The Convolution Theorem

20.5

Introduction

In this Section we introduce the convolution of two functions $f(t),\ g(t)$ which we denote by $(f*g)(t)$. The convolution is an important construct because of the convolution theorem which allows us to find the inverse Laplace transform of a product of two transformed functions:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$$

Prerequisites

Before starting this Section you should ...

- be able to find Laplace transforms and inverse Laplace transforms of simple functions
- be able to integrate by parts
- understand how to use step functions in integration

Learning Outcomes

On completion you should be able to ...

- calculate the convolution of simple functions
- apply the convolution theorem to obtain inverse Laplace transforms

1. Convolution

Let $f(t)$ and $g(t)$ be two functions of t. The **convolution** of $f(t)$ and $g(t)$ is also a function of t, denoted by $(f * g)(t)$ and is defined by the relation

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)\,dx$$

However if f and g are both **causal** functions then (strictly) $f(t), g(t)$ are written $f(t)u(t)$ and $g(t)u(t)$ respectively, so that

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)u(t-x)g(x)u(x)\,dx = \int_0^t f(t-x)g(x)\,dx$$

because of the properties of the step functions: $u(t-x) = 0$ if $x > t$ and $u(x) = 0$ if $x < 0$.

Key Point 14

Convolution

If $f(t)$ and $g(t)$ are causal functions then their convolution is defined by:

$$(f * g)(t) = \int_0^t f(t-x)g(x)\,dx$$

This is an odd looking definition but it turns out to have considerable use both in Laplace transform theory and in the modelling of linear engineering systems. The reader should note that the variable of integration is x. As far as the integration process is concerned the t-variable is (temporarily) regarded as a constant.

Example 3

Find the convolution of f and g if $f(t) = tu(t)$ and $g(t) = t^2u(t)$.

Solution

$$f(t-x) = (t-x)u(t-x) \quad \text{and} \quad g(x) = x^2u(x)$$

Therefore

$$\begin{aligned}(f * g)(t) = \int_0^t (t-x)x^2\,dx &= \left[\tfrac{1}{3}x^3t - \tfrac{1}{4}x^4\right]_0^t \\ &= \tfrac{1}{3}t^4 - \tfrac{1}{4}t^4 = \tfrac{1}{12}t^4\end{aligned}$$

Example 4

Find the convolution of $f(t) = t.u(t)$ and $g(t) = \sin t.u(t)$.

Solution

Here $f(t-x) = (t-x)u(t-x)$ and $g(x) = \sin x.u(x)$ and so

$$(f * g)(t) = \int_0^t (t-x)\sin x\, dx$$

We need to integrate by parts. We find, remembering again that t is a constant in the integration process,

$$\begin{aligned}\int_0^t (t-x)\sin x\, dx &= \Big[-(t-x)\cos x\Big]_0^t - \int_0^t (-1)(-\cos x)\, dx \\ &= [0+t] - \int_0^t \cos x\, dx \\ &= t - \Big[\sin x\Big]_0^t = t - \sin t\end{aligned}$$

so that

$$(f * g)(t) = t - \sin t \quad \text{or, equivalently, in this case} \quad (t * \sin t)(t) = t - \sin t$$

Task

In Example 4 we found the convolution of $f(t) = t.u(t)$ and $g(t) = \sin t.u(t)$. In this Task you are asked to find the convolution $(g * f)(t)$ that is, to reverse the order of f and g.

Begin by writing $(g * f)(t)$ as an appropriate integral:

Your solution

Answer

$g(t-x) = \sin(t-x).u(t-x)$ and $f(x) = xu(x)$, so $(g * f)(t) = \int_0^t \sin(t-x).x\, dx$

Now evaluate the convolution integral:

Your solution

Answer

$$\begin{aligned}(g*f)(t) &= \int_0^t \sin(t-x).x\,dx \\ &= \Big[x\cos(t-x)\Big]_0^t - \int_0^t \cos(t-x)\,dx \\ &= [t-0] + \Big[\ \sin(t-x)\Big]_0^t = t - \sin t\end{aligned}$$

This Task illustrates the general result in the following Key Point:

Key Point 15

Commutativity Property of Convolution

$$(f*g)(t) = (g*f)(t)$$

In words: the convolution of $f(t)$ with $g(t)$ is the same as the convolution of $g(t)$ with $f(t)$.

Task

Obtain the Laplace transforms of $f(t) = t.u(t)$ and $g(t) = \sin t.u(t)$ and $(f*g)(t)$.

Begin by finding $\mathcal{L}\{f(t)\}$, $\mathcal{L}\{g(t)\}$:

Your solution

Answer

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} \qquad \mathcal{L}\{g(t)\} = \frac{1}{s^2+1} \qquad \text{(from Table 1)}$$

Now find $\mathcal{L}\{(f*g)(t)\}$:

Your solution

Answer

From Example 4 $(f*g)(t) = t - \sin t$ and so $\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{t - \sin t\} = \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$

Now compare $\mathcal{L}\{f(t)\} \times \mathcal{L}\{g(t)\}$ with $\mathcal{L}\{f*g(t)\}$. What do you observe?

Your solution

Answer

$$\mathcal{L}\{(f*g)(t)\} = \frac{1}{s^2} - \frac{1}{s^2+1} = \frac{1}{s^2}\left(\frac{1}{s^2+1}\right) = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

We see that the Laplace transform of the convolution of $f(t)$ and $g(t)$ is the product of their separate Laplace transforms. This, in fact, is a general result which is expressed in the statement of the **convolution theorem** which we discuss in the next subsection.

2. The convolution theorem

Let $f(t)$ and $g(t)$ be causal functions with Laplace transforms $F(s)$ and $G(s)$ respectively, i.e. $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then it can be shown that

Key Point 16

The Convolution Theorem

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t) \qquad \text{or equivalently} \qquad \mathcal{L}\{(f*g)(t)\} = F(s)G(s)$$

Example 5

Find the inverse transform of $\dfrac{6}{s(s^2+9)}$.

(a) Using partial fractions (b) Using the convolution theorem.

Solution

(a) $\dfrac{6}{s(s^2+9)} = \dfrac{(2/3)}{s} - \dfrac{(2/3)s}{s^2+9}$ and so

$$\mathcal{L}^{-1}\left\{\frac{6}{s(s^2+9)}\right\} = \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \tfrac{2}{3}u(t) - \tfrac{2}{3}\cos 3t.u(t)$$

(b) Let us choose $F(s) = \dfrac{2}{s}$ and $G(s) = \dfrac{3}{s^2+9}$ then

$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2u(t)$ and $g(t) = \mathcal{L}^{-1}\{G(s)\} = \sin 3t.u(t)$

So

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= (f*g)(t) \qquad \text{(by the convolution theorem)}\\ &= \int_0^t 2u(t-x)\sin 3x.u(x)\,dx\end{aligned}$$

Now the variable t can take any value from $-\infty$ to $+\infty$. If $t<0$ then the variable of integration, x, is negative and so $u(x)=0$. We conclude that

$(f*g)(t) = 0$ if $t<0$

that is, $(f*g)(t)$ is a **causal function**. Let us now consider the other possibility for t, that is the range $t \geq 0$. Now, in the range of integration $0 \leq x \leq t$ and so

$u(t-x) = 1 \qquad u(x) = 1$

since both $t-x$ and x are non-negative. Therefore

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= \int_0^t 2\sin 3x\,dx\\ &= \left[-\frac{2}{3}\cos 3x\right]_0^t = -\frac{2}{3}(\cos 3t - 1) \qquad t \geq 0\end{aligned}$$

Hence

$$\mathcal{L}^{-1}\{\frac{6}{s(s^2+9)}\} = -\frac{2}{3}(\cos 3t - 1)u(t)$$

which agrees with the value obtained above using the partial fraction approach.

Task

Use the convolution theorem to find the inverse transform of $H(s) = \dfrac{s}{(s-1)(s^2+1)}$.

Begin by choosing two functions of s, that is, $F(s)$ and $G(s)$:

Your solution

Answer

Although there are many possibilities it would seem sensible to choose

$$F(s) = \frac{1}{s-1} \qquad \text{and} \qquad G(s) = \frac{s}{s^2+1}$$

since, by inspection, we can write down their inverse Laplace transforms:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathrm{e}^t u(t) \qquad \text{and} \qquad g(t) = \mathcal{L}^{-1}\{G(s)\} = \cos t.u(t)$$

Now construct the convolution integral:

Your solution

$h(t) =$

Answer

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\{H(s)\} \\ &= \mathcal{L}^{-1}\{F(s)G(s)\} \\ &= \int_0^t f(t-x)g(x)\,dx = \int_0^t \mathrm{e}^{t-x}u(t-x)\cos x.u(x)\,dx \end{aligned}$$

Now complete the evaluation of the integral, treating the cases $t < 0$ and $t \geq 0$ separately:

Your solution

Answer

You should find $h(t) = \frac{1}{2}(\sin t - \cos t + \mathrm{e}^t)u(t)$ since $h(t) = 0$ if $t < 0$ and

$$\begin{aligned} h(t) &= \int_0^t \mathrm{e}^{t-x}\cos x\,dx \qquad \text{if} \quad t \geq 0 \\ &= \Big[\mathrm{e}^{t-x}\sin x\Big]_0^t - \int_0^t (-1)\mathrm{e}^{t-x}\sin x\,dx \qquad \text{(integrating by parts)} \\ &= \sin t + \Big[-\mathrm{e}^{t-x}\cos x\Big]_0^t - \int_0^t (-\mathrm{e}^{t-x})(-\cos x)\,dx \\ &= \sin t - \cos t + \mathrm{e}^t - h(t) \\ \text{or} \quad 2h(t) &= \sin t - \cos t + \mathrm{e}^t \qquad t \geq 0 \end{aligned}$$

Finally $h(t) = \frac{1}{2}(\sin t - \cos t + \mathrm{e}^t)u(t)$

Exercises

1. Find the convolution of

   (a) $2tu(t)$ and $t^3u(t)$ (b) $\mathrm{e}^t u(t)$ and $tu(t)$ (c) $\mathrm{e}^{-2t}u(t)$ and $\mathrm{e}^{-t}u(t)$.

   In each case reverse the order to check that $(f * g)(t) = (g * f)(t)$.

2. Use the convolution theorem to determine the inverse Laplace transforms of

   (a) $\dfrac{1}{s^2(s+1)}$ (b) $\dfrac{1}{(s-1)(s-2)}$ (c) $\dfrac{1}{(s^2+1)^2}$

Answers

1. (a) $\frac{1}{10}t^5$ (b) $-t - 1 + \mathrm{e}^t$ (c) $\mathrm{e}^{-t} - \mathrm{e}^{-2t}$
2. (a) $(t - 1 + \mathrm{e}^{-t})u(t)$ (b) $(-\mathrm{e}^t + \mathrm{e}^{2t})u(t)$ (c) $\frac{1}{2}(\sin t - t\cos t)u(t)$