Multiple Integrals over Non-rectangular Regions

27.2

Introduction

In the previous Section we saw how to evaluate double integrals over simple rectangular regions. We now see how to extend this to non-rectangular regions.

In this Section we introduce functions as the limits of integration, these functions define the region over which the integration is performed. These regions can be non-rectangular. Extra care now must be taken when changing the order of integration. Producing a sketch of the region is often very helpful.

Prerequisites

Before starting this Section you should . . .

- have a thorough understanding of the various techniques of integration
- be familiar with the concept of a function of two variables
- have completed Section 27.1
- be able to sketch a function in the plane

Learning Outcomes

On completion you should be able to . . .

- evaluate double integrals over non-rectangular regions

1. Functions as limits of integration

In Section 27.1 double integrals of the form

$$I = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y) \; dydx$$

were considered. They represent an integral over a rectangular region in the xy plane. If the limits of integration of the inner integral are replaced with functions G_1, G_2,

$$I = \int_{x=a}^{x=b} \int_{G_1(x)}^{G_2(x)} f(x, y) \; dydx$$

then the region described will not, in general, be a rectangle. The region will be a shape bounded by the curves (or lines) which these functions G_1 and G_2 describe.
As was indicated in 27.1

$$I = \int_{x=a}^{x=b} \int_{G_1(x)}^{G_2(x)} f(x, y) dy \, dx$$

can be interpreted as the volume lying above the region in the xy plane defined by $G_1(x)$ and $G_2(x)$, bounded above by the surface $z = f(x, y)$. Not all double integrals are interpreted as volumes but this is often the case. If $z = f(x, y) < 0$ anywhere in the relevant region, then the double integral no longer represents a volume.

Key Point 4

Double Integral Over General Region

$$I = \int_{x=a}^{x=b} \int_{G_1(x)}^{G_2(x)} f(x, y) \; dydx$$

1. The functions G_1, G_2 which are the limits for the inner integral are functions of the variable of the outer integral. This must be the case for the integral to make sense.
2. The limits of the outer integral are constant.
3. Integration over rectangular regions can be thought of as the special case where G_1 and G_2 are constant functions.

Example 8

Evaluate the integral $I = \int_{x=0}^{1} \int_{y=0}^{1-x} 2xy \; dydx$

Solution

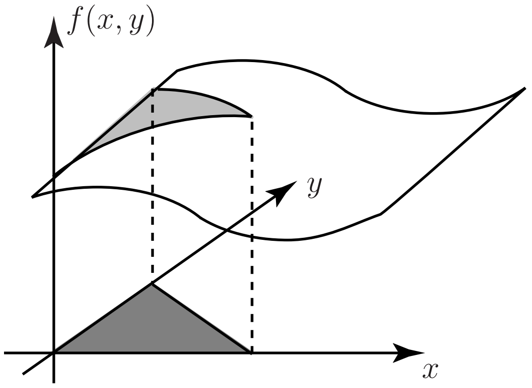


Figure 10

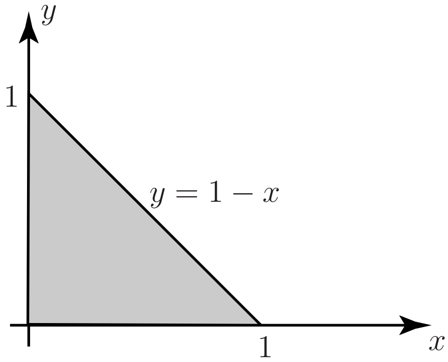


Figure 11

Projecting the relevant part of the surface (Figure 10) down to the xy plane produces the triangle shown in Figure 11. The extremes that x takes are $x = 0$ and $x = 1$ and so these are the limits on the outer integral. For any value of x, the variable y varies between $y = 0$ (at the bottom) and $y = 1 - x$ (at the top). Thus if the volume, shown in the diagram, under the function $f(x, y)$, bounded by this triangle is required then the following integral is to be calculated.

$$\int_{x=0}^{1} \int_{y=0}^{1-x} f\,(x, y)\;\, dydx$$

Once the correct limits have been determined, the integration is carried out in exactly the same manner as in Section 27.1

First consider the inner integral $g(x) = \int_{y=0}^{1-x} 2xy \; dy$

Integrating $2xy$ with respect to y gives $xy^2 + C$ so $g(x) = \Big[xy^2 \Big]_{y=0}^{1-x} = x(1-x)^2$

Note that, as is required, this is a function of x, the variable of the outer integral. Now the outer integral is

$$\begin{aligned} I &= \int_{x=0}^{1} x(1-x)^2 \; dx \\ &= \int_{x=0}^{1} \left(x^3 - 2x^2 + x\right) \; dx = \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_{x=0}^{1} = \frac{1}{4} - \frac{2}{3} + \frac{1}{2} = \frac{1}{12} \end{aligned}$$

Regions do not have to be bounded only by straight lines. Also the integrals may involve other tools of integration, such as substitution or integration by parts. Drawing a sketch of the limit functions in the plane and shading the region is a valuable tools when evaluating such integrals.

Example 9

Evaluate the volume under the surface given by $z = f(x, y) = 2x\sin(y)$, over the region bounded above by the curve $y = x^2$ and below by the line $y = 0$, for $0 \le x \le 1$.

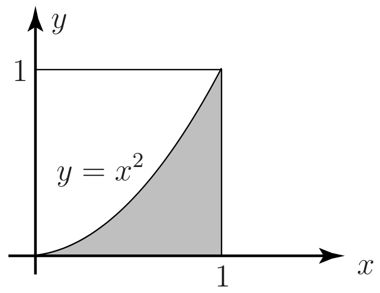


Figure 12

Solution

First sketch the curve $y = x^2$ and identify the region. This is the shaded region in Figure 12. The required integral is

$$\begin{aligned} I &= \int_{x=0}^{1}\int_{y=0}^{x^2} 2x\sin(y)\ dydx \\ &= \int_{x=0}^{1}\Big[-2x\cos(y)\Big]_{y=0}^{x^2}\ dx \\ &= \int_{x=0}^{1}\left(-2x\cos\left(x^2\right)+2x\right)\ dx \\ &= \int_{x=0}^{1}\left(1-\cos\left(x^2\right)\right)2x\ \ dx \end{aligned}$$

Making the substitution $u = x^2$ so $du = 2x\ dx$ and noting that the limits $x = 0, 1$ map to $u = 0, 1$, gives

$$\begin{aligned} I &= \int_{u=0}^{1}(1-\cos(u))\,du \\ &= \Big[u-\sin(u)\Big]_{u=0}^{1} \\ &= 1-\sin(1) \\ &\approx 0.1585 \end{aligned}$$

Example 10

Evaluate the volume under the surface given by $z = f(x,y) = x^2 + \frac{1}{2}y$, over the region bounded by the curves $y = 2x$ and $y = x^2$.

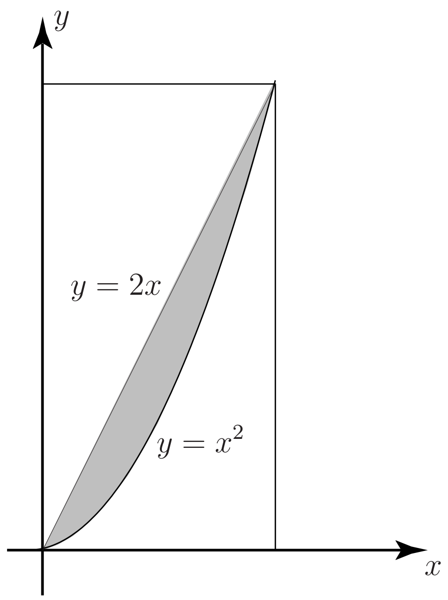


Figure 13

Solution

The sketch of the region is shown in Figure 13. The required integral is

$$I = \int_{x=a}^{b} \int_{y=x^2}^{2x} \left(x^2 + \frac{1}{2}y \right) \, dydx$$

To determine the limits for the integration with respect to x, the points where the curves intersect are required. These points are the solutions of the equation $2x = x^2$, so the required limits are $x = 0$ and $x = 2$. Then the volume is given by

$$\begin{aligned}
I &= \int_{x=0}^{2} \int_{y=x^2}^{2x} \left(x^2 + \frac{1}{2}y \right) \, dydx \\
&= \int_{x=0}^{2} \left[x^2 y + \frac{1}{4}y^2 \right]_{y=x^2}^{2x} \, dx \\
&= \int_{x=0}^{2} \left(x^2 + 2x^3 - \frac{5}{4}x^4 \right) \, dx \\
&= \left[\frac{x^3}{3} + \frac{x^4}{2} - \frac{x^5}{4} \right]_{x=0}^{2} \\
&= \frac{8}{3}
\end{aligned}$$

Example 11

(a) Evaluate the volume under $z = f(x, y) = 5x^2y$, over the half of the unit circle that lies above the x-axis. (Figure 14).

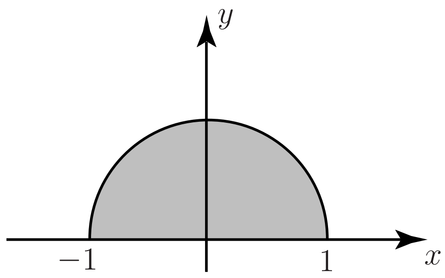


Figure 14

(b) Repeat (a) for $z = f(x, y) = 1$.

Solution

(a) This region is bounded by the circle $y^2 + x^2 = 1$ and the line $y = 0$. Since only positive values of y are required, the equation of the circle can be written $y = \sqrt{1 - x^2}$. Then the required volume is given by

$$\begin{aligned} I &= \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left(5x^2y\right)\, dydx \quad = \int_{x=-1}^{1} \left[\frac{5}{2}x^2y^2\right]_{y=0}^{\sqrt{1-x^2}} dx \\ &= \int_{x=-1}^{1} \frac{5}{2}x^2(1-x^2)\, dx \quad = \frac{5}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{3} \end{aligned}$$

(b)

$$\begin{aligned} I &= \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} 1\, dydx \quad = \int_{x=-1}^{1} \Big[y\Big]_{y=0}^{\sqrt{1-x^2}} dx \\ &= \int_{x=-1}^{1} \sqrt{1-x^2}\, dx \text{ (which by substituting } x = \sin\theta) = \frac{\pi}{2} \end{aligned}$$

Note that by putting $f(x, y) = 1$ we have found the volume of a semi-circular lamina of uniform height 1. This result is numerically the same as the **area** of the region in Figure 14. (This is a general result.)

Task

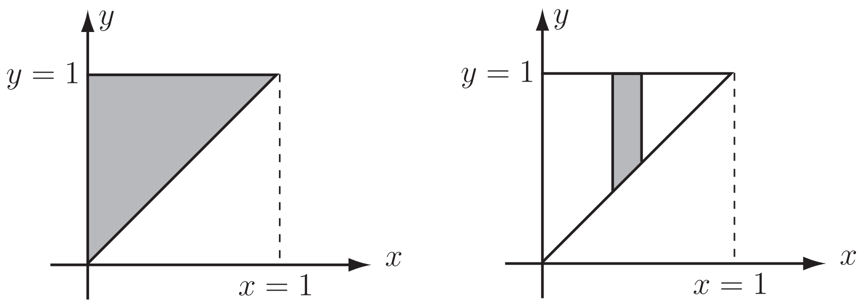
Evaluate the following double integral over a non-rectangular region.

$$\int_{x=0}^{1} \int_{y=x}^{1} \left(x^2 + y^2\right)\ dydx$$

(a) First sketch the region of the xy-plane determined by the limits:

Your solution

Answer



(b) Now evaluate the inner triangle:

Your solution

Answer

In the triangle, x varies between $x = 0$ and $x = 1$. For every value of x, y varies between $y = x$ and $y = 1$.

The inner integral is given by

$$\begin{aligned}\text{Inner Integral} = \int_{y=x}^{1} \left(x^2 + y^2\right) dy &= \left[x^2 y + \frac{1}{3}y^3\right]_x^1 \\ &= x^2 \times 1 + \frac{1}{3} \times 1^3 - (x^2 \times x + \frac{1}{3}x^3) \\ &= x^2 + \frac{1}{3} - \frac{4}{3}x^3\end{aligned}$$

(c) Finally evaluate the outer integral:

Your solution

Answer
The inner integral is placed in the outer integral to give

$$\text{Outer Integral} = \int_0^1 \left(x^2 - \frac{4}{3}x^3 + \frac{1}{3} \right)\, dx \;=\; \left[\frac{1}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{3}x \right]_0^1$$
$$= (\frac{1}{3} - \frac{1}{3} + \frac{1}{3}) - 0$$
$$= \frac{1}{3}$$

Note that the above Task is simply one of integrating a function over a region - there is no reference to a volume here. Another like this now follows.

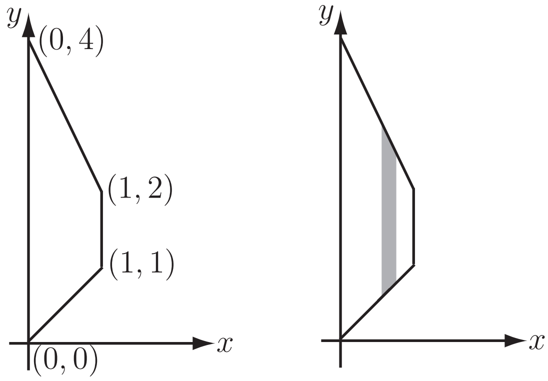
Task

Integrate the function $z = x^2y$ over the trapezium with vertices at $(0, 0)$, $(1, 1)$, $(1, 2)$ and $(0, 4)$.

Your solution

Answer

The integration takes place over the trapezium shown (left)



Considering variable x on the outer integral and variable y on the inner integral, the trapezium has an extent in x of $x = 0$ to $x = 1$. So, the limits on the outer integral (limits on x) are $x = 0$ and $x = 1$.

For each value of x, y varies from $y = x$ (line joining $(0,0)$ to $(1,1)$) to $y = 4 - 2x$ (line joining $(1,2)$ and $(0,4)$). So the limits on the inner integral (limits on y) are $y = x$ to $y = 4 - 2x$.

The double integral thus becomes

$$\int_{x=0}^{1} \int_{y=x}^{4-2x} x^2 y \; dy \; dx$$

The inner integral is

$$\int_{y=x}^{4-2x} x^2 y \; dy = \left[x^2 \frac{y^2}{2} \right]_{y=x}^{4-2x} = x^2 \frac{(4-2x)^2}{2} - x^2 \frac{x^2}{2} = 8x^2 - 8x^3 + \frac{3}{2}x^4$$

Putting this into the outer integral gives

$$\int_{x=0}^{1} (8x^2 - 8x^3 + \frac{3}{2}x^4) \; dx = \left[\frac{8}{3}x^3 - 2x^4 + \frac{3}{10}x^5 \right]_0^1 = (\frac{8}{3} - 2 + \frac{3}{10}) - 0 = \frac{29}{30}$$

Exercises

Evaluate the following integrals

1. $\displaystyle\int_{x=0}^{1} \int_{y=3x}^{x^2+2} xy \; dydx$

2. $\displaystyle\int_{x=1}^{2} \int_{y=x^2+2}^{3x} xy \; dydx$ [Hint: Note how the same curves can define different regions.]

3. $\displaystyle\int_{x=1}^{2} \int_{y=1}^{x^2} \frac{x}{y} \; dydx,$ [Hint: use integration by parts for the outer integral.]

Answers

1. $\dfrac{11}{24}$

2. $\dfrac{9}{8}$

3. $4\ln 2 - \dfrac{3}{2} \approx 1.27$

Splitting the region of integration

Sometimes it is difficult or impossible to represent the region of integration by means of consistent limits on x and y. Instead, it is possible to divide the region of integration into two (or more) sub-regions, carry out a multiple integral on each region and add the integrals together. For example, suppose it is necessary to integrate the function $g(x, y)$ over the triangle defined by the three points $(0, 0)$, $(1, 4)$ and $(2, -2)$.

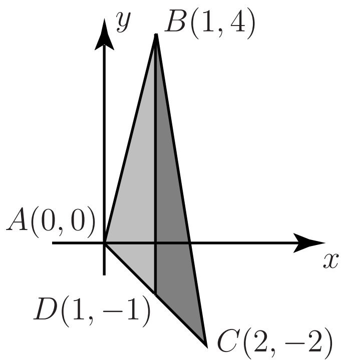


Figure 15

It is not possible to represent the triangle ABC by means of limits on an inner integral and an outer integral. However, it can be split into the triangle ABD and the triangle BCD. D is chosen to be the point on AC directly beneath B, that is, line BD is parallel to the y-axis so that x is constant along it. Note that the sides of triangle ABC are defined by sections of the lines $y = 4x$, $y = -x$ and $y = -6x + 10$.

In triangle ABD, the variable x takes values between $x = 0$ and $x = 1$. For each value of x, y can take values between $y = -x$ (bottom) and $y = 4x$. Hence, the integral of the function $g(x, y)$ over triangle ABD is

$$I_1 = \int_{x=0}^{x=1} \int_{y=-x}^{y=4x} g\,(x, y)\ dydx$$

Similarly, the integral of $g(x, y)$ over triangle BCD is

$$I_2 = \int_{x=1}^{x=2} \int_{y=-x}^{y=-6x+10} g\,(x, y)\ dydx$$

and the integral over the full triangle is

$$I = I_1 + I_2 = \int_{x=0}^{x=1} \int_{y=-x}^{y=4x} g\,(x, y)\ dydx + \int_{x=1}^{x=2} \int_{y=-x}^{y=-6x+10} g\,(x, y)\ dydx$$

Example 12

Integrate the function $g(x,y) = xy$ over the triangle ABC.

Solution

Over triangle ABD, the integral is

$$\begin{aligned} I_1 &= \int_{x=0}^{x=1} \int_{y=-x}^{y=4x} xy \; dydx \\ &= \int_{x=0}^{x=1} \left[\frac{1}{2}xy^2 \right]_{y=-x}^{y=4x} dx = \int_{x=0}^{x=1} \left[8x^3 - \frac{1}{2}x^3 \right] dx \\ &= \int_{x=0}^{x=1} \frac{15}{2}x^3 \; dx = \left[\frac{15}{8}x^4 \right]_0^1 = \frac{15}{8} - 0 = \frac{15}{8} \end{aligned}$$

Over triangle BCD, the integral is

$$\begin{aligned} I_2 &= \int_{x=1}^{x=2} \int_{y=-x}^{y=-6x+10} xy \; dydx \\ &= \int_{x=1}^{x=2} \left[\frac{1}{2}xy^2 \right]_{y=-x}^{y=-6x+10} dx = \int_{x=1}^{x=2} \left[\frac{1}{2}x(-6x+10)^2 - \frac{1}{2}x(-x)^2 \right] dx \\ &= \int_{x=1}^{x=2} \frac{1}{2} \left[36x^3 - 120x^2 + 100x - x^3 \right] dx = \frac{1}{2} \int_{x=1}^{x=2} \left[35x^3 - 120x^2 + 100x \right] dx \\ &= \frac{1}{2} \left[\frac{35}{4}x^4 - 40x^3 + 50x^2 \right]_1^2 = 10 - \frac{75}{8} = \frac{5}{8} \end{aligned}$$

So the total integral is $I_1 + I_2 = \dfrac{15}{8} + \dfrac{5}{8} = \dfrac{5}{2}$

2. Order of integration

All of the preceding Examples and Tasks have been integrals of the form

$$I = \int_{x=a}^{x=b} \int_{G_1(x)}^{G_2(x)} f\,(x,y) \; dydx$$

These integrals represent taking vertical slices through the volume that are parallel to the yz-plane. That is, vertically through the xy-plane.

Just as for integration over rectangular regions, the order of integration can be changed and the region can be sliced parallel to the xz-plane. If the inner integral is taken with respect to x then an integral of the following form is obtained:

$$I = \int_{y=c}^{y=d} \int_{H_1(y)}^{H_2(y)} f\,(x,y) \; dxdy$$

Key Point 5

Changing Order of Integration

1. The integrand $f(x,y)$ is not altered by changing the order of integration.
2. The limits will, in general, be different.

Example 13

The following integral was evaluated in Example 9.

$$I = \int_{x=0}^{1}\int_{y=0}^{x^2} 2x\sin(y)\ dydx = 1-\sin(1)$$

Change the order of integration and confirm that the new integral gives the same result.

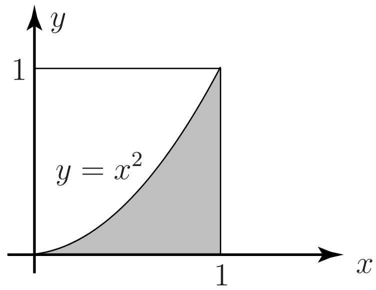


Figure 16

Solution

The integral is taken over the region which is bounded by the curve $y=x^2$. Expressed as a function of y this curve is $x=\sqrt{y}$. Now consider this curve as bounding the region from the left, then the line $x=1$ bounds the region to the right. These then are the limit functions for the inner integral $H_1(y)=\sqrt{y}$ and $H_2(y)=1$. Then the limits for the outer integral are $c=0\le y\le 1=d$. The following integral is obtained

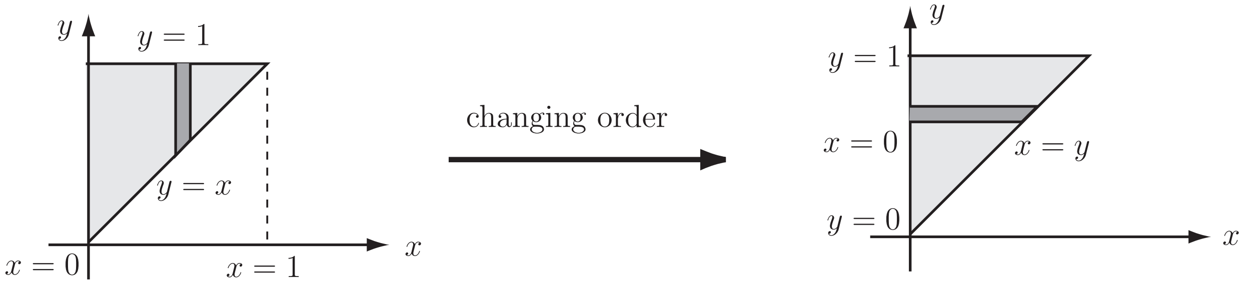
$$\begin{aligned} I &= \int_{y=0}^{1}\int_{x=\sqrt{y}}^{1} 2x\sin(y)\ \ dxdy = \int_{y=0}^{1}\left[x^2\sin(y)\right]_{x=\sqrt{y}}^{x=1} dy = \int_{y=0}^{1}(1-y)\sin(y))\,dy \\ &= \Big[-(1-y)\cos(y)\Big]_{y=0}^{1} - \int_{y=0}^{1}\cos(y)\ dy, \quad \text{using integration by parts} \\ &= 1-\Big[\sin(y)\Big]_{y=0}^{1} = 1-\sin(1) \end{aligned}$$

Task

The double integral $I = \int_0^1 \int_x^1 e^{y^2}\ dydx$ involves an inner integral which is impossible to integrate. Show that if the order of integration is reversed, the integral can be expressed as $I = \int_0^1 \int_0^y e^{y^2}\ dxdy$. Hence evaluate the integral I.

Your solution

Answer
The following diagram shows the changing description of the boundary as the order of integration is changed.



$$I = \int_0^1 \int_x^1 e^{y^2}\ dydx = \int_0^1 \int_0^y e^{y^2} dxdy \quad = \quad \int_0^1 \left[x e^{y^2} \right]_0^y dy$$

$$= \quad \int_0^1 y e^{y^2} dy = \left[\frac{1}{2} e^{y^2} \right]_0^1 = \frac{1}{2}(e - 1)$$

3. Evaluating surface integrals using polar coordinates

Areas with circular boundaries often lead to double integrals with awkward limits, and these integrals can be difficult to evaluate. In such cases it is easier to work with polar (r, θ) rather than Cartesian (x, y) coordinates.

Polar coordinates

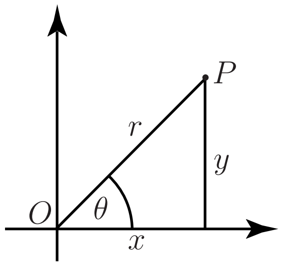


Figure 17

The polar coordinates of the point P are the distance r from P to the origin O and the angle θ that the line OP makes with the positive x axis. The following are used to transform between polar and rectangular coordinates.

1. Given (x, y), (r, θ) are found using $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.
2. Given (r, θ), (x, y) are found using $x = r\cos\theta$ and $y = r\sin\theta$

Note that we also have the relation $r^2 = x^2 + y^2$.

Finding surface integrals with polar coordinates

The area of integration A is covered with coordinate circles given by $r =$ constant and coordinate lines given by $\theta =$ constant.
The elementary areas δA are almost rectangles having width δr and length determined by the length of the part of the circle of radius r between θ and $\delta\theta$, the arc length of this part of the circle is $r\delta\theta$. So $\delta A \approx r\delta r\delta\theta$. Thus to evaluate $\displaystyle\int_A f(x, y)\, dA$ we sum $f(r, \theta)r\delta r\delta\theta$ for all δA.

$$\int_A f(x, y)\, dA = \int_{\theta=\theta_A}^{\theta=\theta_B} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r, \theta)\ r\ drd\theta$$

Key Point 6

Polar Coordinates

In double integration using polar coordinates, the variable r appears in $f(r, \theta)$ and in $rdrd\theta$. As explained above, this r is required because the elementary area element become larger further away from the origin.

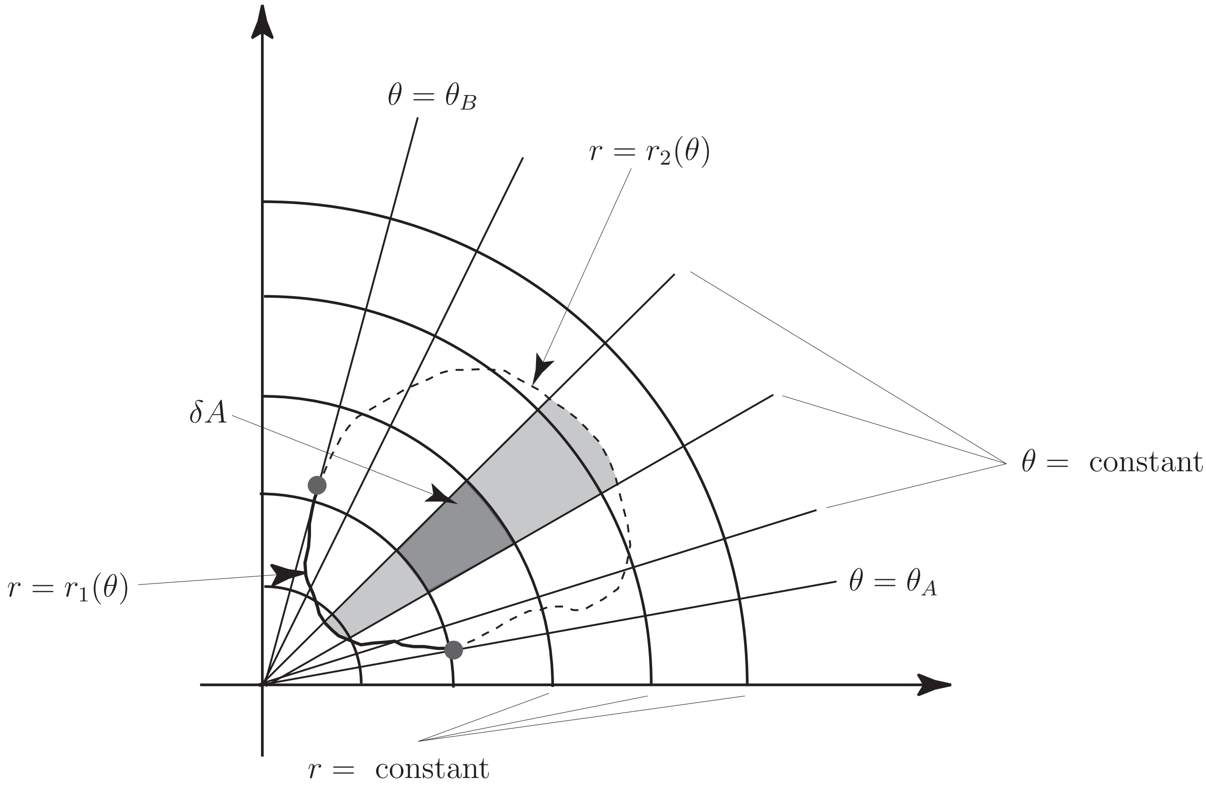


Figure 18

Note that the use of polar coordinates is a special case of the use of a change of variables. Further cases of change of variables will be considered in Section 27.4.

Example 14

Evaluate $\displaystyle\int_0^{\frac{\pi}{3}}\int_0^2 r\cos\theta\; dr d\theta$ and sketch the region of integration. Note that it is the function $\cos\theta$ which is being integrated over the region and the r comes from the $r dr d\theta$.

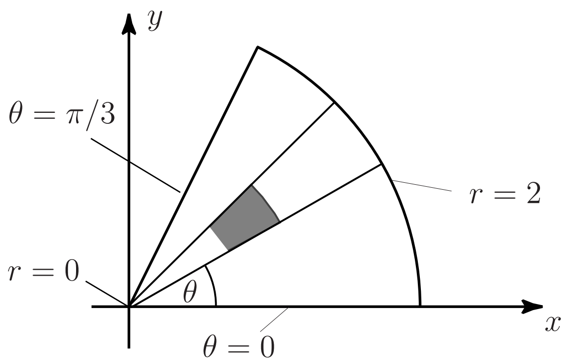


Figure 19

Solution

The evaluation is similar to that for cartesian coordinates. The inner integral with respect to r, is evaluated first with θ constant. Then the outer θ integral is evaluated.

$$\begin{aligned}\int_0^{\frac{\pi}{3}}\int_0^2 r\cos\theta\ d\theta &= \int_0^{\frac{\pi}{3}}\left[\frac{1}{2}r^2\cos\theta\right]_0^2 d\theta \\ &= \int_0^{\frac{\pi}{3}} 2\cos\theta\ d\theta \\ &= \Big[2\sin\theta\Big]_0^{\frac{\pi}{3}} = 2\sin\frac{\pi}{3} = \sqrt{3}\end{aligned}$$

With θ constant r varies between 0 and 2, so the bounding curves of the polar strip start at $r = 0$ and end at $r = 2$. As θ varies between 0 and $\frac{\pi}{3}$ a sector of a circular disc is swept out. This sector is the region of integration shown above.

Example 15

Earlier in this Section, an example concerned integrating the function $f(x, y) = 5x^2y$ over the half of the unit circle which lies above the x-axis. It is also possible to carry out this integration using polar coordinates.

Solution

The semi-circle is characterised by $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$. So the integral may be written (remembering that $x = r\cos\theta$ and $y = r\sin\theta$)

$$\int_0^{\pi}\int_0^1 5(r\cos\theta)^2(r\sin\theta)\ r\ drd\theta$$

which can be evaluated as follows

$$\begin{aligned}&\int_0^{\pi}\int_0^1 5r^4\sin\theta\cos^2\theta\ drd\theta \\ = &\int_0^{\pi}\left[r^5\sin\theta\cos^2\theta\right]_0^1 d\theta \\ = &\int_0^{\pi}\sin\theta\cos^2\theta\ d\theta = \left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi} = -\frac{1}{3}\cos^3\pi + \frac{1}{3}\cos^3 0 = -\frac{1}{3}(-1) + \frac{1}{3}(1) = \frac{2}{3}\end{aligned}$$

This is, of course, the same answer that was obtained using an integration over rectangular coordinates.

4. Applications of surface integration

Force on a dam

Section 27.1 considered the force on a rectangular dam of width 100 m and height 40 m. Instead, imagine that the dam is not rectangular in profile but instead has a width of 100 m at the top but only 80 m at the bottom. The top and bottom of the dam can be given by line segments $y = 0$ (bottom) and $y = 40$ while the sides are parts of the lines $y = 40 - 4x$ i.e. $x = 10 - \dfrac{y}{4}$ (left) and $y = 40 + 4(x - 100) = 4x - 360$ i.e. $x = 90 + \dfrac{y}{4}$(right). (See Figure 20).

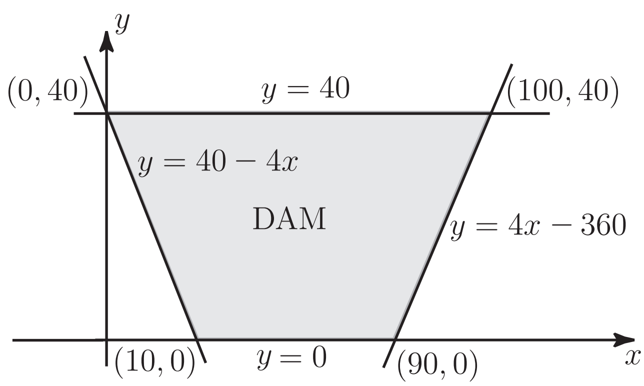


Figure 20

Thus the dam exists at heights y between 0 and 40 while for each value of y, the horizontal coordinate x varies between $x = 10 - \dfrac{y}{4}$ and $x = 90 + \dfrac{y}{4}$. Thus the surface integral representing the total force i.e.

$I = \displaystyle\int_A 10^4(40 - y)\, dA$ becomes the double integral $I = \displaystyle\int_0^{40} \int_{10-\frac{y}{4}}^{90+\frac{y}{4}} 10^4(40 - y)\ dxdy$

which can be evaluated as follows

$$\begin{aligned}
I &= \int_0^{40} \int_{10-\frac{y}{4}}^{90+\frac{y}{4}} 10^4(40 - y)\ dxdy \\
&= 10^4 \int_0^{40} \Big[\ (40 - y)x\Big]_{10-\frac{y}{4}}^{90+\frac{y}{4}}\ dy = 10^4 \int_0^{40} \left[(40 - y)(90 + \frac{y}{4}) - (40 - y)(10 - \frac{y}{4})\right]\ dy \\
&= 10^4 \int_0^{40} \left[(40 - y)(80 + \frac{y}{2})\right]\ dy = 10^4 \int_0^{40} \left[3200 - 60y - \frac{y^2}{2}\right]\ dy \\
&= 10^4 \left[3200y - 30y^2 - \frac{1}{6}y^3\right]_0^{40} = 10^4 \left[(3200 \times 40 - 30 \times 40^2 - \frac{1}{6}40^3) - 0\right] \\
&= 10^4 \times \frac{208000}{3} \approx 6.93 \times 10^8 \text{N}
\end{aligned}$$

i.e. the total force is just under 700 meganewtons.

Centre of pressure

A plane area in the shape of a quadrant of a circle of radius a is immersed vertically in a fluid with one bounding radius in the surface. Find the position of the centre of pressure.

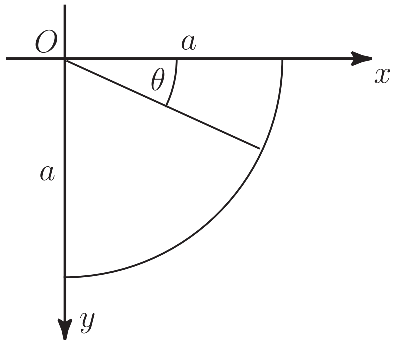


Figure 21

Note: In subsection 6 of Section 27.1 it was shown that the coordinates of the centre of pressure of a (thin) object are

$$x_p = \frac{\displaystyle\int_A xy\ dA}{\displaystyle\int_A y\ dA} \qquad \text{and} \qquad y_p = \frac{\displaystyle\int_A y^2\ dA}{\displaystyle\int_A y\ dA}$$

$$\begin{aligned}
\int_A y\ dA &= \int_0^{\frac{\pi}{2}}\int_0^a r^2\sin\theta\ drd\theta = \int_0^{\frac{\pi}{2}}\left[\frac{1}{3}r^3\sin\theta\right]_0^a\ d\theta \\
&= \frac{1}{3}a^3\int_0^{\frac{\pi}{2}}\sin\theta\ d\theta = \frac{1}{3}a^3\Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} = \frac{1}{3}a^3 \\
\int_A xy\ dA &= \int_0^{\frac{\pi}{2}}\int_0^a r^3\cos\theta\sin\theta\ d\theta = \int_0^{\frac{\pi}{2}}\left[\frac{1}{4}r^4\cos\theta\sin\theta\right]_0^a\ d\theta \\
&= \frac{1}{4}a^4\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\ d\theta = \frac{1}{4}a^4\left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{8}a^4 \\
\int_A y^2\ dA &= \int_0^{\frac{\pi}{2}}\int_0^a r^3\sin^2\theta\ drd\theta = \int_0^{\frac{\pi}{2}}\left[\frac{1}{4}r^4\sin^2\theta\right]_0^a\ d\theta \\
&= \frac{1}{4}a^4\int_0^{\frac{\pi}{2}}\sin^2\theta\ d\theta = \frac{1}{4}a^4\int_0^{\frac{\pi}{2}}\frac{1}{2}\left(1-\cos 2\theta\right)\ d\theta = \frac{1}{8}a^4\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{16}\pi a^4
\end{aligned}$$

Then $x_p = \dfrac{\displaystyle\int_A xy\ dA}{\displaystyle\int_A y\ dA} = \dfrac{\frac{1}{8}a^4}{\frac{1}{3}a^3} = \dfrac{3}{8}a$ and $y_p = \dfrac{\displaystyle\int_A y^2\ dA}{\displaystyle\int_A y\ dA} = \dfrac{\frac{1}{16}\pi a^4}{\frac{1}{3}a^3} = \dfrac{3}{16}\pi a.$

The centre of pressure is at $\left(\dfrac{3}{8}a, \dfrac{3}{16}\pi a\right)$.

Engineering Example 1

Volume of liquid in an elliptic tank

Introduction

A tank in the shape of an elliptic cylinder has a volume of liquid poured into it. It is useful to know in advance how deep the liquid will be. In order to make this calculation, it is necessary to perform a multiple integration.

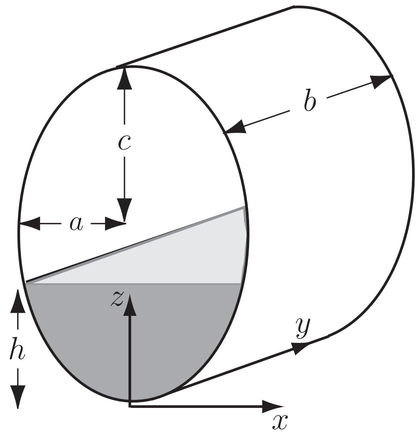


Figure 22

Problem in words

The tank has semi-axes a (horizontal) and c (vertical) and is of constant thickness b. A volume of liquid V is poured in (assuming that $V < \pi abc$, the volume of the tank), filling it to a depth h, which is to be calculated. Assume 3-D coordinate axes based on a point at the bottom of the tank.

Mathematical statement of the problem

Since the tank is of constant thickness b, the volume of liquid is given by the shaded area multiplied by b, i.e.

$$V = b \times \text{shaded area}$$

where the shaded area can be expressed as the double integral

$$\int_{z=0}^{h} \int_{x=x_1}^{x_2} dxdz$$

where the limits x_1 and x_2 on x can be found from the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{(z-c)^2}{c^2} = 1$$

Mathematical analysis

From the equation of the ellipse

$$\begin{aligned} x^2 &= a^2\left[1-\frac{(z-c)^2}{c^2}\right] \\ &= \frac{a^2}{c^2}\left[c^2-(z-c)^2\right] \\ &= \frac{a^2}{c^2}\left[2zc-z^2\right] \qquad \text{so} \quad x=\pm\frac{a}{c}\sqrt{2zc-z^2} \end{aligned}$$

Thus

$$x_1=-\frac{a}{c}\sqrt{2zc-z^2}, \quad \text{and} \quad x_2=+\frac{a}{c}\sqrt{2zc-z^2}$$

Consequently

$$\begin{aligned} V=b\int_{z=0}^{h}\int_{x=x_1}^{x_2} dx dz &= b\int_{z=0}^{h}\Big[x\Big]_{x_1}^{x_2} dz \\ &= b\int_{z=0}^{h} 2\frac{a}{c}\sqrt{2zc-z^2}\, dz \end{aligned}$$

Now use substitution $z-c=c\ \sin\theta$ so that $dz=c\ \cos\theta\ d\theta$

$$z=0 \quad \text{gives} \quad \theta=-\frac{\pi}{2}$$

$$z=h \quad \text{gives} \quad \theta=\sin^{-1}\left(\frac{h}{c}-1\right)=\theta_0 \quad \text{(say)}$$

$$\begin{aligned} V &= b\int_{-\frac{\pi}{2}}^{\theta_0} 2\frac{a}{c}\ c\cos\theta\ c\cos\theta\ d\theta \\ &= 2abc\int_{-\frac{\pi}{2}}^{\theta_0}\cos^2\theta\ d\theta \\ &= abc\int_{-\frac{\pi}{2}}^{\theta_0}\left[1+\cos 2\theta\right]\ d\theta \\ &= abc\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\theta_0} \\ &= abc\left[\theta_0+\frac{1}{2}\sin 2\theta_0-\left(-\frac{\pi}{2}+0\right)\right] \\ &= abc\left[\theta_0+\frac{1}{2}\sin 2\theta_0+\frac{\pi}{2}\right] \qquad \ldots(*) \end{aligned}$$

which can also be expressed in the form

$$V=abc\left[\sin^{-1}\left(\frac{h}{c}-1\right)+\left(\frac{h}{c}-1\right)\sqrt{1-\left(\frac{h}{c}-1\right)^2}+\frac{\pi}{2}\right]$$

While $(*)$ expresses V as a function of θ_0 (and therefore h) to find θ_0 as a function of V requires a numerical method. For a given a, b, c and V, solve equation $(*)$ by a numerical method to find θ_0 and find h from $h = c(1 + \sin\theta_0)$.

Interpretation

If $a = 2$ m, $b = 1$ m, $c = 3$ m (so the total volume of the tank is 6π m$^3 \approx 18.85$ m^3), and a volume of 7 m^3 is to be poured into the tank then

$$V = abc\left[\theta_0 + \frac{1}{2}\sin 2\theta_0 + \frac{\pi}{2}\right]$$

which becomes

$$7 = 6\left[\theta_0 + \frac{1}{2}\sin 2\theta_0 + \frac{\pi}{2}\right]$$

and has solution $\theta_0 = -0.205$ (3 decimal places).

Finally

$$\begin{aligned} h &= c(1 + \sin\theta_0) \\ &= 3(1 + \sin(-0.205)) \\ &= 2.39 \text{ m to 2 d.p} \end{aligned}$$

compared to the maximum height of 6 m.

Exercises

1. Evaluate the functions (a) xy and (b) $xy + 3y^2$

   over the quadrilateral with vertices at $(0,0)$, $(3,0)$, $(2,2)$ and $(0,4)$.

2. Show that $\displaystyle\int\int_A f(x,y)\,dy\,dx = \int\int_A f(x,y)\,dx\,dy$ for $f(x,y) = xy^2$ when A is the interior of the triangle with vertices at $(0,0)$, $(2,0)$ and $(2,4)$.

3. By reversing the order of the two integrals, evaluate the integral $\displaystyle\int_{y=0}^{4}\int_{x=y^{1/2}}^{2} \sin x^3\,dx\,dy$

4. Integrate the function $f(x,y) = x^3 + xy^2$ over the quadrant $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 1$.

Answers

1. $\displaystyle\int_{x=0}^{2}\int_{y=0}^{4-x} f(x,y)\,dy\,dx + \int_{x=2}^{3}\int_{y=0}^{6-2x} f(x,y)\,dy\,dx;\qquad \frac{22}{3} + \frac{3}{2} = \frac{53}{6}\ ;\ \frac{202}{3} + \frac{7}{2} = \frac{425}{6}$

2. Both equal $\dfrac{256}{15}$

3. $\displaystyle\int_{x=0}^{2}\int_{y=0}^{x^2} \sin x^3\,dy\,dx = \frac{1}{3}(1 - \cos 8) \approx 0.382$

4. $\displaystyle\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^4\cos\theta\,dr\,d\theta = \frac{1}{5}$