We wish to find an expression for the area under a curve based on the sum of many rectangles. Firstly, we note that the distance from $x = a$ to $x = b$ is $b - a$. In Figure 2a the area has been divided into n rectangles. If n rectangles span the distance from a to b the width of each rectangle is $\dfrac{b-a}{n}$:

It is conventional to label the width of each rectangle as δx, i.e. $\delta x = \dfrac{b-a}{n}$. We label the x coordinates at the left-hand side of the rectangles as x_1, x_2 up to x_n (here $x_1 = a$ and $x_{n+1} = b$). A typical rectangle, the kth rectangle, is shown in Figure 2b. Note that its height is $y(x_k)$, so its area is $y(x_k) \times \delta x$.

The sum of the areas of all n rectangles is then

$$y(x_1)\delta x + y(x_2)\delta x + y(x_3)\delta x + \cdots + y(x_n)\delta x$$

which we write concisely using sigma notation as

$$\sum_{k=1}^{n} y(x_k)\delta x$$

This quantity gives us an estimate of the area under the curve but it is not exact. To improve the estimate we must take a large number of very thin rectangles. So, what we want to find is the value of this sum when n tends to infinity and δx tends to zero. We write this value as

$$\lim_{n\to\infty}\sum_{k=1}^{n} y(x_k)\delta x$$

The lower and upper limits on the sum correspond to the first rectangle and last rectangle where $x = a$ and $x = b$ respectively and so we can write this limit in the equivalent form

$$\lim_{\delta x\to 0}\sum_{x=a}^{x=b} y(x)\delta x \qquad (1)$$

Here, as the number of rectangles increases without bound we drop the subscript k from x_k and write $y(x)$ which is the value of y at a 'typical' value of x. If this sum can actually be found, it is called the **definite integral** of $y(x)$, from $x = a$ to $x = b$ and it is written $\int_a^b y(x)dx$. You are already familiar with the technique for evaluating definite integrals which was studied in Section 14.2. Therefore we have the following definition:

Key Point 1

The definite integral $\int_a^b y(x)dx$ is defined as $\lim_{\delta x\to 0}\sum_{x=a}^{x=b} y(x)\delta x$

Note that the quantity δx represents the thickness of a small but finite rectangle. When we have taken the limit as δx tends to zero to obtain the integral, we write dx, which reminds us of the variable of integration.

This process of dividing an area into very small regions, performing a calculation on each region, and then adding the results by means of an integral is very important. This will become apparent when finding volumes, centres of gravity, moments of inertia etc in the following Sections where similar procedures are followed.

Example 1

The area under the graph of $y = x^2$ between $x = 0$ and $x = 1$ is to be found by approximating it by a large number of thin rectangles and finding the limit of the sum of their areas. From Equation (1) this is $\lim_{\delta x \to 0} \sum_{x=0}^{x=1} y(x)\,\delta x$. Write down the integral which this sum defines and evaluate it to obtain the area under the curve.

Solution

The limit of the sum defines the integral $\int_0^1 y(x)dx$. Here $y = x^2$ and so $\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

To show that the process of taking the limit of a sum actually works we investigate the problem in detail. We use the idea of the limit of a sum to find the area under the graph of $y = x^2$ between $x = 0$ and $x = 1$, as illustrated in Figure 3.

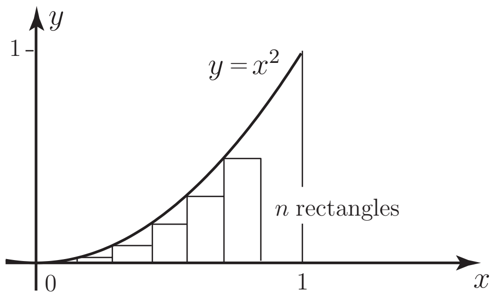
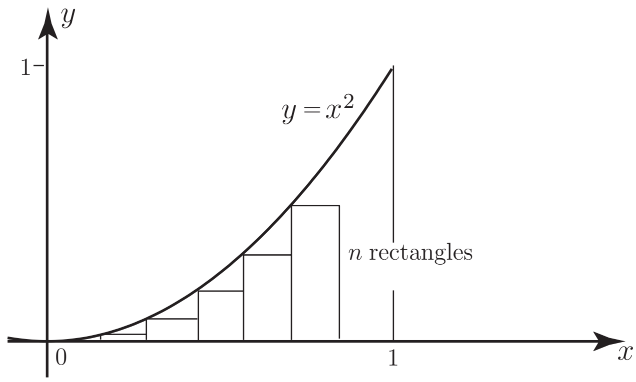


Figure 3: The area under $y = x^2$ is approximated by a number of thin rectangles

Task

Refer to the diagram below to help you answer the questions below.



If the interval between $x = 0$ and $x = 1$ is divided into n rectangles what is the width of each rectangle?

Your solution

Answer

$1/n$

Mark this on the diagram. What is the x coordinate at the left-hand side of the first rectangle ?

Your solution

Answer

0

What is the x coordinate at the left-hand side of the second rectangle ?

Your solution

Answer

$1/n$

What is the x coordinate at the left-hand side of the third rectangle ?

Your solution

Answer

$2/n$

Mark these coordinates on the diagram.
What is the x coordinate at the left-hand side of the kth rectangle ?

Your solution

Answer

$(k-1)/n$

Given that $y = x^2$, what is the y coordinate at the left-hand side of the kth rectangle ?

Your solution

Answer

$$\left(\frac{k-1}{n}\right)^2$$

The area of the kth rectangle is its height $\times$ its width. Write down the area of the kth rectangle:

Your solution

Answer

$$\left(\frac{k-1}{n}\right)^2 \times \frac{1}{n} = \frac{(k-1)^2}{n^3}$$

To find the total area A_n of the n rectangles we must add up all these individual rectangular areas:

$$A_n = \sum_{k=1}^{n} \frac{(k-1)^2}{n^3}$$

This sum can be simplified and then calculated as follows. You will need to make use of the formulas for the sum of the first n integers, and the sum of the squares of the first n integers:

$$\sum_{k=1}^{n} 1 = n, \qquad \sum_{k=1}^{n} k = \frac{1}{2}n(n+1), \qquad \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Then, the total area of the rectangles is given by

$$\begin{aligned}
A_n &= \sum_{k=1}^{n} \frac{(k-1)^2}{n^3} \\
&= \frac{1}{n^3}\sum_{k=1}^{n}(k-1)^2 \\
&= \frac{1}{n^3}\sum_{k=1}^{n}(k^2-2k+1) \\
&= \frac{1}{n^3}\left(\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right) \\
&= \frac{1}{n^3}\left(\frac{n}{6}(n+1)(2n+1) - 2\frac{n}{2}(n+1) + n\right) \\
&= \frac{1}{n^2}\left(\frac{(n+1)(2n+1)}{6} - (n+1) + 1\right) \\
&= \frac{1}{n^2}\left(\frac{(n+1)(2n+1)}{6} - n\right) \\
&= \frac{1}{6n^2}\left(2n^2-3n+1\right) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}
\end{aligned}$$

Note that this is a formula for the **exact** total area of the n rectangles. It is an **estimate** of the area under the graph of $y = x^2$. However, as n gets larger, the terms $\dfrac{1}{2n}$ and $\dfrac{1}{6n^2}$ become small and will eventually tend to zero. If we let n tend to infinity we obtain the exact answer of $\dfrac{1}{3}$.

The required area is $\dfrac{1}{3}$. It has been found as **the limit of a sum** and of course agrees with that calculated by integration.

In the calculations which follow in subsequent Sections the need to evaluate complicated limits like this is avoided by performing the integration using the techniques of HELM 13. Nevertheless it will sometimes be necessary to go through the process of dividing a region into small sections, performing a calculation on each section and then adding the results, in order to formulate the integral required. When numerical methods of integration are studied (HELM 31) this summation method will prove fundamental.

Engineering Example 1

Pulley belt tension

Problem

Consider that a belt is partially wound around a pulley so that there is a difference in the tension either side of the pulley (see Figure 4). The pulley will be stationary as long as the friction between belt and pulley is sufficient. The frictional force on the pulley will depend on the extent of the contact between belt and pulley i.e. on the angle θ shown in Figure 4. Given that the tensions on either side of the belt are T_2 and T_1 and that the coefficient of friction between belt and pulley is μ, find an expression for T_2 in terms of T_1, μ and θ.

Solution

Consider a small element of the belt, at angle θ where the tension is T. Changing the angle by a small amount $\Delta\theta$ changes the tension from T to $T + \Delta T$.

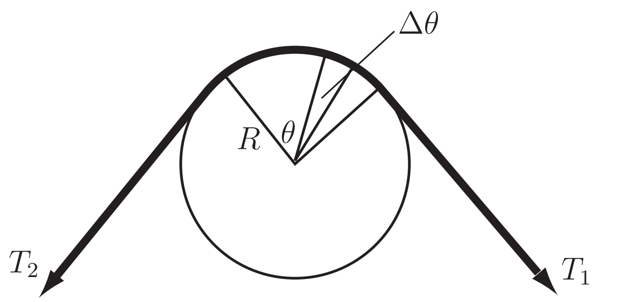


Figure 4

Take moments about the centre of the pulley, denoting the radius of the pulley by R and assuming that the frictional force is μT per unit length. For the pulley to remain stationary,

$$R\Delta\theta\mu T = R(T + \Delta T) - RT \quad \text{or} \quad \Delta\theta = \frac{\Delta T}{\mu T}.$$

Using integration as the limit of a sum,

$$\theta = \int_{T_1}^{T_2} \frac{dT}{\mu T} = \frac{1}{\mu}\Big[\ \ln T\Big]_{T_1}^{T_2} = \frac{1}{\mu}\ln\left(\frac{T_2}{T_1}\right). \quad \text{So } T_2 = T_1 e^{\mu\theta}.$$

Exercises

1. Find the area under $y = x + 1$ from $x = 0$ to $x = 10$ using the limit of a sum.

2. Find the area under $y = 3x^2$ from $x = 0$ to $x = 2$ using the limit of a sum.

3. Write down, but do not evaluate, the integral defined by the limit as $\delta x \to 0$, or $\delta t \to 0$ of the following sums:

   (a) $\displaystyle\sum_{x=0}^{x=1} x^3\delta x$, (b) $\displaystyle\sum_{x=0}^{x=4} 4\pi x^2\delta x$, (c) $\displaystyle\sum_{t=0}^{t=1} t^3\delta t$, (d) $\displaystyle\sum_{x=0}^{x=1} 6mx^2\delta x$.

Answers

1. 60,

2. 8,

3. (a) $\displaystyle\int_0^1 x^3 dx$, (b) $4\pi\displaystyle\int_0^4 x^2 dx$, (c) $\displaystyle\int_0^1 t^3 dt$, (d) $6m\displaystyle\int_0^1 x^2 dx$.