Volumes of Revolution 14.3

Introduction

In this Section we show how the concept of integration as the limit of a sum, introduced in Section 14.1, can be used to find volumes of solids formed when curves are rotated around the x or y axis.

Prerequisites

Before starting this Section you should ...

- be able to calculate definite integrals
- understand integration as the limit of a sum

Learning Outcomes

On completion you should be able to ...

- calculate volumes of revolution

1. Volumes generated by rotating curves about the x-axis

Figure 8 shows a graph of the function $y = 2x$ for x between 0 and 3.

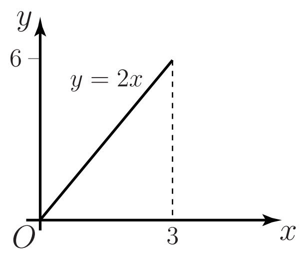


Figure 8: A graph of the function $y = 2x$, for $0 \leq x \leq 3$

Imagine rotating the line $y = 2x$ by one complete revolution (360^0 or 2π radians) around the x-axis. The surface so formed is the surface of a cone as shown in Figure 9. Such a three-dimensional shape is known as a **solid of revolution**. We now discuss how to obtain the volumes of such solids of revolution.

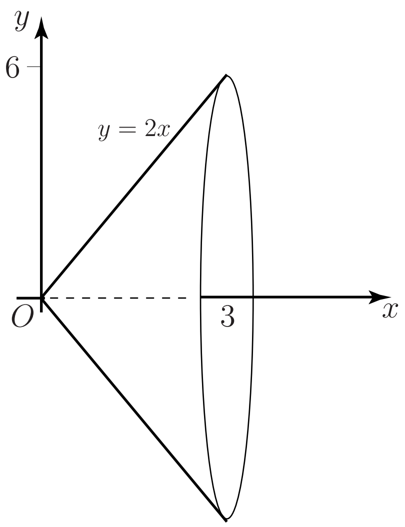
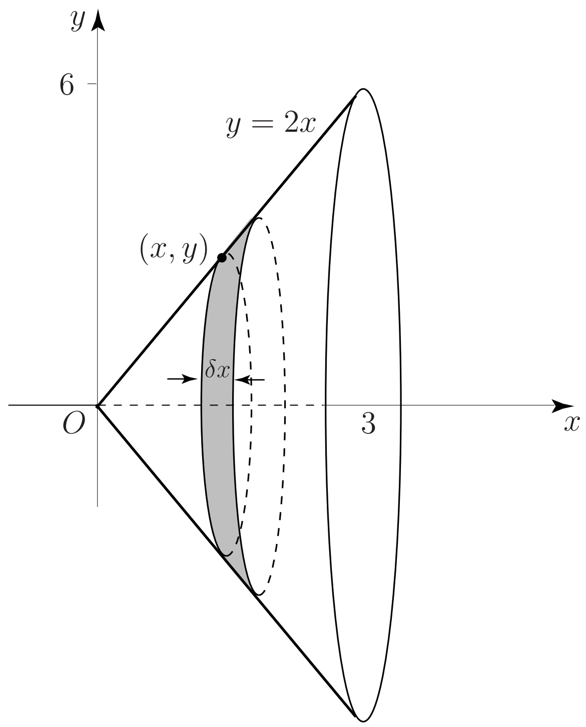


Figure 9: When the line $y = 2x$ is rotated around the axis, a solid is generated

Task

Find the volume of the cone generated by rotating $y = 2x$, for $0 \leq x \leq 3$, around the x-axis, as shown in Figure 9.

In order to find the volume of this solid we assume that it is composed of lots of thin circular discs all aligned perpendicular to the x-axis, such as that shown in the diagram. From the diagram below we note that a typical disc has radius y, which in this case equals $2x$, and thickness δx.



The cone is divided into a number of thin circular discs.

The volume of a circular disc is the circular area multiplied by the thickness.

Write down an expression for the volume of this typical disc:

Your solution

Answer

$\pi(2x)^2\delta x = 4\pi x^2 \delta x$

To find the total volume we must sum the contributions from all discs and find the limit of this sum as the number of discs tends to infinity and δx tends to zero. That is

$$\lim_{\delta x \to 0} \sum_{x=0}^{x=3} 4\pi x^2 \delta x$$

This is the definition of a definite integral. Write down the corresponding integral:

Your solution

Answer

$$\int_0^3 4\pi x^2\, dx$$

Find the required volume by performing the integration:

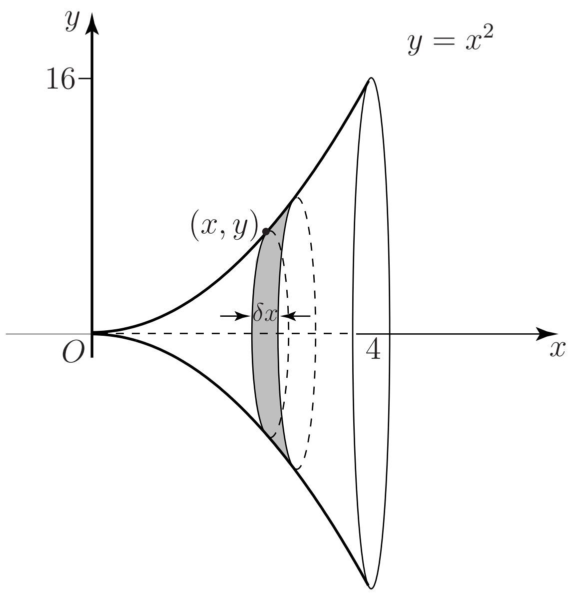
Your solution

Answer

$$\left[\frac{4\pi x^3}{3}\right]_0^3 = 36\pi$$

Task

A graph of the function $y = x^2$ for x between 0 and 4 is shown in the diagram. The graph is rotated around the x-axis to produce the solid shown. Find its volume.



The solid of revolution is divided into a number of thin circular discs.

As in the previous Task, the solid is considered to be composed of lots of circular discs of radius y, (which in this example is equal to x^2), and thickness δx.

Write down the volume of each disc:

Your solution

Answer

$\pi(x^2)^2\,\delta x = \pi x^4 \delta x$

Write down the expression which represents summing the volumes of all such discs:

Your solution

Answer

$$\sum_{x=0}^{x=4} \pi x^4 \delta x$$

Write down the integral which results from taking the limit of the sum as $\delta x \to 0$:

Your solution

Answer

$\int_0^4 \pi x^4 \, dx$

Perform the integration to find the volume of the solid:

Your solution

Answer

$\frac{4^5 \pi}{5} = 204.8\pi$

Task

In general, suppose the graph of $y(x)$ between $x = a$ and $x = b$ is rotated about the x-axis, and the solid so formed is considered to be composed of lots of circular discs of thickness δx.

Write down an expression for the radius of a typical disc:

Your solution

Answer

y

Write down an expression for the volume of a typical disc:

Your solution

Answer

$\pi y^2 \delta x$

The total volume is found by summing these individual volumes and taking the limit as δx tends to zero:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x$$

Write down the definite integral which this sum defines:

Your solution

Answer

$\int_a^b \pi y^2 \, dx$

Key Point 5

If the graph of $y(x)$, between $x = a$ and $x = b$, is rotated about the x-axis the volume of the solid formed is

$$\int_a^b \pi y^2\, dx$$

Exercises

1. Find the volume of the solid formed when that part of the curve between $y = x^2$ between $x = 1$ and $x = 2$ is rotated about the x-axis.
2. The parabola $y^2 = 4x$ for $0 \leq x \leq 1$, is rotated around the x-axis. Find the volume of the solid formed.

Answers 1. $31\pi/5$, 2. 2π.

2. Volumes generated by rotating curves about the y-axis

We can obtain a different solid of revolution by rotating a curve around the y-axis instead of around the x-axis. See Figure 10.

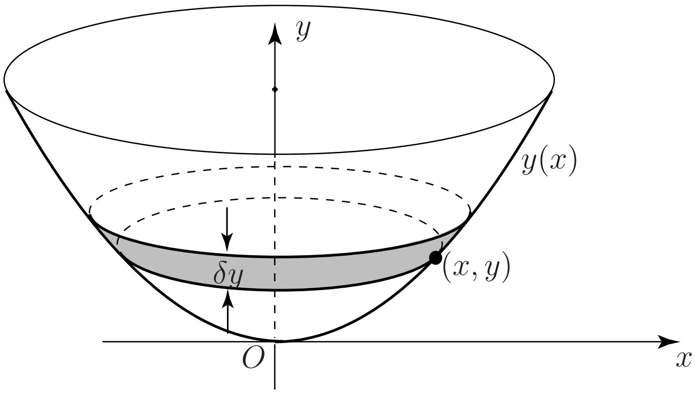


Figure 10: A solid generated by rotation around the y-axis

To find the volume of this solid it is divided into a number of circular discs as before, but this time the discs are horizontal. The radius of a typical disc is x and its thickness is δy. The volume of the disc will be $\pi x^2 \delta y$.

The total volume is found by summing these individual volumes and taking the limit as $\delta y \to 0$. If the lower and upper limits on y are c and d, we obtain for the volume:

$$\lim_{\delta y \to 0} \sum_{y=c}^{y=d} \pi x^2 \delta y \quad \text{which is the definite integral} \quad \int_c^d \pi x^2\, dy$$

Key Point 6

If the graph of $y(x)$, between $y = c$ and $y = d$, is rotated about the y-axis the volume of the solid formed is

$$\int_c^d \pi x^2\, dy$$

Task

Find the volume generated when the graph of $y = x^2$ between $x = 0$ and $x = 1$ is rotated around the y-axis.

Using Key Point 6 write down the required integral:

Your solution

Answer

$$\int_0^1 \pi x^2\, dy$$

This integral can be written entirely in terms of y, using the fact that $y = x^2$ to eliminate x. Do this now, and then evaluate the integral:

Your solution

Answer

$$\int_0^1 \pi x^2\, dy = \int_0^1 \pi y\, dy = \left[\frac{\pi y^2}{2}\right]_0^1 = \frac{\pi}{2}$$

Exercises

1. The curve $y = x^2$ for $1 < x < 2$ is rotated about the y-axis. Find the volume of the solid formed.
2. The line $y = 2 - 2x$ for $0 \leq x \leq 2$ is rotated around the y-axis. Find the volume of revolution.

Answers

1. $\dfrac{15\pi}{2}$ 2. $\dfrac{16\pi}{3}$.