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Conics and Polar Coordinates

Learning outcomes

In this Workbook you will learn about some of the most important curves in the whole of mathematics - the conic sections: the ellipse, the parabola and the hyperbola. You will learn how to recognise these curves and how to describe them in Cartesian and in polar form. In the final block you will learn how to describe cruves using a parametric approach and, in particular, how the conic sections are described in parametric form.

Conic Sections

17.1

Introduction

The **conic sections** (or **conics**) - the **ellipse**, the **parabola** and the **hyperbola** - play an important role both in mathematics and in the application of mathematics to engineering. In this Section we look in detail at the equations of the conics in both standard form and general form.

Although there are various ways that can be used to define a conic, we concentrate in this Section on defining conics using Cartesian coordinates (x, y). However, at the end of this Section we examine an alternative way to obtain the conics.

Prerequisites

Before starting this Section you should ...

- be able to factorise simple algebraic expressions
- be able to change the subject in simple algebraic equations
- be able to complete the square in quadratic expressions

Learning Outcomes

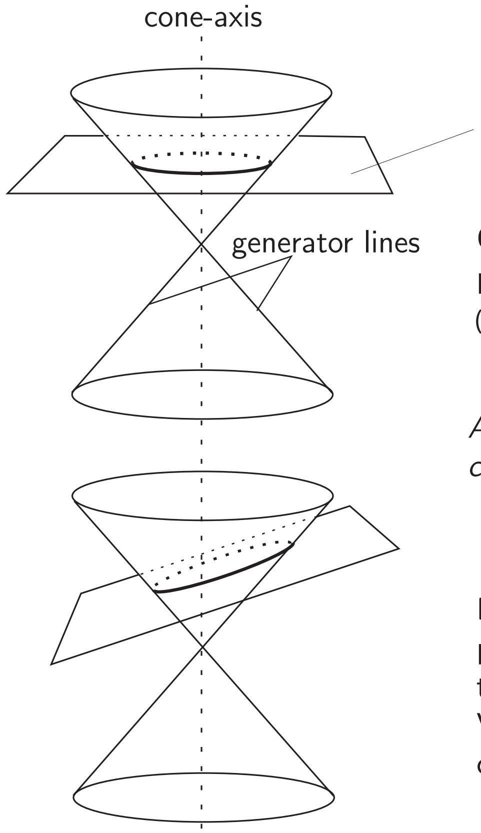
On completion you should be able to ...

- understand how conics are obtained as curves of intersection of a double-cone with a plane
- state the standard form of the equations of the ellipse, the parabola and the hyperbola
- classify quadratic expressions in $x,\ y$ in terms of conics

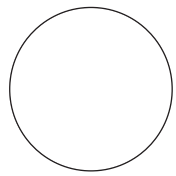
1. The ellipse, parabola and hyperbola

Mathematicians, engineers and scientists encounter numerous functions in their work: polynomials, trigonometric and hyperbolic functions amongst them. However, throughout the history of science one group of functions, the conics, arise time and time again not only in the development of mathematical theory but also in practical applications. The conics were first studied by the Greek mathematician Apollonius more than 200 years BC.

Essentially, the conics form that class of curves which are obtained when a double cone is intersected by a plane. There are three main types: the **ellipse**, the **parabola** and the **hyperbola**. From the ellipse we obtain the **circle** as a special case, and from the hyperbola we obtain the **rectangular hyperbola** as a special case. These curves are illustrated in the Figures 1 and 2.



Circle: obtained by intersection of a plane perpendicular to the cone-axis with cone.
(A degenerate case is a single point.)



As the plane of intersection tilts the other conics are obtained:

Ellipse: obtained by a plane, which is not perpendicular to the cone-axis, but cutting the cone in a closed curve.
Various ellipses are obtained as the plane continues to rotate.

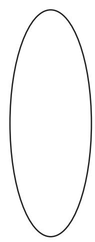
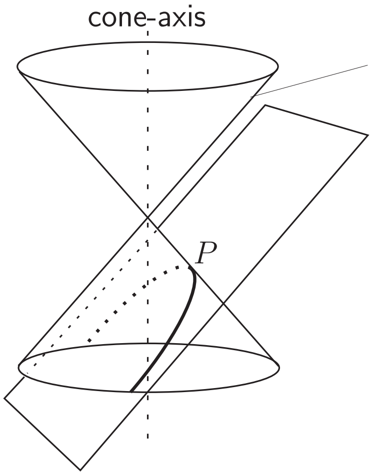
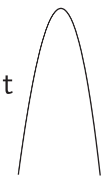
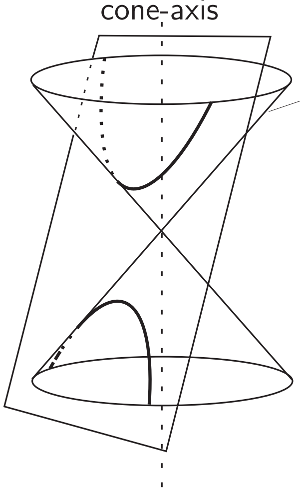


Figure 1: Circle and ellipse



Parabola: obtained when the plane is parallel to the generator of the cone. Different parabolas are obtained as the point P moves along a generator.





Hyperbola: obtained when the plane intersects both parts of the cone. The **rectangular hyperbola** is obtained when the plane is parallel to the cone-axis.
(A degenerate case is two straight lines.)

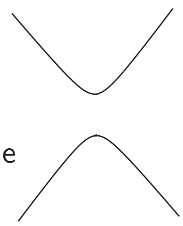


Figure 2: Parabola and hyperbola

The ellipse

We are all aware that the paths followed by the planets around the sun are elliptical. However, more generally the ellipse occurs in many areas of engineering. The standard form of an ellipse is shown in Figure 3.

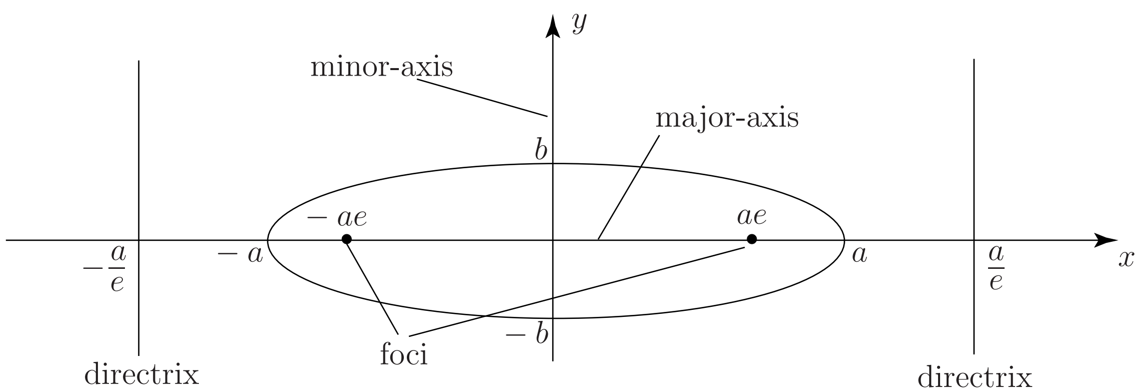


Figure 3

If $a > b$ (as in Figure 1) then the x-axis is called the **major-axis** and the y-axis is called the **minor-axis**. On the other hand if $b > a$ then the y-axis is called the major-axis and the x-axis is then the minor-axis. Two points, inside the ellipse are of importance; these are the **foci**. If $a > b$ these are located at coordinate positions $\pm ae$ (or at $\pm be$ if $b > a$) on the major-axis, with e, called the **eccentricity**, given by

$$e^2 = 1 - \frac{b^2}{a^2} \qquad (b < a) \qquad \text{or by} \qquad e^2 = 1 - \frac{a^2}{b^2} \qquad (a < b)$$

The foci of an ellipse have the property that if light rays are emitted from one focus then on reflection at the elliptic curve they pass through at the other focus.

Key Point 1

The standard Cartesian equation of the **ellipse** with its centre at the origin is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

This ellipse has intercepts on the x-axis at $x = \pm a$ and on the y-axis at $\pm b$. The curve is also symmetrical about both axes. The curve reduces to a circle in the special case in which $a = b$.

Example 1

(a) Sketch the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ (b) Find the eccentricity e

(c) Locate the positions of the foci.

Solution

(a) We can calculate the values of y as x changes from 0 to 2:

x	0	0.30	0.60	0.90	1.20	1.50	1.80	2
y	3	2.97	2.86	2.68	2.40	1.98	1.31	0

From this table of values, and using the symmetry of the curve, a sketch can be drawn (see Figure 4). Here $b = 3$ and $a = 2$ so the y-axis is the major axis and the x-axis is the minor axis.

Here $b = 3$ and $a = 2$ so the y-axis is the major axis and the x-axis is the minor axis.

(b) $e^2 = 1 - a^2/b^2 = 1 - 4/9 = 5/9 \qquad \therefore \quad e = \sqrt{5}/3$

(c) Since $b > a$ and $be = \sqrt{5}$, the foci are located at $\pm\sqrt{5}$ on the y-axis.

Solution (contd.)

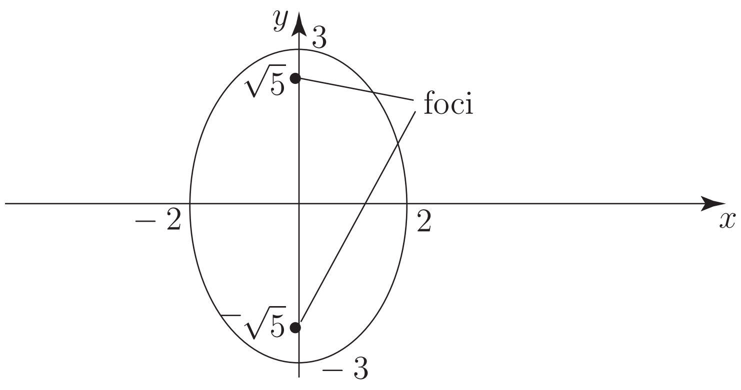


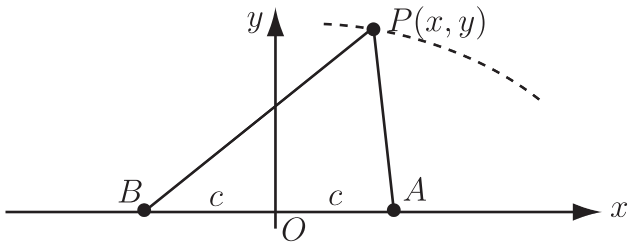
Figure 4

Key Point 1 gives the equation of the ellipse with its centre at the origin. If the centre of the ellipse has coordinates $(\alpha,\ \beta)$ and still has its axes parallel to the x- and y-axes the standard equation becomes

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1.$$

Task

Consider the points A and B with Cartesian coordinates $(c, 0)$ and $(-c, 0)$ respectively. A curve has the property that for every point P on it the sum of the distances PA and PB is a constant (which we will call $2a$). Derive the Cartesian form of the equation of the curve and show that it is an ellipse.



Your solution

Answer
We use Pythagoras's theorem to work out the distances PA and PB:

Let $R_1 = PB = [(x+c)^2 + y^2]^{1/2}$ and let $R_2 = PA = [(c-x)^2 + y^2]^{1/2}$

We now take the given equation $R_1 + R_2 = 2a$ and multiply both sides by $R_1 - R_2$. The quantity $R_1^2 - R_2^2$ on the left is calculated to be $4cx$, and $2a(R_1 - R_2)$ is on the right. We thus obtain a pair of equations: $R_1 + R_2 = 2a$ and $R_1 - R_2 = \dfrac{2cx}{a}$

Adding these equations together gives $R_1 = a + \dfrac{cx}{a}$ and squaring this equation gives

$$x^2 + c^2 + 2cx + y^2 = a^2 + \frac{c^2x^2}{a^2} + 2cx$$

Simplifying: $x^2(1 - \dfrac{c^2}{a^2}) + y^2 = a^2 - c^2$ whence $\dfrac{x^2}{a^2} + \dfrac{y^2}{(a^2-c^2)} = 1$

This is the standard equation of an ellipse if we set $b^2 = a^2 - c^2$, which is the traditional equation which relates the two semi-axis lengths a and b to the distance c of the foci from the centre of the ellipse.

The foci A and B have **optical** properties; a beam of light travelling from A along AP and undergoing a mirror reflection from the ellipse at P will return along the path PB to the other focus B.

The circle

The circle is a special case of the ellipse; it occurs when $a = b = r$ so the equation becomes

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \qquad \text{or, more commonly} \qquad x^2 + y^2 = r^2$$

Here, the **centre** of the circle is located at the origin $(0,0)$ and the **radius** of the circle is r. If the centre of the circle at a point (α, β) then the equation takes the form:

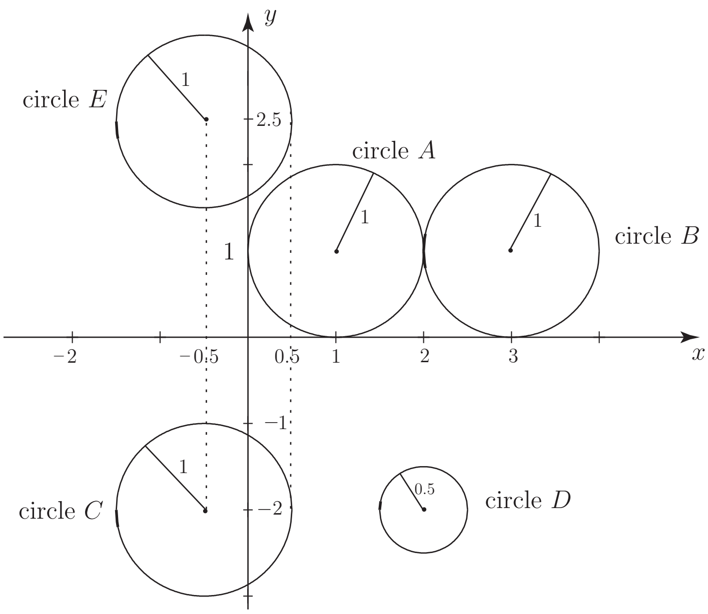
$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$

Key Point 2

The equation of a circle with centre at (α, β) and radius r is $(x-\alpha)^2 + (y-\beta)^2 = r^2$

Task

Write down the equations of the five circles (A to E) below:



Your solution

Answer

$A \quad (x-1)^2 + (y-1)^2 = 1$

$B \quad (x-3)^2 + (y-1)^2 = 1$

$C \quad (x+0.5)^2 + (y+2)^2 = 1$

$D \quad (x-2)^2 + (y+2)^2 = (0.5)^2$

$E \quad (x+0.5)^2 + (y-2.5)^2 = 1$

Example 2

Show that the expression

$$x^2 + y^2 - 2x + 6y + 6 = 0$$

represents the equation of a circle. Find its centre and radius.

Solution

We shall see later how to recognise this as the equation of a circle simply by examination of the coefficients of the quadratic terms x^2, y^2 and xy. However, in the present example we will use the process of completing the square, for x and for y, to show that the expression can be written in standard form.

Now $\quad x^2 + y^2 - 2x + 6y + 6 \equiv x^2 - 2x + y^2 + 6y + 6$.

Also,

$$x^2 - 2x \equiv (x-1)^2 - 1 \quad \text{and} \quad y^2 + 6y \equiv (y+3)^2 - 9.$$

Hence we can write

$$x^2 + y^2 - 2x + 6y + 6 \equiv (x-1)^2 - 1 + (y+3)^2 - 9 + 6 = 0$$

or, taking the free constants to the right-hand side:

$$(x-1)^2 + (y+3)^2 = 4.$$

By comparing this with the standard form we conclude this represents the equation of a circle with centre at $(1, -3)$ and radius 2.

Task

Find the centre and radius of each of the following circles:

(a) $x^2 + y^2 - 4x - 6y = -12$ $\qquad$ (b) $2x^2 + 2y^2 + 4x + 1 = 0$

Your solution

Answer

(a) centre: $(2, 3)$ radius 1 $\qquad$ (b) centre: $(-1, 0)$ radius $\sqrt{2}/2$.

Engineering Example 1

A circle-cutting machine

Introduction

A cutting machine creates circular holes in a piece of sheet-metal by starting at the centre of the circle and cutting its way outwards until a hole of the correct radius exists. However, prior to cutting, the circle is characterised by three points on its circumference, rather than by its centre and radius. Therefore, it is necessary to be able to find the centre and radius of a circle given three points that it passes through.

Problem in words

Given three points on the circumference of a circle, find its centre and radius

(a) for three general points

(b) (i) for $(-6, 5)$, $(-3, 6)$ and $(2, 1)$ (ii) for $(-0.7, 0.6)$, $(5.9, 1.4)$ and $(0.8, -2.8)$

where coordinates are in cm.

Mathematical statement of problem

A circle passes through the three points. Find the centre (x_0, y_0) and radius R of this circle when the three circumferential points are

(a) (x_1, y_1), (x_2, y_2) and (x_3, y_3)

(b) (i) $(-6, 5)$, $(-3, 6)$ and $(2, 1)$

(ii) $(-0.7, 0.6)$, $(5.9, 1.4)$ and $(0.8, -2.8)$

Measurements are in centimetres; give answers correct to 2 decimal places.

Mathematical analysis

(a) The equation of a circle with centre at (x_0, y_0) and radius R is

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

and, if this passes through the 3 points (x_1, y_1), (x_2, y_2) and (x_3, y_3) then

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = R^2 \qquad (1)$$
$$(x_2 - x_0)^2 + (y_2 - y_0)^2 = R^2 \qquad (2)$$
$$(x_3 - x_0)^2 + (y_3 - y_0)^2 = R^2 \qquad (3)$$

Eliminating the R^2 term between (1) and (2) gives

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2$$

so that

$$x_1^2 - 2x_0x_1 + y_1^2 - 2y_0y_1 = x_2^2 - 2x_0x_2 + y_2^2 - 2y_0y_2 \qquad (4)$$

Similarly, eliminating R^2 between (1) and (3) gives

$$x_1^2 - 2x_0x_1 + y_1^2 - 2y_0y_1 = x_3^2 - 2x_0x_3 + y_3^2 - 2y_0y_3 \qquad (5)$$

Re-arranging (4) and (5) gives a system of two equations in x_0 and y_0.

$$2(x_2 - x_1)x_0 + 2(y_2 - y_1)y_0 = x_2^2 + y_2^2 - x_1^2 - y_1^2 \qquad (6)$$

$$2(x_3 - x_1)x_0 + 2(y_3 - y_1)y_0 = x_3^2 + y_3^2 - x_1^2 - y_1^2 \qquad (7)$$

Multiplying (6) by $(y_3 - y_1)$, and multiplying (7) by $(y_2 - y_1)$, subtracting and re-arranging gives

$$x_0 = \frac{1}{2}\left(\frac{(y_3 - y_1)(x_2^2 + y_2^2) + (y_1 - y_2)(x_3^2 + y_3^2) + (y_2 - y_3)(x_1^2 + y_1^2)}{x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1}\right) \qquad (8)$$

while a similar procedure gives

$$y_0 = \frac{1}{2}\left(\frac{(x_1 - x_3)(x_2^2 + y_2^2) + (x_2 - x_1)(x_3^2 + y_3^2) + (x_3 - x_2)(x_1^2 + y_1^2)}{x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1}\right) \qquad (9)$$

Knowing x_0 and y_0, the radius R can be found from

$$R = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \qquad (10)$$

(or alternatively using x_2 and y_2 (or x_3 and y_3) as appropriate).

Equations (8), (9) and (10) can now be used to analyse the two particular circles above.

(i) Here $x_1 = -6$ cm, $y_1 = 5$ cm, $x_2 = -3$ cm, $y_2 = 6$ cm, $x_3 = 2$ cm and $y_3 = 1$ cm, so that

$$x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1 = -3 - 12 + 10 + 6 - 36 + 15 = -20$$

and

$$x_1^2 + y_1^2 = 61 \qquad x_2^2 + y_2^2 = 45 \qquad x_3^2 + y_3^2 = 5$$

From (8)

$$x_0 = \frac{1}{2}\left(\frac{-4 \times 45 + (-1) \times 5 + 5 \times 61}{-20}\right) = \frac{-180 - 5 + 305}{-40} = -3$$

while (9) gives

$$y_0 = \frac{1}{2}\left(\frac{-8 \times 45 + 3 \times 5 + 5 \times 61}{-20}\right)$$
$$= \frac{-360 + 15 + 305}{-40} = 1$$

The radius can be found from (10)

$$R = \sqrt{(-6 - (-3))^2 + (5 - 1)^2} = \sqrt{25} = 5$$

so that the circle has centre at $(-3, 1)$ and a radius of 5 cm.

(ii) Now $x_1 = -0.7$ cm, $y_1 = 0.6$ cm, $x_2 = 5.9$ cm, $y_2 = 1.4$ cm, $x_3 = 0.8$ cm and $y_3 = -2.8$ cm, so that

$$x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1 = -16.52 - 1.12 + 0.48 - 1.96 - 0.98 - 3.54 = -23.64$$

and

$$x_1^2 + y_1^2 = 0.85 \qquad x_2^2 + y_2^2 = 36.77 \qquad x_3^2 + y_3^2 = 8.48$$

so from (8)

$$x_0 = \frac{1}{2}\left(\frac{-125.018 - 6.784 + 3.57}{-23.64}\right) = \frac{-128.232}{-47.28} = 2.7121827$$

and from (9)

$$y_0 = \frac{1}{2}\left(\frac{-55.155 + 55.968 - 4.335}{-23.64}\right) = \frac{-3.522}{-47.28} = 0.0744924$$

and from (10)

$$R = \sqrt{(-0.7 - 2.7121827)^2 + (0.6 - 0.0744924)^2} = \sqrt{11.9191490} = 3.4524121$$

so that, to 2 d.p., the circle has centre at $(2.71, 0.07)$ and a radius of 3.45 cm.

Mathematical comment

Note that the expression

$$x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1$$

appears in the denominator for both x_0 and y_0. If this expression is equal to zero, the calculation will break down. Geometrically, this corresponds to the three points being in a straight line so that no circle can be drawn, or not all points being distinct so no unique circle is defined.

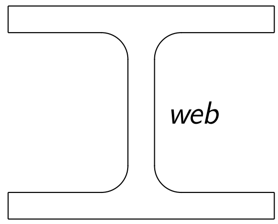
Engineering Example 2

The web-flange junction

Introduction

In problems of torsion, the **torsion constant**, J, which is a function of the shape and structure of the element under consideration, is an important quantity.

A common beam section is the **thick I-section** shown here, for which the torsion constant is given by

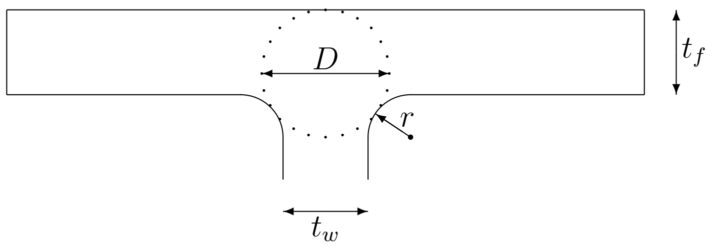


$$J = 2J_1 + J_2 + 2\alpha D^4$$

where the J_1 and J_2 terms refer to the flanges and web respectively, and the D^4 term refers to the web-flange junction. In fact

$$\alpha = \min\left[\frac{t_f}{t_w}, \frac{t_w}{t_f}\right]\left(0.15 + 0.1\,\frac{r}{t_f}\right)$$

where t_f and t_w are the thicknesses of the flange and web respectively, and r is the radius of the concave circle element between the flange and the web. D is the diameter of the circle of the web-flange junction.



As D occurs in the form D^4, the torsion constant is very sensitive to it. Calculation of D is therefore a crucial part of the calculation of J.

Problem in words

Find D, the diameter of the circle within the web–flange junction as a function of the other dimensions of the structural element.

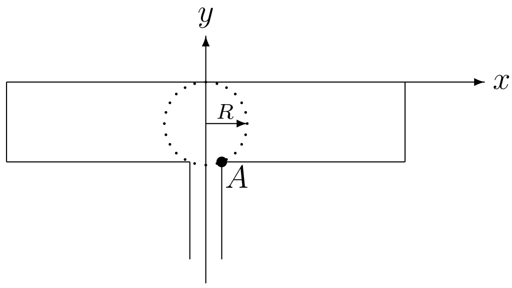
Mathematical statement of problem

(a) Find D, the diameter of the circle, in terms of t_f and t_w (the thicknesses of the flange and the web respectively) in the case where $r = 0$. When $t_f = 3$cm and $t_w = 2$cm, find D.

(b) For $r \neq 0$, find D in terms of t_f, t_w and r. In the special case where $t_f = 3$ cm, $t_w = 2$ cm and $r = 0.4$ cm, find D.

Mathematical analysis

(a) Consider a co-ordinate system based on the midpoint of the outer surface of the flange.



The centre of the circle will lie at $(0, -R)$ where R is the radius of the circle, i.e. $R = D/2$. The equation of the circle is

$$x^2 + (y + R)^2 = R^2 \tag{1}$$

In addition, the circle passes through the 'corner' at point A $(t_w/2, -t_f)$, so

$$\left(\frac{t_w}{2}\right)^2 + (-t_f + R)^2 = R^2 \tag{2}$$

On expanding

$$\frac{t_w^2}{4} + t_f^2 - 2Rt_f + R^2 = R^2$$

giving

$$2Rt_f = \frac{t_w^2}{4} + t_f^2 \qquad \Rightarrow \qquad R = \frac{(t_w^2/4) + t_f^2}{2t_f} = \frac{t_w^2}{8t_f} + \frac{t_f}{2}$$

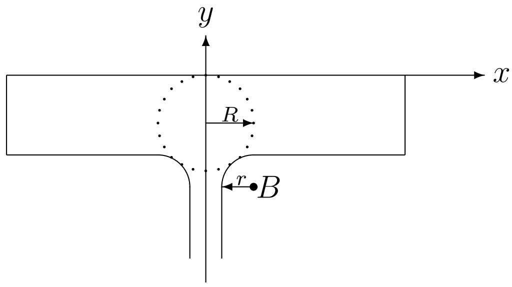
so that

$$D = 2R = \frac{t_w^2}{4t_f} + t_f \tag{3}$$

Setting $t_f = 3$ cm, $t_w = 2$ cm gives

$$D = \frac{2^2}{4 \times 3} + 3 = 3.33 \text{ cm}$$

(b) Again using a co-ordinate system based on the mid-point of the outer surface of the flange, consider now the case $r \neq 0$.



Point B $(t_w/2 + r, -t_f - r)$ lies, not on the circle described by (1), but on the slightly larger circle with the same centre, and radius $R + r$. The equation of this circle is

$$x^2 + (y + R)^2 = (R + r)^2 \qquad (4)$$

Putting the co-ordinates of point B into equation (4) gives

$$\left(\frac{t_w}{2} + r\right)^2 + (-t_f - r + R)^2 = (R + r)^2 \qquad (5)$$

which, on expanding gives

$$\frac{t_w^2}{4} + t_w r + r^2 + t_f^2 + r^2 + R^2 + 2t_f r - 2t_f R - 2rR = R^2 + 2rR + r^2$$

Cancelling and gathering terms gives

$$\begin{aligned} \frac{t_w^2}{4} + t_w r + r^2 + t_f^2 + 2t_f r &= 4rR + 2t_f R \\ &= 2R\,(2r + t_f) \end{aligned}$$

so that

$$2R = D = \frac{(t_w^2/4) + t_w r + r^2 + t_f^2 + 2t_f r}{(2r + t_f)}$$

so $$D = \frac{t_w^2 + 4t_w r + 4r^2 + 4t_f^2 + 8t_f r}{(8r + 4t_f)} \qquad (6)$$

Now putting $t_f = 3$ cm, $t_w = 2$ cm and $r = 0.4$ cm makes

$$D = \frac{2^2 + (4 \times 2 \times 0.4) + (4 \times 0.4^2) + (4 \times 3^2) + (8 \times 3 \times 0.4)}{(8 \times 0.4) + (4 \times 3)} = \frac{53.44}{15.2} = 3.52 \text{ cm}$$

Interpretation

Note that setting $r = 0$ in Equation (6) recovers the special case of $r = 0$ given by equation (3). The value of D is now available to be used in calculations of the torsion constant, J.

The parabola

The standard form of the parabola is shown in Figure 5. Here the x-axis is the line of symmetry of the parabola.

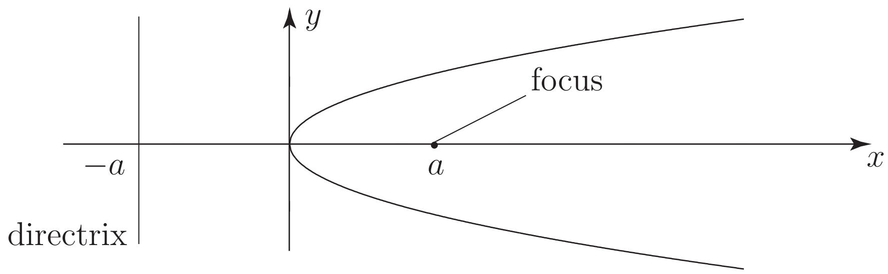


Figure 5

Key Point 3

The standard equation of the **parabola** with focus at $(a, 0)$ is

$$y^2 = 4ax$$

It can be shown that light rays parallel to the x-axis will, on reflection from the parabolic curve, come together at the focus. This is an important property and is used in the construction of some kinds of telescopes, satellite dishes and car headlights.

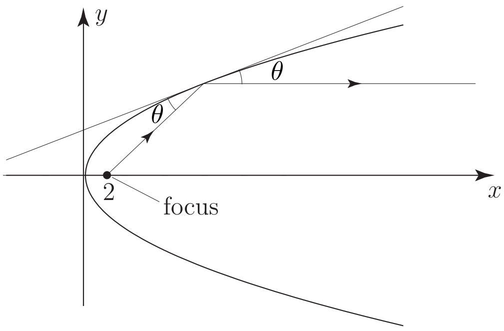
Task

Sketch the curve $y^2 = 8x$. Find the position of the focus and confirm its light-focusing property.

Your solution

Answer

This is a standard parabola ($y^2 = 4ax$) with $a = 2$. Thus the focus is located at coordinate position $(2, 0)$.



If your sketch is sufficiently accurate you should find that light-rays (lines) parallel to the x-axis when reflected off the parabolic surface pass through the focus. (Draw a tangent at the point of reflection and ensure that the angle of incidence (θ say) is the same as the angle of reflection.)

By changing the equation of the parabola slightly we can change the position of the parabola along the x-axis. See Figure 6.

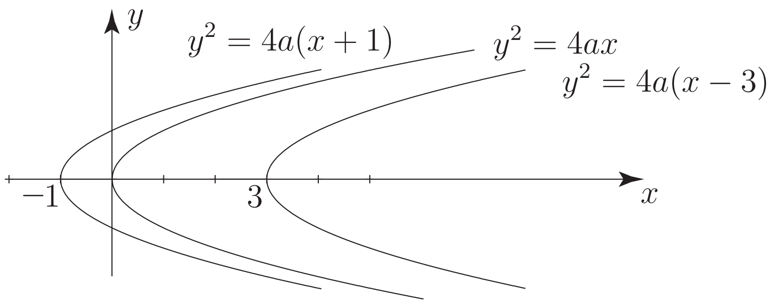


Figure 6: Parabola $y = 4a(x - b)$ with vertex at $x = b$

We can also have parabolas where the y-axis is the line of symmetry (see Figure 7). In this case the standard equation is

$$x^2 = 4ay \qquad \text{or} \qquad y = \frac{x^2}{4a}$$

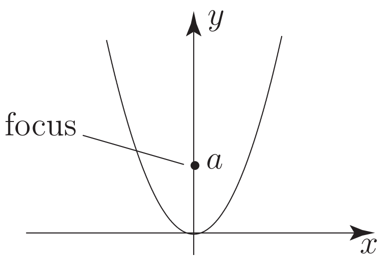


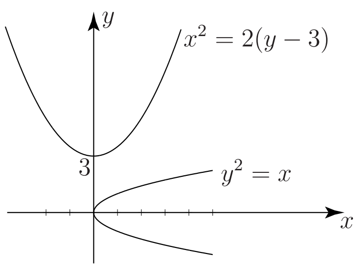
Figure 7

Task

Sketch the curves $y^2 = x$ and $x^2 = 2(y-3)$.

Your solution

Answer



The focus of the parabola $y^2 = 4a(x-b)$ is located at coordinate position $(a+b,\ 0)$. Changing the value of a changes the convexity of the parabola (see Figure 8).

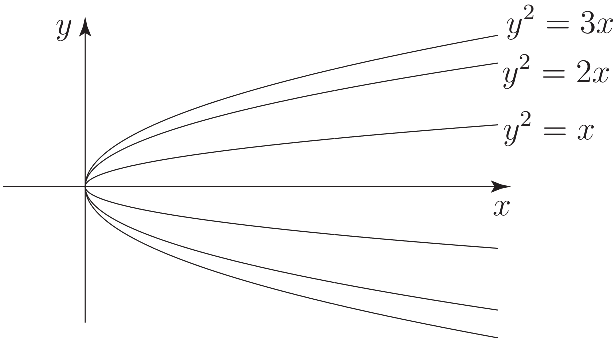


Figure 8

The hyperbola

The standard form of the hyperbola is shown in Figure 9(a).
This has standard equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The eccentricity, e, is defined by

$$e^2 = 1 + \frac{b^2}{a^2} \qquad (e > 1)$$

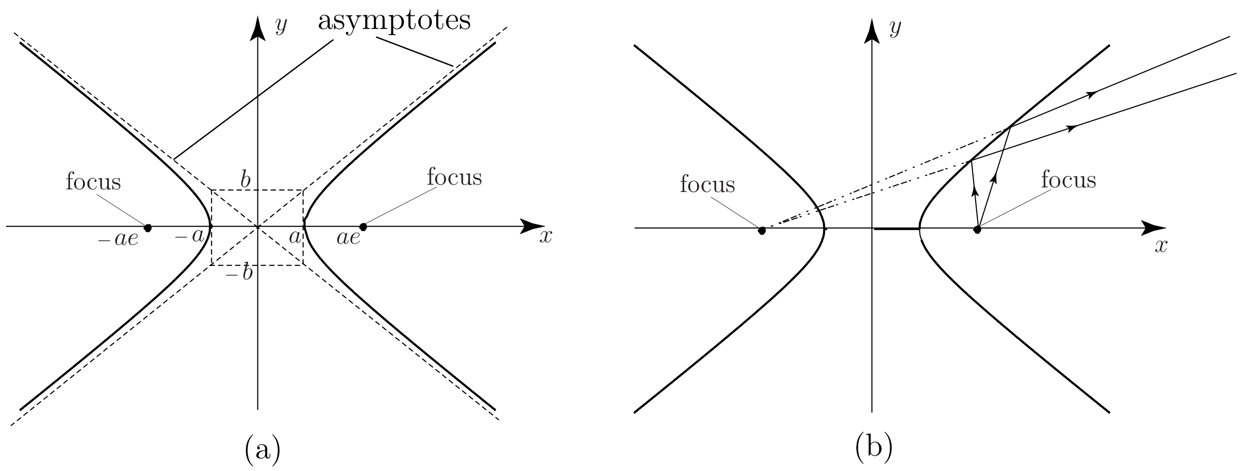


Figure 9

Note the change in sign compared to the equivalent expressions for the ellipse. The lines $y = \pm\frac{b}{a}x$ are **asymptotes** to the hyperbola (these are the lines to which each branch of the hyperbola approach as $x \to \pm\infty$).

If light is emitted from one focus then on hitting the hyperbolic curve it is reflected in such a way as to appear to be coming from the *other* focus. See Figure 9(b). The hyperbola has fewer uses in applications than the other conic sections and so we will not dwell here on its properties.

Key Point 4

The standard equation of the hyperbola with foci at $(\pm ae, 0)$ is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{with eccentricity } e \text{ given by} \qquad e^2 = 1 + \frac{b^2}{a^2} \quad (e > 1)$$

General conics

The conics we have considered above - the ellipse, the parabola and the hyperbola - have all been presented in standard form:- their axes are parallel to either the x- or y-axis. However, conics may be rotated to any angle with respect to the axes: they clearly remain conics, but what equations do they have?

It can be shown that the equation of **any conic**, can be described by the quadratic expression

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, F are constants.

If not all of A, B, C are zero (and F is a suitable number) the graph of this equation is

(i) an ellipse if $B^2 < 4AC$ (circle if $A = C$ **and** $B = 0$)

(ii) a parabola if $B^2 = 4AC$

(iii) a hyperbola if $B^2 > 4AC$

Example 3

Classify each of the following equations as ellipse, parabola or hyperbola:

(a) $x^2 + 2xy + 3y^2 + x - 1 = 0$

(b) $x^2 + 2xy + y^2 - 3y + 7 = 0$

(c) $2x^2 + xy + 2y^2 - 2x + 3y = 6$

(d) $3x^2 + 2x - 5y + 3y^2 - 10 = 0$

Solution

(a) Here $A = 1,\ B = 2,\ C = 3$ $\quad\therefore\quad B^2 < 4AC$. This is an ellipse.

(b) Here $A = 1,\ B = 2,\ C = 1$ $\quad\therefore\quad B^2 = 4AC$. This is a parabola.

(c) Here $A = 2,\ B = 1,\ C = 2$ $\quad\therefore\quad B^2 < 4AC$ also $A = C$ but $B \neq 0$. This is an ellipse.

(d) Here $A = 3,\ B = 0,\ C = 3$ $\quad\therefore\quad B^2 < 4AC$. Also $A = C$ and $B = 0$. This is a circle.

Task

Classify each of the following conics:

(a) $x^2 - 2xy - 3y^2 + x - 1 = 0$

(b) $2x^2 + xy - y^2 - 2x + 3y = 0$

(c) $4x^2 - y + 3 = 0$

(d) $-x^2 - xy - y^2 + 3x = 0$

(e) $2x^2 + 2y^2 - x + 3y = 7$

Your solution

Answer

(a) $A = 1, B = -2, C = -3$ $\quad B^2 > 4AC \quad \therefore$ hyperbola

(b) $A = 2, B = 1, C = -1$ $\quad B^2 > 4AC \quad \therefore$ hyperbola

(c) $A = 4, B = 0, C = 0$ $\quad B^2 = 4AC \quad \therefore$ parabola

(d) $A = -1, B = -1, C = -1$ $\quad B^2 < 4AC,\ A = C,\ \ B \neq 0 \quad \therefore$ ellipse

(e) $A = 2,\ B = 0,\ C = 2$ $\quad B^2 < 4AC,\ \ A = C$ and $B = 0 \quad \therefore$ circle

Exercises

1. The equation $9x^2 + 4y^2 - 36x + 24y - 1 = 0$ represents an ellipse. Find its centre, the semi-major and semi-minor axes and the coordinate positions of the foci.

2. Find the equation of a circle of radius 3 which has its centre at $(-1, 2.2)$

3. Find the centre and radius of the circle $x^2 + y^2 - 2x - 2y - 5 = 0$

4. Find the position of the focus of the parabola $y^2 - x + 3 = 0$

5. Classify each of the following conics

   (a) $x^2 + 2x - y - 3 = 0$

   (b) $8x^2 + 12xy + 17y^2 - 20 = 0$

   (c) $x^2 + xy - 1 = 0$

   (d) $4x^2 - y^2 - 4y = 0$

   (e) $6x^2 + 9y^2 - 24x - 54y + 51 = 0$

6. An asteroid has an elliptical orbit around the Sun. The major axis is of length 5×10^8 km. If the distance between the foci is 4×10^8 km find the equation of the orbit.

Answers

1. centre: $(2, -3)$, semi-major 3, semi-minor 2, foci: $(2, -3 \pm \sqrt{5})$

2. $(x + 1)^2 + (y - 2.2)^2 = 9$

3. centre: $(1, 1)$ radius $\sqrt{7}$

4. $y^2 = (x - 3) \qquad \therefore \qquad a = 1, \quad b = -3$. Hence focus is at coordinate position $(4, 0)$.

5. (a) parabola with vertex $(-1, -4)$

   (b) ellipse

   (c) hyperbola

   (d) hyperbola

   (e) ellipse with centre $(2, 3)$

6. $9x^2 + 25y^2 = 5.625 \times 10^7$