Half-Range Series 23.5

Introduction

In this Section we address the following problem:

Can we find a Fourier series expansion of a function defined over a finite interval?

Of course we recognise that such a function could not be periodic (as periodicity demands an infinite interval). The answer to this question is yes but we must first convert the given non-periodic function into a periodic function. There are many ways of doing this. We shall concentrate on the most useful extension to produce a so-called **half-range Fourier series**.

Prerequisites

Before starting this Section you should . . .

- know how to obtain a Fourier series
- be familiar with odd and even functions and their properties
- have knowledge of integration by parts

Learning Outcomes

On completion you should be able to . . .

- choose to expand a non-periodic function either as a series of sines or as a series of cosines

1. Half-range Fourier series

So far we have shown how to represent given periodic functions by Fourier series. We now consider a slight variation on this theme which will be useful in HELM 25 on solving Partial Differential Equations.

Suppose that instead of specifying a periodic function we begin with a function $f(t)$ defined only over a **limited range of values** of t, say $0 < t < \pi$. Suppose further that we wish to represent this function, over $0 < t < \pi$, by a Fourier series. (This situation may seem a little artificial at this point, but this is precisely the situation that will arise in solving differential equations.)
To be specific, suppose we define $f(t) = t^2 \qquad 0 < t < \pi$

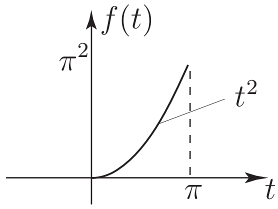


Figure 21

We shall consider the interval $0 < t < \pi$ to be half a period of a 2π periodic function. We must therefore define $f(t)$ for $-\pi < t < 0$ to complete the specification.

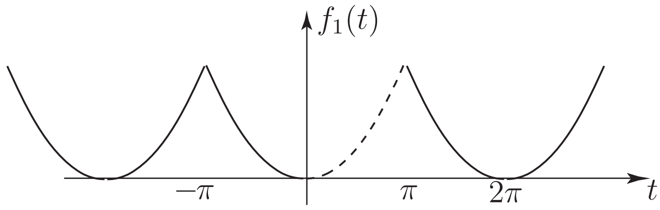
Task

Complete the definition of the above function $f(t) = t^2, \quad 0 < t < \pi$
by defining it over $-\pi < t < 0$ such that the resulting functions will have a Fourier series containing

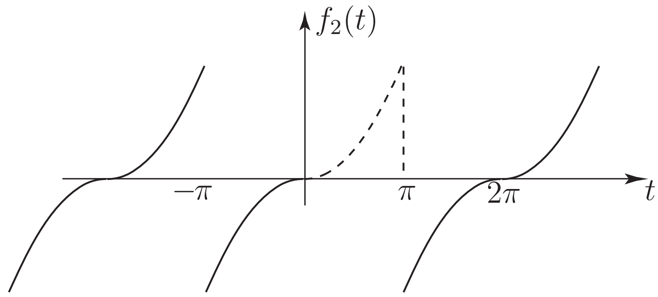
(a) only cosine terms, (b) only sine terms, (c) both cosine and sine terms.

Your solution

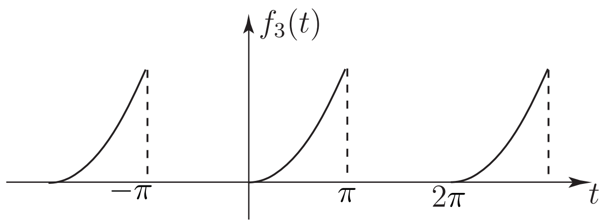
Answer
(a) We must complete the definition so as to have an **even** periodic function:
$f(t) = t^2, \quad -\pi < t < 0$



(b) We must complete the definition so as to have an **odd** periodic function:
$f(t) = -t^2, \quad -\pi < t < 0$



(c) We may define $f(t)$ in any way we please (other than (a) and (b) above). For example we might define $f(t) = 0$ over $-\pi < t < 0$:



The point is that all three periodic functions $f_1(t), f_2(t), f_3(t)$ will give rise to a **different** Fourier series but all will represent the function $f(t) = t^2$ over $0 < t < \pi$. Fourier series obtained by extending functions in this sort of way are often referred to as **half-range** series.

Normally, in applications, we require either a Fourier Cosine series (so we would complete a definition as in (i) above to obtain an **even** periodic function) or a Fourier Sine series (for which, as in (ii) above, we need an **odd** periodic function.)

The above considerations apply equally well for a function defined over any interval.

Example 3

Obtain the half range Fourier Sine series to represent $f(t) = t^2 \quad 0 < t < 3.$

Solution

We first extend $f(t)$ as an odd periodic function $F(t)$ of **period 6**: $f(t) = -t^2, \quad -3 < t < 0$

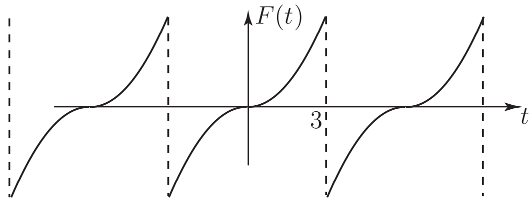


Figure 22

We now evaluate the Fourier series of $F(t)$ by standard techniques but take advantage of the symmetry and put $a_n = 0, \; n = 0, 1, 2, \ldots$.

Using the results for the Fourier Sine coefficients for period T from HELM 23.2 subsection 5,

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(t)\sin\left(\frac{2n\pi t}{T}\right)\,dt,$$

we put $T = 6$ and, since the integrand is even (a product of 2 odd functions), we can write

$$b_n = \frac{2}{3}\int_0^3 F(t)\sin\left(\frac{2n\pi t}{6}\right)\,dt = \frac{2}{3}\int_0^3 t^2\sin\left(\frac{n\pi t}{3}\right)\,dt.$$

(Note that we always integrate over the originally defined range, in this case $0 < t < 3$.)

We now have to integrate by parts (twice!)

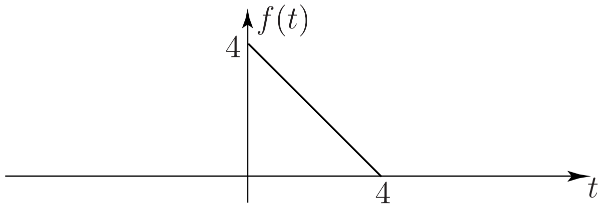
$$\begin{aligned}
b_n &= \frac{2}{3}\left\{\left[-\frac{3t^2}{n\pi}\cos\left(\frac{n\pi t}{3}\right)\right]_0^3 + 2\left(\frac{3}{n\pi}\right)\int_0^3 t\cos\left(\frac{n\pi t}{3}\right)\,dt\right\} \\
&= \frac{2}{3}\left\{-\frac{27}{n\pi}\cos n\pi + \frac{6}{n\pi}\left[\frac{3}{n\pi}t\sin\frac{n\pi t}{3}\right]_0^3 - \left(\frac{6}{n\pi}\right)\left(\frac{3}{n\pi}\right)\int_0^3 \sin\left(\frac{n\pi t}{3}\right)\,dt\right\} \\
&= \frac{2}{3}\left\{-\frac{27}{n\pi}\cos n\pi - \frac{18}{n^2\pi^2}\left[-\frac{3}{n\pi}\cos\left(\frac{n\pi t}{3}\right)\right]_0^3\right\} = \frac{2}{3}\left\{-\frac{27}{n\pi}\cos n\pi + \frac{54}{n^3\pi^3}(\cos n\pi - 1)\right\} \\
&= \begin{cases} -\dfrac{18}{n\pi} & n = 2, 4, 6, \ldots \\[2ex] \dfrac{18}{n\pi} - \dfrac{72}{n^3\pi^3} & n = 1, 3, 5, \ldots \end{cases}
\end{aligned}$$

So the required Fourier Sine series is

$$F(t) = 18\left(\frac{1}{\pi} - \frac{4}{\pi^3}\right)\sin\left(\frac{\pi t}{3}\right) - \frac{18}{2\pi}\sin\left(\frac{2\pi t}{3}\right) + 18\left(\frac{1}{3\pi} - \frac{4}{27\pi^3}\right)\sin(\pi t) - \ldots$$

Obtain a half-range Fourier Cosine series to represent the function

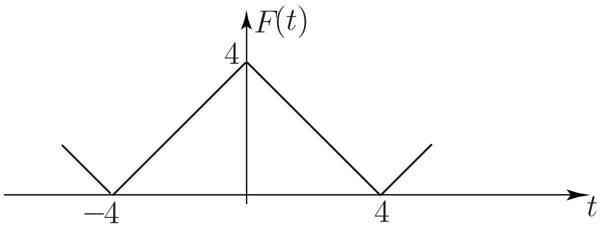
$$f(t) = 4 - t \qquad 0 < t < 4.$$



First complete the definition to obtain an even periodic function $F(t)$ of period 8. Sketch $F(t)$:

Your solution

Answer



Now formulate the integral from which the Fourier coefficients a_n can be calculated:

Your solution

Answer

We have with $T = 8$

$$a_n = \frac{2}{8}\int_{-4}^{4} F(t)\cos\left(\frac{2n\pi t}{8}\right)\,dt$$

Utilising the fact that the integrand here is even we get

$$a_n = \frac{1}{2}\int_{0}^{4}(4-t)\cos\left(\frac{n\pi t}{4}\right)\,dt$$

Now integrate by parts to obtain a_n and also obtain a_0:

Your solution

Answer

Using integration by parts we obtain for $n = 1, 2, 3, \ldots$

$$\begin{aligned} a_n &= \frac{1}{2}\left\{\left[(4-t)\frac{4}{n\pi}\sin\left(\frac{n\pi t}{4}\right)\right]_0^4 + \frac{4}{n\pi}\int_0^4 \sin\left(\frac{n\pi t}{4}\right)\,dt\right\} \\ &= \frac{1}{2}\left(\frac{4}{n\pi}\right)\left(\frac{4}{n\pi}\right)\left[-\cos\left(\frac{n\pi t}{4}\right)\right]_0^4 = \frac{8}{n^2\pi^2}\left[-\cos(n\pi)+1\right] \end{aligned}$$

i.e. $$a_n = \begin{cases} 0 & n = 2, 4, 6, \ldots \\ \dfrac{16}{n^2\pi^2} & n = 1, 3, 5, \ldots \end{cases}$$

Also $a_0 = \dfrac{1}{2}\displaystyle\int_0^4 (4-t)\,dt = 4$. So the constant term is $\dfrac{a_0}{2} = 2$.

Now write down the required Fourier series:

Your solution

Answer

We get $$2 + \frac{16}{\pi^2}\left\{\cos\left(\frac{\pi t}{4}\right) + \frac{1}{9}\cos\left(\frac{3\pi t}{4}\right) + \frac{1}{25}\cos\left(\frac{5\pi t}{4}\right) + \ldots\right\}$$

Note that the form of the Fourier series (a constant of 2 together with odd harmonic cosine terms) could be predicted if, in the sketch of $F(t)$, we imagine raising the t-axis by 2 units i.e. writing

$$F(t) = 2 + G(t)$$

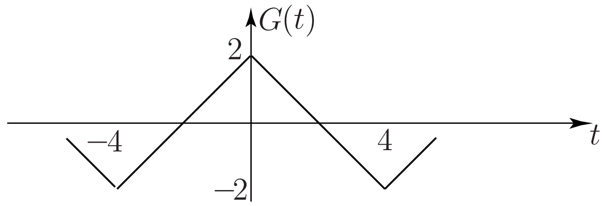


Figure 23

Clearly $G(t)$ possesses half-period symmetry

$$G(t+4) = -G(t)$$

and hence its Fourier series must contain only odd harmonics.

Exercises

Obtain the half-range Fourier series specified for each of the following functions:

1. $f(t) = 1 \qquad 0 \leq t \leq \pi$ (sine series)
2. $f(t) = t \qquad 0 \leq t \leq 1$ (sine series)
3. (a) $f(t) = \mathrm{e}^{2t} \qquad 0 \leq t \leq 1$ (cosine series)

   (b) $f(t) = \mathrm{e}^{2t} \qquad 0 \leq t \leq \pi$ (sine series)
4. (a) $f(t) = \sin t \qquad 0 \leq t \leq \pi$ (cosine series)

   (b) $f(t) = \sin t \qquad 0 \leq t \leq \pi$ (sine series)

Answers

1. $\dfrac{4}{\pi}\left\{\sin t + \dfrac{1}{3}\sin 3t + \dfrac{1}{5}\sin 5t + \cdots\right\}$
2. $\dfrac{2}{\pi}\{\sin \pi t - \dfrac{1}{2}\sin 2\pi t + \dfrac{1}{3}\sin 3\pi t - \cdots\}$
3. (a) $\dfrac{\mathrm{e}^2 - 1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{4}{4+n^2\pi^2}\{\mathrm{e}^2\cos(n\pi) - 1\}\cos n\pi t$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n\pi}{4+n^2\pi^2}\{1 - \mathrm{e}^2\cos(n\pi)\}\sin n\pi t$
4. (a) $\dfrac{2}{\pi} + \displaystyle\sum_{n=2}^{\infty} \frac{1}{\pi}\left\{\frac{1}{1-n}(1-\cos(1-n)\pi) + \frac{1}{1+n}(1-\cos(1+n)\pi)\right\}\cos nt$

   (b) $\sin t$ itself (!)