Lengths of Curves and Surfaces of Revolution 14.4

Introduction

Integration can be used to find the length of a curve and the area of the surface generated when a curve is rotated around an axis. In this Section we state and use formulae for doing this.

Prerequisites

Before starting this Section you should ...

- be able to calculate definite integrals

Learning Outcomes

On completion you should be able to ...

- find the length of curves
- find the area of the surface generated when a curve is rotated about an axis

1. The length of a curve

To find the length of a curve in the xy plane we first divide the curve into a large number of pieces. We measure (or, at least, approximate) the length of each piece and then by an obvious summation process obtain an estimate for the length of the curve. Theoretically, we allow the number of pieces to increase without bound, implying that the length of each piece will tend to zero. In this limit the summation process becomes an integration process.

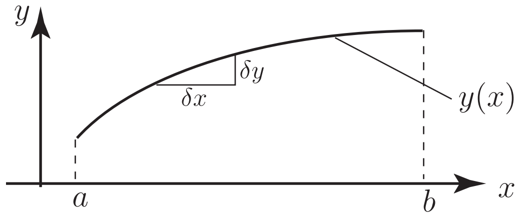


Figure 11

Figure 11 shows the portion of the curve $y(x)$ between $x = a$ and $x = b$. A small piece of this curve has been selected and can be considered as the hypotenuse of a triangle with base δx and height δy. (Here δx and δy are intended to be 'small' so that the **curved segment** can be regarded as a **straight segment**.)

Using Pythagoras' theorem, the length of the hypotenuse is: $\sqrt{\delta x^2 + \delta y^2} = \sqrt{1 + \left(\dfrac{\delta y}{\delta x}\right)^2}\ \delta x$

By summing all such contributions between $x = a$ and $x = b$, and letting $\delta x \to 0$ we obtain an expression for the total length of the curve:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\ \delta x$$

But we already know how to write such an expression in terms of an integral. We obtain the following result:

Key Point 7

Given a curve with equation $y = f(x)$, then the length of the curve between the points where $x = a$ and $x = b$ is given by the formula:

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Because of the complicated form of the integrand, and in particular the square root, integrals of this type are often difficult to calculate. In practice, approximate numerical methods rather than exact methods are normally needed to perform the integration. We shall first illustrate the application of the formula in Key Point 7 by a problem which could be calculated in a much simpler way, before looking at some harder problems.

Example 5

Find the length of the curve $y = 3x + 2$ between $x = 1$ and $x = 5$.

Solution

In this Example, the curve is in fact a straight line segment, and its length could be obtained using Pythagoras' theorem without the need for integration.

Notice from the formula in Key Point 7 that it is necessary to find $\dfrac{dy}{dx}$, which in this case is 3.

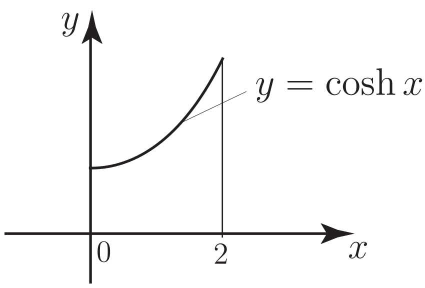
Applying the formula we find

$$\begin{aligned}\text{length of curve} &= \int_1^5 \sqrt{1+(3)^2}\,dx \\ &= \int_1^5 \sqrt{10}\,dx \\ &= \left[\sqrt{10}x\right]_1^5 \\ &= (5-1)\sqrt{10} = 4\sqrt{10} = 12.65 \quad \text{to 2 d.p.}\end{aligned}$$

Thus the length of the curve $y = 3x + 2$ between the points where $x = 1$ and $x = 5$ is 12.65 units.

Task

Find the length of the curve $y = \cosh x$ between $x = 0$ and $x = 2$ shown in the diagram.



First write down $\dfrac{dy}{dx}$:

Your solution

$\dfrac{dy}{dx} =$

Answer

$\dfrac{dy}{dx} = \sinh x$

Hence write down the required integral:

Your solution

Answer

$$\int_0^2 \sqrt{1+\sinh^2 x}\,dx$$

This integral can be evaluated by making use of the hyperbolic identity $\cosh^2 x - \sinh^2 x \equiv 1$.

Write down the integral which results after applying this identity:

Your solution

Answer

$$\int_0^2 \cosh x\,dx$$

Perform the integration to find the required length:

Your solution

Answer

$\Big[\sinh x\Big]_0^2 = 3.63$ to 2 d.p.

Thus the length of $y = \cosh x$ between $x = 0$ and $x = 2$ is 3.63 units.

The next Task is more complicated still and requires the use of a hyperbolic substitution and knowledge of the hyperbolic identities.

Task

Find the length of the curve $y = x^2$ between $x = 0$ and $x = 3$.

Given $y = x^2$ then $\dfrac{dy}{dx} = 2x$. Use this result and apply the formula in Key Point 7 to obtain the integral required:

Your solution

Answer

$$\int_0^3 \sqrt{1+4x^2}\,dx$$

Make the substitution $x = \frac{1}{2}\sinh u$, giving $\dfrac{dx}{du} = \dfrac{1}{2}\cosh u$, to obtain an integral in terms of u:

Your solution

Answer

$$\int_0^{\sinh^{-1} 6} \sqrt{1+\sinh^2 u}\,\frac{1}{2}\cosh u\,du$$

Use the hyperbolic identity $\cosh^2 u - \sinh^2 u \equiv 1$ to eliminate $\sinh^2 u$:

Your solution

Answer

$$\frac{1}{2}\int_0^{\sinh^{-1} 6} \cosh^2 u\,du$$

Use the hyperbolic identity $\cosh^2 u \equiv \frac{1}{2}(\cosh 2u + 1)$ to rewrite the integrand in terms of $\cosh 2u$:

Your solution

Answer

$$\frac{1}{4}\int_0^{\sinh^{-1} 6} (\cosh 2u + 1)\,du$$

Finally, perform the integration to complete the calculation:

Your solution

Answer

$$\begin{aligned} \frac{1}{4}\int_0^{\sinh^{-1} 6} (\cosh 2u + 1)\,du &= \frac{1}{4}\left[\frac{\sinh 2u}{2} + u\right]_0^{\sinh^{-1} 6} \\ &= 9.75 \quad \text{to 2 d.p.} \end{aligned}$$

Thus the length of the curve $y = x^2$ between $x = 0$ and $x = 3$ is 9.75 units.

Exercises

1. Find the length of the line $y = 2x + 7$ between $x = 1$ and $x = 3$ using the technique of this Section. Verify your result from your knowledge of the straight line.

2. Find the length of $y = x^{3/2}$ between $x = 0$ and $x = 5$.

3. Calculate the length of the curve $y^2 = 4x^3$ between $x = 0$ and $x = 2$, in the first quadrant.

Answers

1. $2\sqrt{5} \approx 4.47$. The distance is from (1.9) to $(3, 13)$ along the line. This is given using Pythagoras' theorem as $\sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$.

2. 12.41

3. 6.06 (first quadrant only).

2. The area of a surface of revolution

In Section 14.2 we found an expression for the volume of a solid of revolution. Here we consider the more complicated problem of formulating an expression for the surface area of a solid of revolution.

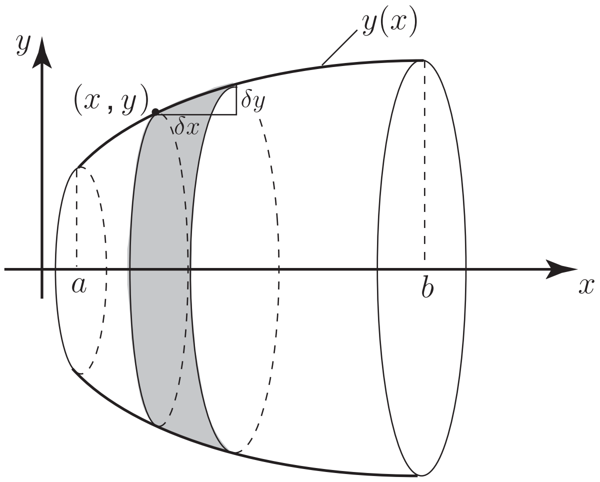


Figure 12

Figure 12 shows the portion of the curve $y(x)$ between $x = a$ and $x = b$ which is rotated around the x axis through $360°$. A small disc, of thickness δx, of the solid of revolution has been selected. Its radius is y and so its circumference has length $2\pi y$. (As usual we assume δx is 'small' so that the **curved** part of $y(x)$ representing the hypotenuse of the highlighted 'triangle' can be regarded as **straight**). This surface 'ribbon', shown shaded, has a length $2\pi y$ and a width $\sqrt{(\delta x)^2 + (\delta y)^2}$ and so its area is, to a good approximation, $2\pi y\sqrt{(\delta x)^2 + (\delta y)^2}$. We now let $\delta x \to 0$ to obtain the result in Key Point 8:

Key Point 8

Given a curve with equation $y = f(x)$, then the surface area of the solid generated by rotating that part of the curve between the points where $x = a$ and $x = b$ around the x axis is given by the formula:

$$\text{area of surface} \;=\; \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Task

Find the area of the surface generated when the part of the curve $y = x^3$ between $x = 0$ and $x = 4$ is rotated around the x axis.

Using Key Point 8 write down the integral:

Your solution

Answer

$$\text{area} \;=\; \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^4 2\pi x^3\sqrt{1+(3x^2)^2}\,dx = \int_0^4 2\pi x^3\sqrt{1+9x^4}\,dx$$

Use the substitution $u = 1 + 9x^4$ so $\dfrac{du}{dx} = 36x^3$ to write down the integral in terms of u:

Your solution

Answer

$$\frac{\pi}{18}\int_1^{2305} \sqrt{u}\,du$$

Perform the integration:

Your solution

Answer

$$\frac{\pi}{18}\left[\frac{2u^{3/2}}{3}\right]_1^{2305}$$

Apply the limits of integration to find the area:

Your solution

Answer

$\frac{\pi}{27}\left((2305)^{3/2}-1\right)$

Exercises

1. The line $y = x$ between $x = 0$ and $x = 1$ is rotated around the x axis.
   (a) Find the area of the surface generated.
   (b) Verify this result by finding the curved surface area of the corresponding cone. (The curved surface area of a cone of radius r and slant height ℓ is $\pi r \ell$.)
2. Find the area of the surface generated when $y = \sqrt{x}$ in the interval $1 \leq x \leq 2$ is rotated about the x axis.

Answers

1. $\pi\sqrt{2}$
2. 8.28