Power Series

16.4

Introduction

In this Section we consider power series. These are examples of infinite series where each term contains a variable, x, raised to a positive integer power. We use the ratio test to obtain the **radius of convergence** R, of the power series and state the important result that the series is absolutely convergent if $|x| < R$, divergent if $|x| > R$ and may or may not be convergent if $x = \pm R$. Finally, we extend the work to apply to general power series when the variable x is replaced by $(x - x_0)$.

Prerequisites

Before starting this Section you should ...

- have knowledge of infinite series and of the ratio test
- have knowledge of inequalities and of the factorial notation.

Learning Outcomes

On completion you should be able to ...

- explain what a power series is
- obtain the radius of convergence for a power series
- explain what a general power series is

1. Power series

A power series is simply a sum of terms each of which contains a variable raised to a non-negative integer power. To illustrate:

$$x - x^3 + x^5 - x^7 + \cdots$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

are examples of power series. In HELM 16.3 we encountered an important example of a power series, the binomial series:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

which, as we have already noted, represents the function $(1+x)^p$ as long as the variable x satisfies $|x| < 1$.

A power series has the general form

$$b_0 + b_1x + b_2x^2 + \cdots = \sum_{p=0}^{\infty} b_p x^p$$

where $b_0, b_1, b_2, \cdots$ are constants. Note that, in the summation notation, we have chosen to start the series at $p = 0$. This is to ensure that the power series can include a constant term b_0 since $x^0 = 1$.

The convergence, or otherwise, of a power series, clearly depends upon the value of x chosen. For example, the power series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

is convergent if $x = -1$ (for then it is the alternating harmonic series) and divergent if $x = +1$ (for then it is the harmonic series).

2. The radius of convergence

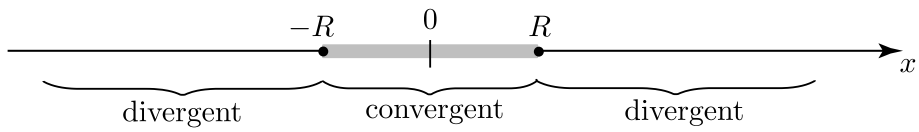
The most important statement one can make about a power series is that there exists a number, R, called the radius of convergence, such that if $|x| < R$ the power series is absolutely convergent and if $|x| > R$ the power series is divergent. At the two points $x = -R$ and $x = R$ the power series may be convergent or divergent.

Key Point 11

Convergence of Power Series

For a power series $\sum_{p=0}^{\infty} b_p x^p$ with radius of convergence R then

- the series converges absolutely if $|x| < R$
- the series diverges if $|x| > R$
- the series may be convergent or divergent at $x = \pm R$



For any particular power series $\sum_{p=0}^{\infty} b_p x^p$ the value of R can be obtained using the ratio test. We know, from the ratio test that $\sum_{p=0}^{\infty} b_p x^p$ is absolutely convergent if

$$\lim_{p\to\infty} \frac{|b_{p+1}x^{p+1}|}{|b_p x^p|} = \lim_{p\to\infty} \left|\frac{b_{p+1}}{b_p}\right| |x| < 1 \quad \text{implying} \quad |x| < \lim_{p\to\infty} \left|\frac{b_p}{b_{p+1}}\right| \quad \text{and so} \quad R = \lim_{p\to\infty} \left|\frac{b_p}{b_{p+1}}\right|.$$

Example 2

(a) Find the radius of convergence of the series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

(b) Investigate what happens at the end-points $x = -1$, $x = +1$ of the region of absolute convergence.

Solution

(a) Here $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots \quad = \sum_{p=0}^{\infty} \frac{x^p}{p+1}$

so

$$b_p = \frac{1}{p+1} \qquad \therefore \qquad b_{p+1} = \frac{1}{p+2}$$

In this case,

$$R = \lim_{p\to\infty} \left| \frac{p+2}{p+1} \right| = 1$$

so the given series is absolutely convergent if $|x| < 1$ and is divergent if $|x| > 1$.

(b) At $x = +1$ the series is $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ which is divergent (the harmonic series). However, at $x = -1$ the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which is convergent (the alternating harmonic series).

Finally, therefore, the series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

is convergent if $-1 \leq x < 1$.

Task

Find the range of values of x for which the following power series converges:

$$1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots$$

First find the coefficient of x^p:

Your solution

$b_p =$

Answer

$b_p = \dfrac{1}{3^p}$

Now find R, the radius of convergence:

Your solution

$$R = \lim_{p\to\infty} \left| \frac{b_p}{b_{p+1}} \right| =$$

Answer

$$R = \lim_{p\to\infty} \left| \frac{b_p}{b_{p+1}} \right| = \lim_{p\to\infty} \left| \frac{3^{p+1}}{3^p} \right| = \lim_{p\to\infty} (3) = 3.$$

When $x = \pm 3$ the series is clearly divergent. Hence the series is convergent only if $-3 < x < 3$.

3. Properties of power series

Let P_1 and P_2 represent two power series with radii of convergence R_1 and R_2 respectively. We can combine P_1 and P_2 together by addition and multiplication. We find the following properties:

Key Point 12

If P_1 and P_2 are power series with respective radii of convergence R_1 and R_2 then the sum (P_1+P_2) and the product (P_1P_2) are each power series with the radius of convergence being the **smaller** of R_1 and R_2.

Power series can also be differentiated and integrated on a term by term basis:

Key Point 13

If P_1 is a power series with radius of convergence R_1 then

$$\frac{d}{dx}(P_1) \quad \text{and} \quad \int (P_1)\,dx$$

are each power series with radius of convergence R_1

Example 3

Using the known result that $\quad (1+x)^p = 1 + px + \dfrac{p(p-1)}{2!}x^2 + \cdots \qquad |x| < 1,$

choose $p = \frac{1}{2}$ and by differentiating obtain the power series expression for $(1+x)^{-\frac{1}{2}}$.

Solution

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$

Differentiating both sides: $\quad \dfrac{1}{2}(1+x)^{-\frac{1}{2}} = \dfrac{1}{2} + \dfrac{1}{2}\left(-\dfrac{1}{2}\right)x + \dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \cdots$

Multiplying through by 2: $\quad (1+x)^{-\frac{1}{2}} = 1 - \dfrac{1}{2}x + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \cdots$

This result can, of course, be obtained directly from the expansion for $(1+x)^p$ with $p = -\frac{1}{2}$.

Task

Using the known result that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1,$$

(a) Find an expression for $\ln(1+x)$

(b) Use the expression to obtain an approximation to $\ln(1.1)$

(a) Integrate both sides of $\dfrac{1}{1+x} = 1 - x + x^2 - \cdots$ and so deduce an expression for $\ln(1+x)$:

Your solution

$$\int \frac{dx}{1+x} =$$

$$\int (1 - x + x^2 - \cdots)\, dx =$$

Answer

$\displaystyle\int \frac{dx}{1+x} = \ln(1+x) + c$ where c is a constant of integration,

$\displaystyle\int (1 - x + x^2 - \cdots)\, dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + k$ where k is a constant of integration.

So we conclude $\qquad \displaystyle\ln(1+x) + c = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + k \qquad$ if $\quad |x| < 1$

Choosing $x = 0$ shows that $c = k$ so they cancel from this equation.

(b) Now choose $x = 0.1$ to approximate $\ln(1 + 0.1)$ using terms up to cubic:

Your solution

$$\ln(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \cdots \simeq$$

Answer

$\ln(1.1) \simeq 0.0953$ which is easily checked by calculator.

4. General power series

A general power series has the form

$$b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots = \sum_{p=0}^{\infty} b_p(x - x_0)^p$$

Exactly the same considerations apply to this general power series as apply to the 'special' series $\sum_{p=0}^{\infty} b_p x^p$ except that the variable x is replaced by $(x - x_0)$. The radius of convergence of the general series is obtained in the same way:

$$R = \lim_{p\to\infty} \left| \frac{b_p}{b_{p+1}} \right|$$

and the interval of convergence is now shifted to have centre at $x = x_0$ (see Figure 4 below). The series is absolutely convergent if $|x - x_0| < R$, diverges if $|x - x_0| > R$ and may or may not converge if $|x - x_0| = R$.

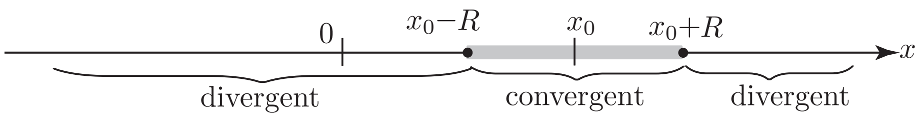


Figure 4

Find the radius of convergence of the general power series

$$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots$$

First find an expression for the general term:

Your solution

$$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots = \sum_{p=0}^{\infty}$$

Answer

$\sum_{p=0}^{\infty} (x - 1)^p (-1)^p$ so $b_p = (-1)^p$

Now obtain the radius of convergence:

Your solution

$\lim_{p\to\infty} \left| \frac{b_p}{b_{p+1}} \right| =$ $\therefore$ $R =$

Answer

$$\lim_{p\to\infty} \left| \frac{b_p}{b_{p+1}} \right| = \lim_{p\to\infty} \left| \frac{(-1)^p}{(-1)^{p+1}} \right| = 1.$$

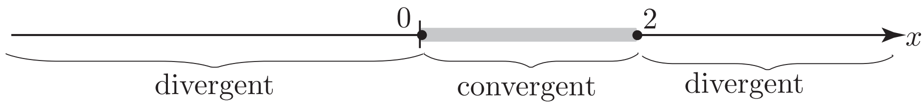
Hence $R = 1$, so the series is absolutely convergent if $|x - 1| < 1$.

Finally, decide on the convergence at $|x-1|=1$ (i.e. at $x-1=-1$ and $x-1=1$ i.e. $x=0$ and $x=2$):

Your solution

Answer

At $x=0$ the series is $1+1+1+\cdots$ which diverges and at $x=2$ the series is $1-1+1-1\cdots$ which also diverges. Thus the given series only converges if $|x-1|<1$ i.e. $0<x<2$.



Exercises

1. From the result $\dfrac{1}{1-x} = 1+x+x^2+x^3+\ldots\ , \quad |x|<1$

   (a) Find an expression for $\ln(1-x)$

   (b) Use this expression to obtain an approximation to $\ln(0.9)$ to 4 d.p.

2. Find the radius of convergence of the general power series $1-(x+2)+(x+2)^2-(x+2)^3+\ldots$

3. Find the range of values of x for which the power series $1+\dfrac{x}{4}+\dfrac{x^2}{4^2}+\dfrac{x^3}{4^3}+\ldots$ converges.

4. By differentiating the series for $(1+x)^{1/3}$ find the power series for $(1+x)^{-2/3}$ and state its radius of convergence.

5. (a) Find the radius of convergence of the series $1+\dfrac{x}{3}+\dfrac{x^2}{4}+\dfrac{x^3}{5}+\ldots$

   (b) Investigate what happens at the points $x=-1$ and $x=+1$

Answers

1. $\ln(1-x) = -x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\dfrac{x^4}{4}-\ldots \qquad \ln(0.9)\approx -0.1054$ (4 d.p.)

2. $R=1$. Series converges if $-3<x<-1$. If $x=-1$ series diverges. If $x=-3$ series diverges.

3. Series converges if $-4<x<4$.

4. $(1+x)^{-2/3} = 1-\dfrac{2}{3}x+\dfrac{5}{3}x^2+\ldots$ valid for $|x|<1$.

5. (a) $R=1$. (b) At $x=+1$ series diverges. At $x=-1$ series converges.