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Functions of a Complex Variable

Learning outcomes

By studying the Workbook you will understand the concept of a complex function and its derivative and learn what is meant by an analytic function and why analytic functions are important.
You will learn about the Cauchy-Riemann equations and the concept of conformal mapping and be able to solve complex problems involving standard complex functions and evaluate simple complex integrals.
You will learn Cauchy's theorem and be able to use it to evaluate complex integrals.
You will learn how to develop simple Laurent series and classify singularities of a complex function.
Finally, you will learn about the residue theorem and how to use it to solve problems.

Complex Functions 26.1

Introduction

In this introduction to functions of a complex variable we shall show how the operations of taking a limit and of finding a derivative, which we are familiar with for functions of a real variable, extend in a natural way to the complex plane. In fact the **notation** used for functions of a complex variable and for complex operations is almost identical to that used for functions of a real variable. In effect, the **real variable x** is simply replaced by the **complex variable** z. However, it is the **interpretation** of functions of a complex variable and of complex operations that differs significantly from the real case. In effect, a function of a complex variable is equivalent to **two** functions of a real variable and our standard interpretation of a function of a real variable as being a curve on an xy plane no longer holds.

There are many situations in engineering which are described quite naturally by specifying two harmonic functions of a real variable: a harmonic function is one satisfying the two-dimensional Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Fluids and heat flow in two dimensions are particular examples. It turns out that knowledge of functions of a complex variable can significantly ease the calculations involved in this area.

Prerequisites

Before starting this Section you should ...

- understand how to use the polar and exponential forms of a complex number
- be familiar with trigonometric relations, hyperbolic and logarithmic functions
- be able to form a partial derivative
- be able to take a limit

Learning Outcomes

On completion you should be able to ...

- explain the meaning of the term analytic function

1. Complex functions

Let the complex variable z be defined by $z = x + \mathrm{i}y$ where x and y are real variables and i is, as usual, given by $\mathrm{i}^2 = -1$. Now let a second complex variable w be defined by $w = u + \mathrm{i}v$ where u and v are real variables. If there is a relationship between w and z such that to each value of z in a given region of the $z-$plane there is assigned one, and only one, value of w then w is said to be a **function** of z, defined on the given region. In this case we write

$$w = f(z).$$

As a example consider $w = z^2 - z$, which is defined for all values of z (that is, the right-hand side can be computed for **every** value of z). Then, remembering that $z = x + \mathrm{i}y$,

$$w = u + \mathrm{i}v = (x + \mathrm{i}y)^2 - (x + \mathrm{i}y) = x^2 + 2\mathrm{i}xy - y^2 - x - \mathrm{i}y.$$

Hence, equating real and imaginary parts: $\quad u = x^2 - x - y^2 \quad$ and $\quad v = 2xy - y.$

If $z = 2 + 3\mathrm{i}$, for example, then $x = 2, y = 3$ so that $u = 4 - 2 - 9 = -7$ and $v = 12 - 3 = 9$, giving $w = -7 + 9\mathrm{i}$.

Example 1

(a) For which values of z is $w = \dfrac{1}{z}$ defined?

(b) For these values obtain u and v and evaluate w when $z = 2 - \mathrm{i}$.

Solution

(a) w is defined for all $z \neq 0$.

(b) $u + \mathrm{i}v = \dfrac{1}{x + \mathrm{i}y} = \dfrac{1}{x + iy} \cdot \dfrac{x - \mathrm{i}y}{x - \mathrm{i}y} = \dfrac{x - \mathrm{i}y}{x^2 + y^2}$. Hence $u = \dfrac{x}{x^2 + y^2}$ and $v = \dfrac{-y}{x^2 + y^2}$.

If $z = 2 - \mathrm{i}$ then $x = 2, y = -1$ so that $x^2 + y^2 = 5$. Then $u = \dfrac{2}{5}, v = -\dfrac{1}{5}$and $w = \dfrac{2}{5} - \dfrac{1}{5}\mathrm{i}$.

2. The limit of a function

The limit of $w = f(z)$ as $z \to z_0$ is a number ℓ such that $|f(z) - \ell|$ can be made as small as we wish by making $|z - z_0|$ sufficiently small. In some cases the limit is simply $f(z_0)$, as is the case for $w = z^2 - z$. For example, the limit of this function as $z \to \mathrm{i}$ is $f(\mathrm{i}) = \mathrm{i}^2 - \mathrm{i} = -1 - \mathrm{i}$.

There is a fundamental difference from functions of a real variable: there, we could approach a point on the curve $y = g(x)$ either from the left or from the right when considering limits of $g(x)$ at such points. With the function $f(z)$ we are allowed to approach the point $z = z_0$ along **any** path in the z-plane; we require merely that the distance $|z - z_0|$ decreases to zero.

Suppose that we want to find the limit of $f(z) = z^2 - z$ as $z \to 2 + \mathrm{i}$ along each of the paths (a), (b) and (c) indicated in Figure 1.

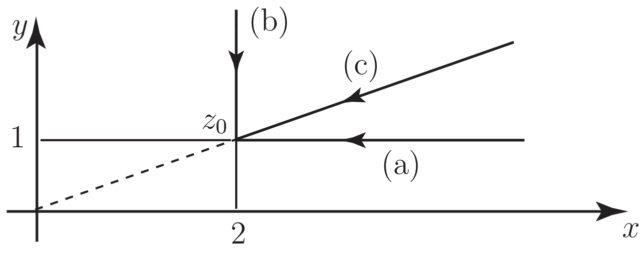

Figure 1

(a) Along this path $z = x + \mathrm{i}$ (for any x) and $z^2 - z = x^2 + 2x\mathrm{i} - 1 - x - i$

That is: $z^2 - z = x^2 - 1 - x + (2x - 1)\mathrm{i}$.

As $z \to 2 + \mathrm{i}$, then $x \to 2$ so that the limit of $z^2 - z$ is $2^2 - 1 - 2 + (4 - 1)\mathrm{i} = 1 + 3\mathrm{i}$.

(b) Here $z = 2 + y\mathrm{i}$ (for any y) so that $z^2 - z = 4 - y^2 - 2 + (4y - y)\mathrm{i}$.

As $z \to 2 + \mathrm{i}, y \to 1$ so that the limit of $z^2 - z$ is $4 - 1 - 2 + (4 - 1)\mathrm{i} = 1 + 3\mathrm{i}$.

(c) Here $z = k(2 + \mathrm{i})$ where k is a real number. Then

$$z^2 - z = k^2(4 + 4i - 1) - k(2 + \mathrm{i}) = 3k^2 - 2k + (4k^2 - k)\mathrm{i}.$$

As $z \to 2 + \mathrm{i}, k \to 1$ so that the limit of $z^2 - z$ is $3 - 2 + (4 - 1)\mathrm{i} = 1 + 3\mathrm{i}$.

In each case the limit is the same.

Task

Evaluate the limit of $f(z) = z^2 + z + 1$ as $z \to 1 + 2\mathrm{i}$ along the paths

(a) parallel to the x-axis coming from the right,

(b) parallel to the y-axis, coming from above,

(c) the line joining the point $1 + 2\mathrm{i}$ to the origin, coming from the origin.

Your solution

Answer

(a) Along this path $z = x + 2\text{i}$ and $z^2 + z + 1 = x^2 - 4 + x + 1 + (4x + 2)\text{i}$. As $z \to 1 + 2\text{i}, x \to 1$ and $z^2 + z + 1 \to -1 + 6\text{i}$.

(b) Along this path $z = 1 + y\text{i}$
and $z^2 + z + 1 = 1 - y^2 + 1 + 1 + (2y + y)\text{i}$. As $z \to 1 + 2\text{i}, y \to 2$ and $z^2 + z + 1 \to -1 + 6\text{i}$.

(c) If $z = k(1 + 2\text{i})$ then $z^2 + z + 1 = k^2 + k + 1 - 4k^2 + (4k^2 + 2k)\text{i}$. As $z \to 1 + 2\text{i}, k \to 1$ and $z^2 + z + 1 \to -1 + 6\text{i}$.

Not all functions of a complex variable are as straightforward to analyse as the last two examples. Consider the function $f(z) = \dfrac{\bar{z}}{z}$. Along the x-axis moving towards the origin from the right

$$z = x \quad \text{and} \quad \bar{z} = x \quad \text{so that} \quad f(z) = 1 \quad \text{which is the limit as} \quad z \to 0 \textbf{ along this path}.$$

However, we can approach the origin along any path. If instead we approach the origin along the positive y-axis $z = \text{i}y$ then

$$\bar{z} = -\text{i}y \quad \text{and} \quad f(z) = \frac{\bar{z}}{z} = -1, \quad \text{which is the limit as } z \to 0 \textbf{ along this path}.$$

Since these two limits are distinct then $\lim\limits_{z\to 0} \dfrac{\bar{z}}{z}$ **does not exist**.

We cannot assume that the limit of a function $f(z)$ as $z \to z_0$ is independent of the path chosen.

Definition of continuity

The function $f(z)$ is **continuous** as $z \to z_0$ if the following two statements are true:

(a) $f(z_0)$ exists;

(b) $\lim\limits_{z\to z_0} f(z)$ exists and is equal to $f(z_0)$.

As an example consider $f(z) = \dfrac{z^2 + 4}{z^2 + 9}$. As $z \to \text{i}$, then $f(z) \to f(\text{i}) = \dfrac{\text{i}^2 + 4}{\text{i}^2 + 9} = \dfrac{3}{8}$. Thus $f(z)$ is continuous at $z = \text{i}$.

However, when $z^2 + 9 = 0$ then $z = \pm 3\text{i}$ and neither $f(3\text{i})$ nor $f(-3\text{i})$ exists. Thus $\dfrac{z^2 + 4}{z^2 + 9}$ is discontinuous at $z = \pm 3\text{i}$. It is easily shown that these are the *only* points of discontinuity.

Task

State where $f(z) = \dfrac{z}{z^2 + 4}$ is discontinuous. Find $\lim\limits_{z\to \text{i}} f(z)$.

Your solution

Answer

$z^2 + 4 = 0$ where $z = \pm 2i$; at these points $f(z)$ is discontinuous as $f(\pm 2\mathsf{i})$ does not exist.

$$\lim_{z \to \mathsf{i}} f(z) = f(\mathsf{i}) = \frac{\mathsf{i}}{\mathsf{i}^2 + 4} = \frac{1}{3}\mathsf{i}.$$

It is easily shown that any polynomial in z is continuous everywhere whilst any rational function is continuous everywhere except at the zeroes of the denominator.

Exercises

1. For which values of z is $w = \dfrac{1}{z - \mathsf{i}}$ defined? For these values obtain u and v and evaluate w when $z = 1 - 2\mathsf{i}$.
2. Find the limit of $f(z) = z^3 + z$ as $z \to \mathsf{i}$ along the paths (a) parallel to the x-axis coming from the right, (b) parallel to the y-axis coming from above.
3. Where is $f(z) = \dfrac{z}{z^2 + 9}$ discontinuous?. Find the $\lim\limits_{z \to -\mathsf{i}} f(z)$.

Answers

1. w is defined for all $z \neq \mathsf{i}$ $\quad w = \dfrac{1}{x + y\mathsf{i} - \mathsf{i}} = \dfrac{1}{x + (y-1)\mathsf{i}} \times \dfrac{x - (y-1)\mathsf{i}}{x - (y-1)\mathsf{i}} = \dfrac{x - (y-1)\mathsf{i}}{x^2 + (y-1)^2}.$

$$\therefore \qquad u = \frac{x}{x^2 + (y-1)^2}, \qquad v = \frac{-(y-1)}{x^2 + (y-1)^2}.$$

When $z = 1 - 2\mathsf{i}$, $x = 1$, $y = -2$ so that $u = \dfrac{1}{1+9} = \dfrac{1}{10}$, $v = \dfrac{3}{10}$, $z = \dfrac{1}{10} + \dfrac{3}{10}\mathsf{i}$

2. (a) $z = x + \mathsf{i}$, $z^3 + z = x^3 + 3x^2\mathsf{i} - 2x$. As $z \to \mathsf{i}$, $x \to 0$ and $z^3 + z \to 0$

   (b) $z = y\mathsf{i}$, $z^3 + z = -y^3\mathsf{i} + y\mathsf{i}$. As $z \to \mathsf{i}$, $y \to 1$ and $z^3 + z \to -\mathsf{i} + \mathsf{i} = 0$.

3. $f(z)$ is discontinuous at $z = \pm 3\mathsf{i}$. The limit is $f(-\mathsf{i}) = \dfrac{-\mathsf{i}}{-1 + 9} = -\dfrac{1}{8}\mathsf{i}$.

3. Differentiating functions of a complex variable

The function $f(z)$ is said to be **differentiable** at $z = z_0$ if

$$\lim_{\Delta z \to 0} \left\{ \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right\} \quad \text{exists.} \qquad \text{Here } \Delta z = \Delta x + \mathsf{i}\Delta y.$$

Apart from a change of notation this is precisely the same as the definition of the derivative of a function of a real variable. Not surprisingly then, the rules of differentiation used in functions of a real variable can be used to differentiate functions of a complex variable. The value of the limit is the **derivative** of $f(z)$ at $z = z_0$ and is often denoted by $\frac{df}{dz}\big|_{z=z_0}$ or by $f'(z_0)$.

A point at which the derivative does not exist is called a **singular point** of the function.

A function $f(z)$ is said to be **analytic** at a point z_0 if it is differentiable throughout a neighbourhood of z_0, however small. (A neighbourhood of z_0 is the region contained within some circle $|x - z_0| = r$.)

For example, the function $f(z) = \dfrac{1}{z^2+1}$ has singular points where $z^2+1=0$, i.e. at $z = \pm\text{i}$.

For all other points the usual rules for differentiation apply and hence

$$f'(z) = -\frac{2z}{(z^2+1)^2} \qquad \text{(quotient rule)}$$

So, for example, at $z = 3\text{i}$, $f'(z) = -\dfrac{6\text{i}}{(-9+1)^2} = -\dfrac{3}{32}\text{i}$.

Example 2

Find the singular point of the rational function $f(z) = \dfrac{z}{z+\text{i}}$. Find $f'(z)$ at other points and evaluate $f'(2\text{i})$.

Solution

$z+\text{i} = 0$ when $z = -\text{i}$ and this is the singular point: $f(-\text{i})$ does not exist. Elsewhere, differentiating using the quotient rule:

$f'(z) = \dfrac{(z+\text{i})\cdot 1 - z\cdot 1}{(z+\text{i})^2} = \dfrac{\text{i}}{(z+\text{i})^2}$. Thus at $z = 2\text{i}$, we have $f'(z) = \dfrac{\text{i}}{(3\text{i})^2} = -\dfrac{1}{9}\text{i}$.

The simple function $f(z) = \bar{z} = x - \text{i}y$ is not analytic anywhere in the complex plane. To see this consider looking at the derivative at an *arbitrary* point z_0. We easily see that

$$\begin{aligned} R &= \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \\ &= \frac{(x_0+\Delta x) - \text{i}(y_0+\Delta y) - (x_0 - \text{i}y_0)}{\Delta x + \text{i}\Delta y} = \frac{\Delta x - \text{i}\,\Delta y}{\Delta x + \text{i}\,\Delta y} \end{aligned}$$

Hence $f(z)$ will fail to have a derivative at z_0 if we can show that this expression has no limit. To do this we consider looking at the limit of the function along two distinct paths.

Along a path parallel to the x-axis:

$$\Delta y = 0 \quad \text{so that} \quad R = \frac{\Delta x}{\Delta x} = 1, \quad \text{and this is the limit as} \quad \Delta z = \Delta x \to 0.$$

Along a path parallel to the y-axis:

$$\Delta x = 0 \quad \text{so that} \quad R = \frac{-\text{i}\,\Delta y}{\text{i}\Delta y} = -1, \quad \text{and this is the limit as } \Delta z = \Delta y \to 0.$$

As these two values of R are distinct, the limit of $\dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ as $z \to z_0$ does not exist and so $f(z)$ fails to be differentiable at *any* point. Hence it is not analytic anywhere.