Surface and Volume Integrals

29.2

Introduction

A vector or scalar field - including one formed from a vector derivative (div, grad or curl) - can be integrated over a surface or volume. This Section shows how to carry out such operations.

Prerequisites

Before starting this Section you should ...

- be familiar with vector derivatives
- be familiar with double and triple integrals

Learning Outcomes

On completion you should be able to ...

- carry out operations involving integration of scalar and vector fields

1. Surface integrals involving vectors

The unit normal

For the surface of any three-dimensional shape, it is possible to find a vector lying perpendicular to the surface and with magnitude 1. The unit vector points outwards from a closed surface and is usually denoted by $\hat{\underline{n}}$.

Example 17

If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ find the unit normal $\hat{\underline{n}}$.

Solution

The unit normal at the point $P(x, y, z)$ points away from the centre of the sphere i.e. it lies in the direction of $x\underline{i} + y\underline{j} + z\underline{k}$. To make this a unit vector it must be divided by its magnitude $\sqrt{x^2 + y^2 + z^2}$ i.e. the unit vector is

$$\begin{aligned}
\hat{\underline{n}} &= \frac{x}{\sqrt{x^2+y^2+z^2}}\underline{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\underline{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\underline{k} \\
&= \frac{x}{a}\underline{i} + \frac{y}{a}\underline{j} + \frac{z}{a}\underline{k}
\end{aligned}$$

where $a = \sqrt{x^2 + y^2 + z^2}$ is the radius of the sphere.

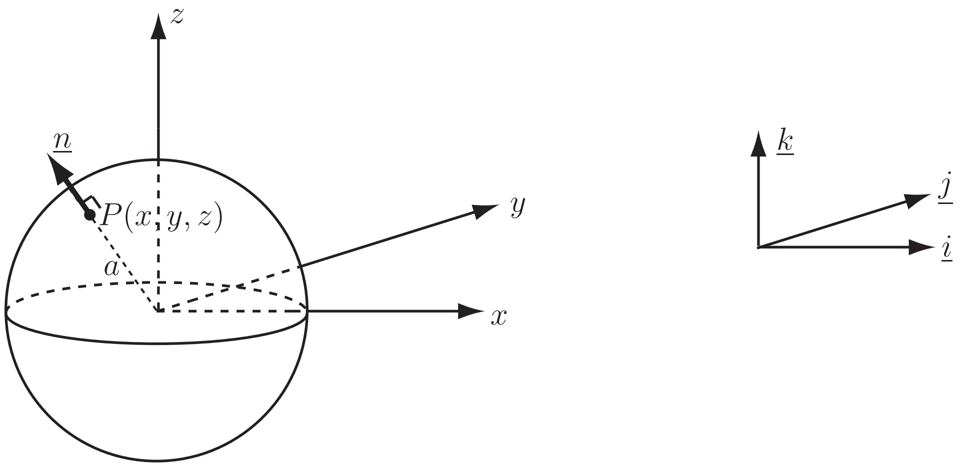

Figure 6: A unit normal $\hat{\underline{n}}$ to a sphere

Example 18

For the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, find the unit outward normal $\underline{\hat{n}}$ for each face.

Solution

On the face given by $x = 0$, the unit normal points in the negative x-direction. Hence the unit normal is $-\underline{i}$. Similarly :-

On the face $x = 1$ the unit normal is $\underline{i}$. On the face $y = 0$ the unit normal is $-\underline{j}$.

On the face $y = 1$ the unit normal is $\underline{j}$. On the face $z = 0$ the unit normal is $-\underline{k}$.

On the face $z = 1$ the unit normal is $\underline{k}$.

$\underline{dS}$ and the unit normal

The vector $\underline{dS}$ is a vector, being an element of the surface with magnitude $du\,dv$ and direction perpendicular to the surface.

If the plane in question is the Oxy plane, then $\underline{dS} = \underline{\hat{n}}\,du\,dv = \underline{k}\,dx\,dy$.

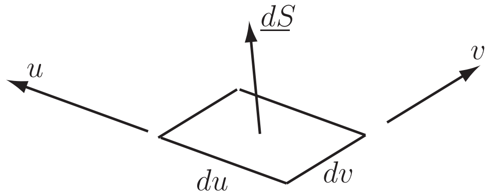

Figure 7: The vector $\underline{dS}$ as an element of a surface, with magnitude $du\,dv$

If the plane in question is not one of the three coordinate planes (Oxy, Oxz, Oyz), appropriate adjustments must be made to express $\underline{dS}$ in terms of two of dx and dy and dz.

Example 19

The rectangle $OABC$ lies in the plane $z = y$ (Figure 8).
The vertices are $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 1, 1)$.
Find a unit vector $\underline{\hat{n}}$ normal to the plane and an appropriate vector $\underline{dS}$ expressed in terms of dx and dy.

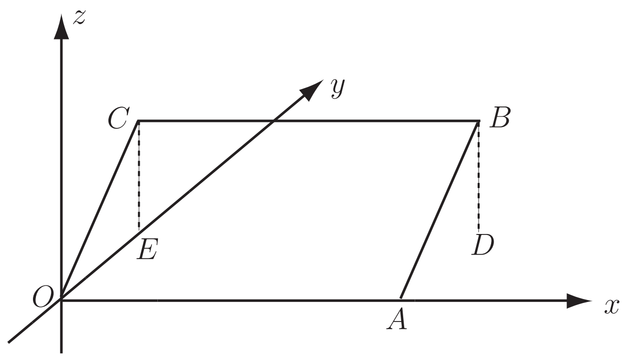

Figure 8: The plane $z = y$ passing through $OABC$

Solution

Note that two vectors in the rectangle are $\overrightarrow{OA} = \underline{i}$ and $\overrightarrow{OC} = \underline{j} + \underline{k}$. A vector perpendicular to the plane is $\underline{i} \times (\underline{j} + \underline{k}) = -\underline{j} + \underline{k}$. However, this vector is of magnitude $\sqrt{2}$ so the unit normal vector is $\hat{\underline{n}} = \dfrac{1}{\sqrt{2}}(-\underline{j} + \underline{k}) = -\dfrac{1}{\sqrt{2}}\underline{j} + \dfrac{1}{\sqrt{2}}\underline{k}$.

The vector $\underline{dS}$ is therefore $(-\dfrac{1}{\sqrt{2}}\underline{j} + \dfrac{1}{\sqrt{2}}\underline{k})\, du\, dv$ where du and dv are increments in the plane of the rectangle $OABC$. Now, one increment, say du, may point in the x-direction while dv will point in a direction up the plane, parallel to OC. Thus $du = dx$ and (by Pythagoras) $dv = \sqrt{(dy)^2 + (dz)^2}$. However, as $z = y$, $dz = dy$ and hence $dv = \sqrt{2}dy$.
Thus, $\underline{dS} = (-\dfrac{1}{\sqrt{2}}\underline{j} + \dfrac{1}{\sqrt{2}}\underline{k})\, dx\, \sqrt{2}\, dy = (-\underline{j} + \underline{k})\, dx\, dy$.

Note :- the factor of $\sqrt{2}$ could also have been found by comparing the area of rectangle $OABC$, i.e. 1, with the area of its projection in the Oxy plane i.e. $OADE$ with area $\dfrac{1}{\sqrt{2}}$.

Integrating a scalar field

A function can be integrated over a surface by constructing a double integral and integrating in a manner similar to that shown in HELM 27.1 and HELM 27.2. Often, such integrals can be carried out with respect to an element containing the unit normal.

Example 20

Evaluate the integral

$$\int_A \frac{1}{1+x^2}\, \underline{dS}$$

over the area A where A is the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$.

Solution

In this integral, $\underline{dS}$ becomes $\underline{k}\, dx\, dy$ i.e. the unit normal times the surface element. Thus the integral is

$$\begin{aligned}
\int_{y=0}^{1}\int_{x=0}^{1} \frac{\underline{k}}{1+x^2}\, dx\, dy &= \underline{k}\int_{y=0}^{1} \Big[\tan^{-1} x\Big]_0^1 dy \\
&= \underline{k}\int_{y=0}^{1} \Big[(\frac{\pi}{4} - 0)\Big]_0^1 dy = \frac{\pi}{4}\underline{k}\int_{y=0}^{1} dy \\
&= \frac{\pi}{4}\underline{k}
\end{aligned}$$

Example 21

Find $\displaystyle\iint_S u\,\underline{dS}$ where $u = x^2 + y^2 + z^2$ and S is the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

The unit cube has six faces and the unit normal vector $\underline{\hat{n}}$ points in a different direction on each face; see Example 18. The surface integral must be evaluated for each face separately and the results summed.

On the face $x = 0$, the unit normal $\underline{\hat{n}} = -\underline{i}$ and the surface integral is

$$\begin{aligned}\int_{y=0}^{1}\int_{z=0}^{1}(0^2 + y^2 + z^2)(-\underline{i})dzdy &= -\underline{i}\int_{y=0}^{1}\left[y^2 z + \frac{1}{3}z^3\right]_{z=0}^{1} dy \\ &= -\underline{i}\int_{y=0}^{1}\left(y^2 + \frac{1}{3}\right)dy = -\underline{i}\left[\frac{1}{3}y^3 + \frac{1}{3}y\right]_0^1 = -\frac{2}{3}\underline{i}\end{aligned}$$

On the face $x = 1$, the unit normal $\underline{\hat{n}} = \underline{i}$ and the surface integral is

$$\begin{aligned}\int_{y=0}^{1}\int_{z=0}^{1}(1^2 + y^2 + z^2)(\underline{i})dzdy &= \underline{i}\int_{y=0}^{1}\left[z + y^2 z + \frac{1}{3}z^3\right]_{z=0}^{1} dy \\ &= \underline{i}\int_{y=0}^{1}\left(y^2 + \frac{4}{3}\right)dy = \underline{i}\left[\frac{1}{3}y^3 + \frac{4}{3}y\right]_0^1 = \frac{5}{3}\underline{i}\end{aligned}$$

The net contribution from the faces $x = 0$ and $x = 1$ is $-\frac{2}{3}\underline{i} + \frac{5}{3}\underline{i} = \underline{i}$.

Due to the symmetry of the scalar field u and the unit cube, the net contribution from the faces $y = 0$ and $y = 1$ is $\underline{j}$ while the net contribution from the faces $z = 0$ and $z = 1$ is $\underline{k}$.

Adding, we obtain $\displaystyle\iint_S u\underline{dS} = \underline{i} + \underline{j} + \underline{k}$

Key Point 4

A scalar function integrated with respect to a normal vector $\underline{dS}$ gives a vector quantity.

When the surface does not lie in one of the planes Oxy, Oxz, Oyz, extra care must be taken when finding $\underline{dS}$.

Example 22

Find $\displaystyle\iint_S (\underline{\nabla} \cdot \underline{F})\underline{dS}$ where $\underline{F} = 2x\underline{i} + yz\underline{j} + xy\underline{k}$ and S is the surface of the triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$.

Solution

Note that $\underline{\nabla} \cdot \underline{F} = 2 + z = 2$ as $z = 0$ everywhere along S. As the triangle lies in the Oxy plane, the normal vector $\underline{n} = \underline{k}$ and $\underline{dS} = \underline{k}dydx$.
Thus,

$$\iint_S (\underline{\nabla} \cdot \underline{F})\underline{dS} = \int_{x=0}^{1}\int_{y=0}^{x} 2dydx\underline{k} = \int_0^1 \Big[2y\Big]_0^x dx\underline{k} = \int_0^1 2xdx\underline{k} = \Big[x^2\Big]_0^1 \underline{k} = \underline{k}$$

Here the scalar function being integrated was the divergence of a vector function.

Example 23

Find $\displaystyle\iint_S f\, \underline{dS}$ where f is the function $2x$ and S is the surface of the triangle bounded by $(0,0,0)$, $(0,1,1)$ and $(1,0,1)$. (See Figure 9.)

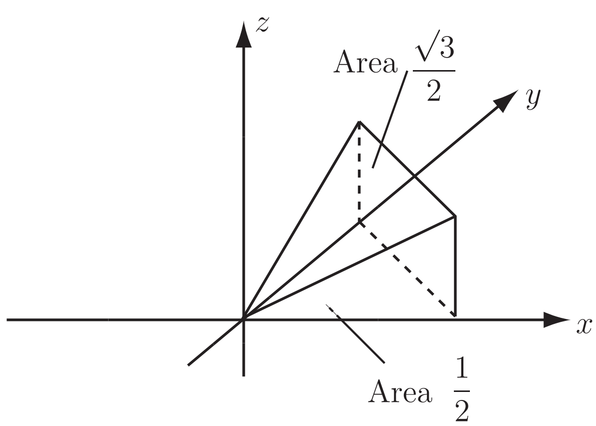

Figure 9: The triangle defining the area S

Solution

The unit vector $\underline{n}$ is perpendicular to two vectors in the plane e.g. $(\underline{j}+\underline{k})$ and $(\underline{i}+\underline{k})$. The vector $(\underline{j}+\underline{k})\times(\underline{i}+\underline{k}) = \underline{i}+\underline{j}-\underline{k}$ which has magnitude $\sqrt{3}$. Hence the unit normal vector $\hat{\underline{n}} = \frac{1}{\sqrt{3}}\underline{i}+\frac{1}{\sqrt{3}}\underline{j}-\frac{1}{\sqrt{3}}\underline{k}$.

As the area of the triangle S is $\frac{\sqrt{3}}{2}$ and the area of its projection in the Oxy plane is $\frac{1}{2}$, the vector $\underline{dS} = \dfrac{\sqrt{3}/2}{1/2}\hat{\underline{n}}\; dydx = (\underline{i}+\underline{j}+\underline{k})dydx.$

Thus

$$\begin{aligned}
\iint_S f\underline{dS} &= (\underline{i}+\underline{j}+\underline{k})\int_{x=0}^{1}\int_{y=0}^{1-x} 2x\; dydx \\
&= (\underline{i}+\underline{j}+\underline{k})\int_{x=0}^{1}\Big[2xy\Big]_{y=0}^{1-x} dx \\
&= (\underline{i}+\underline{j}+\underline{k})\int_{x=0}^{1}(2x-2x^2)dx \\
&= (\underline{i}+\underline{j}+\underline{k})\left[x^2-\frac{2}{3}x^3\right]_0^1 = \frac{1}{3}(\underline{i}+\underline{j}+\underline{k})
\end{aligned}$$

Task

Evaluate the integral $\displaystyle\iint_S 4x\;\underline{dS}$ where S represents the trapezium with vertices at $(0,0)$, $(3,0)$, $(2,1)$ and $(0,1)$.

(a) Find the vector $\underline{dS}$:

Your solution

Answer

$\underline{k}\; dx\; dy$

(b) Write the surface integral as a double integral:

Your solution

Answer

It is easier to integrate first with respect to x. This gives $\displaystyle\int_{y=0}^{1}\int_{x=0}^{3-y} 4x \; dx\, dy\, \underline{k}$.

The range of values of y is $y = 0$ to $y = 1$.

For each value of y, x varies from $x = 0$ to $x = 3 - y$

(c) Evaluate this double integral:

Your solution

Answer

$\dfrac{38}{3}\underline{k}$

Exercises

1. Evaluate the integral $\displaystyle\int\!\!\int_S xy\underline{dS}$ where S is the triangle with vertices at $(0,0,4)$, $(0,2,0)$ and $(1,0,0)$.

2. Find the integral $\displaystyle\int\!\!\int_S xyz\underline{dS}$ where S is the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

3. Evaluate the integral $\displaystyle\int\!\!\int_S \left[\underline{\nabla} \cdot (x^2\underline{i} + yz\underline{j} + x^2y\underline{k})\right] \underline{dS}$ where S is the rectangle with vertices at $(1,0,0)$, $(1,1,0)$, $(1,1,1)$ and $(1,0,1)$.

Answers 1. $\dfrac{2}{3}\underline{i} + \dfrac{1}{3}\underline{j} + \dfrac{1}{6}\underline{k}$ 2. $\dfrac{1}{4}(\underline{x} + \underline{y} + \underline{z})$, 3. $\dfrac{5}{2}\underline{i}$

Integrating a vector field

In a similar manner to the case of a scalar field, a vector field may be integrated over a surface. Two common types of integral are $\displaystyle\int_S \underline{F}(\underline{r}) \cdot \underline{dS}$ and $\displaystyle\int_S \underline{F}(\underline{r}) \times \underline{dS}$ which integrate to a scalar and a vector respectively. Again, when $\underline{dS}$ is expressed appropriately, the expression will reduce to a double integral. The form $\displaystyle\int_S \underline{F}(r) \cdot \underline{dS}$ has many important applications, e.g. the flux of a vector field such as an electric or magnetic field.

Example 24

Evaluate the integral

$$\int_A (x^2y\underline{i} + z\underline{j} + (2x+y)\underline{k}) \cdot \underline{dS}$$

over the area A where A is the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$.

Solution

On A, the unit normal is $dx\,dy\,\underline{k}$

$$\therefore \quad \int_A (x^2y\underline{i} + z\underline{j} + (2x+y)\underline{k}) \cdot (\underline{k}\ dx\,dy)$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} (2x+y)\ dx\,dy = \int_{y=0}^{1} \Big[x^2 + xy\Big]_{x=0}^{1} dy$$

$$= \int_{y=0}^{1} (1+y)dy = \left[y + \frac{1}{2}y^2\right]_0^1 = \frac{3}{2}$$

Example 25

Evaluate $\displaystyle\int_A \underline{r}\cdot \underline{dS}$ where A represents the surface of the unit cube

$0 \leq x \leq 1,\quad 0 \leq y \leq 1,\quad 0 \leq z \leq 1$ and $\underline{r}$ represents the vector $x\underline{i} + y\underline{j} + z\underline{k}$.

Solution

The vector $\underline{dS}$ (in the direction of the normal vector) will be a constant vector on each face, but will be different for each face.

On the face $x = 0$, $\underline{dS} = -dy\,dz\,\underline{i}$ and the integral on this face is

$$\int_{z=0}^{1}\int_{y=0}^{1} (0\underline{i} + y\underline{j} + z\underline{k})\cdot(-dy\,dz\,\underline{i}) = \int_{z=0}^{1}\int_{y=0}^{1} 0\;dy\,dz = 0$$

Similarly on the face $y = 0$, $\underline{dS} = -dx\,dz\,\underline{j}$ and the integral on this face is

$$\int_{z=0}^{1}\int_{x=0}^{1} (x\underline{i} + 0\underline{j} + z\underline{k})\cdot(-dx\,dz\,\underline{j}) = \int_{z=0}^{1}\int_{x=0}^{1} 0\;dx\,dz = 0$$

Furthermore on the face $z = 0$, $\underline{dS} = -dx\,dy\,\underline{k}$ and the integral on this face is

$$\int_{x=0}^{1}\int_{y=0}^{1} (x\underline{i} + y\underline{j} + 0\underline{k})\cdot(-dx\,dy\,\underline{k}) = \int_{x=0}^{1}\int_{y=0}^{1} 0\;dx\,dy = 0$$

On these three faces, the contribution to the integral is thus zero.

However, on the face $x = 1$, $\underline{dS} = +dy\,dz\,\underline{i}$ and the integral on this face is

$$\int_{z=0}^{1}\int_{y=0}^{1} (1\underline{i} + y\underline{j} + z\underline{k})\cdot(+dy\,dz\,\underline{i}) = \int_{z=0}^{1}\int_{y=0}^{1} 1\;dy\,dz = 1$$

Similarly, on the face $y = 1$, $\underline{dS} = +dx\,dz\,\underline{j}$ and the integral on this face is

$$\int_{z=0}^{1}\int_{x=0}^{1} (x\underline{i} + 1\underline{j} + z\underline{k})\cdot(+dx\,dz\,\underline{j}) = \int_{z=0}^{1}\int_{x=0}^{1} 1\;dx\,dz = 1$$

Finally, on the face $z = 1$, $\underline{dS} = +dx\,dy\,\underline{k}$ and the integral on this face is

$$\int_{y=0}^{1}\int_{x=0}^{1} (x\underline{i} + y\underline{j} + 1\underline{k})\cdot(+dx\,dy\,\underline{k}) = \int_{y=0}^{1}\int_{x=0}^{1} 1\;dx\,dy = 1$$

Adding together the contributions gives $\displaystyle\int_A \underline{r}\cdot\underline{dS} = 0+0+0+1+1+1 = 3$

Engineering Example 3

Magnetic flux

Introduction

The magnetic flux through a surface is given by $\displaystyle\iint_S \underline{B}\cdot\underline{dS}$ where S is the surface under consideration, $\underline{B}$ is the magnetic field and $\underline{dS}$ is the vector normal to the surface.

Problem in words

The magnetic field generated by an infinitely long vertical wire on the z-axis, carrying a current I, is given by:

$$\underline{B} = \frac{\mu_0 I}{2\pi}\left(\frac{-y\underline{i} + x\underline{j}}{x^2+y^2}\right)$$

Find the flux through a rectangular region (with sides parallel to the axes) on the plane $y = 0$.

Mathematical statement of problem

Find the integral $\displaystyle\iint_S \underline{B}\cdot\underline{dS}$ over the surface, $x_1 \leq x \leq x_2, \quad z_1 \leq z \leq z_2$. (see Figure 10 which shows part of the plane $y = 0$ for which the flux is to be found and a single magnetic field line. The strength of the field is inversely proportional to the distance from the axis.)

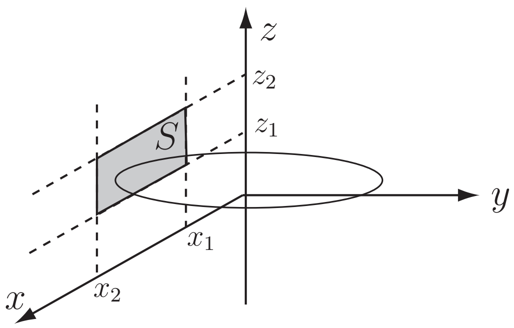

Figure 10: The surface S defined by $x_1 \leq x \leq x_2,\ z_1 \leq z \leq z_2$

Mathematical analysis

On $y = 0$, $\underline{B} = \dfrac{\mu_0 I}{2\pi x}\ \underline{j}$ and $\underline{dS} = dx\ dz\ \underline{j}$ so $\underline{B}\cdot\underline{dS} = \dfrac{\mu_0 I}{2\pi x}\ dx\ dz$

The flux is given by the double integral:

$$\begin{aligned}
\int_{z=z_1}^{z_2}\int_{x=x_1}^{x_2} \frac{\mu_0 I}{2\pi x}\ dx\ dz &= \frac{\mu_0 I}{2\pi}\int_{z=z_1}^{z_2}\Big[\ln x\Big]_{x_1}^{x_2} dz \\
&= \frac{\mu_0 I}{2\pi}\int_{z=z_1}^{z_2}\Big(\ln x_2 - \ln x_1\Big)\ dz \\
&= \frac{\mu_0 I}{2\pi}\Big[z\big(\ln x_2 - \ln x_1\big)\Big]_{z=z_1}^{z_2} = \frac{\mu_0 I}{2\pi}\ (z_2 - z_1)\ln\left(\frac{x_2}{x_1}\right)
\end{aligned}$$

Interpretation

The magnetic flux increases in direct proportion to the extent of the side parallel to the axis (i.e. along the z-direction) but logarithmically with respect to the extent of the side perpendicular to the axis (i.e. along the x-axis).

Example 26

If $\underline{F} = x^2\underline{i} + y^2\underline{j} + z^2\underline{k}$, evaluate $\displaystyle\iint_S \underline{F} \times \underline{dS}$ where S is the part of the plane $z = 0$ bounded by $x = \pm 1$, $y = \pm 1$.

Solution

Here $\underline{dS} = dx\,dy\,\underline{k}$ and hence $\quad \underline{F} \times \underline{dS} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x^2 & y^2 & z^2 \\ 0 & 0 & dx\,dy \end{vmatrix} = y^2\,dx\,dy\,\underline{i} - x^2\,dx\,dy\,\underline{j}$

$$\iint_S \underline{F} \times \underline{dS} = \int_{y=-1}^{1}\int_{x=-1}^{1} y^2\,dx\,dy\,\underline{i} - \int_{y=-1}^{1}\int_{x=-1}^{1} x^2\,dx\,dy\,\underline{j}$$

The first integral is

$$\int_{y=-1}^{1}\int_{x=-1}^{1} y^2\,dx\,dy = \int_{y=-1}^{1}\left[y^2x\right]_{x=-1}^{1}dy = \int_{y=-1}^{1} 2y^2 dy = \left[\frac{2}{3}y^3\right]_{-1}^{1} = \frac{4}{3}$$

Similarly $\displaystyle\int_{y=-1}^{1}\int_{x=-1}^{1} x^2\,dx\,dy = \frac{4}{3}$.

Thus $\displaystyle\iint_S \underline{F} \times \underline{dS} = \frac{4}{3}\underline{i} - \frac{4}{3}\underline{j}$

Key Point 5

(a) An integral of the form $\displaystyle\int_S \underline{F}(\underline{r}) \cdot \underline{dS}$ evaluates to a scalar.

(b) An integral of the form $\displaystyle\int_S \underline{F}(\underline{r}) \times \underline{dS}$ evaluates to a vector.

The vector function involved may be the gradient of a scalar or the curl of a vector.

Example 27

Integrate $\displaystyle\iint_S (\underline{\nabla}\phi).\underline{dS}$ where $\phi = x^2 + 2yz$ and S is the area between $y = 0$ and $y = x^2$ for $0 \leq x \leq 1$ and $z = 0$. (See Figure 11.)

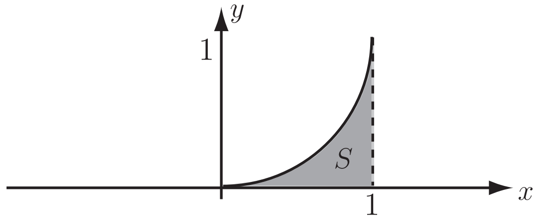

Figure 11: The area S between $y = 0$ and $y = x^2$, for $0 \leq x \leq 1$ and $z = 0$

Solution

Here $\underline{\nabla}\phi = 2x\underline{i} + 2z\underline{j} + 2y\underline{k}$ and $\underline{dS} = \underline{k}\ dydx$. Thus $(\underline{\nabla}\phi).\underline{dS} = 2ydydx$ and

$$\begin{aligned}
\iint_S (\underline{\nabla}\phi).\underline{dS} &= \int_{x=0}^{1}\int_{y=0}^{x^2} 2y\ dydx \\
&= \int_{x=0}^{1}\Big[y^2\Big]_{y=0}^{x^2} dx = \int_{x=0}^{1} x^4 dx \\
&= \left[\frac{1}{5}x^5\right]_0^1 = \frac{1}{5}
\end{aligned}$$

For integrals of the form $\displaystyle\iint_S \underline{F}\cdot\underline{dS}$, non-Cartesian coordinates e.g. cylindrical polar or spherical polar coordinates may be used. Once again, it is necessary to include any scale factors along with the unit normal.

Example 28

Using cylindrical polar coordinates, (see HELM 28.3), find the integral $\displaystyle\int_S \underline{F}(\underline{r})\cdot\underline{dS}$ for $\underline{F} = \rho z\underline{\hat{\rho}} + z\sin^2\phi\underline{\hat{z}}$ and S being the complete surface (including ends) of the cylinder $\rho \leq a$, $0 \leq z \leq 1$. (See Figure 12.)

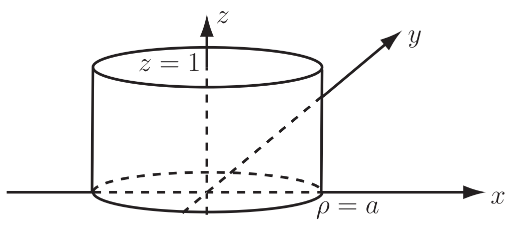

Figure 12: The cylinder $\rho \leq a,\ 0 \leq z \leq 1$

Solution

The integral $\int_S \underline{F}(\underline{r}) \cdot \underline{dS}$ must be evaluated separately for the curved surface and the ends.
For the curved surface, $\underline{dS} = \underline{\hat{\rho}} a d\phi dz$ (with the a coming from ρ the scale factor for ϕ and the fact that $\rho = a$ on the curved surface.) Thus, $\underline{F} \cdot \underline{dS} = a^2 z \, d\phi dz$ and

$$\begin{aligned}\iint_S \underline{F}(\underline{r}) \cdot \underline{dS} &= \int_{z=0}^{1} \int_{\phi=0}^{2\pi} a^2 z \, d\phi dz \\ &= 2\pi a^2 \int_{z=0}^{1} z \, dz = 2\pi a^2 \left[\frac{1}{2} z^2\right]_0^1 = \pi a^2\end{aligned}$$

On the bottom surface, $z = 0$ so $\underline{F} = \underline{0}$ and the contribution to the integral is zero.
On the top surface, $z = 1$ and $\underline{dS} = \underline{\hat{z}} \rho \, d\rho d\phi$ and $\underline{F} \cdot \underline{dS} = \rho z \sin^2 \phi \, d\phi d\rho = \rho \sin^2 \phi \, d\phi d\rho$ and

$$\begin{aligned}\iint_S \underline{F}(\underline{r}) \cdot \underline{dS} &= \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \rho \sin^2 \phi \, d\phi d\rho \\ &= \pi \int_{\rho=0}^{a} \rho \, d\rho = \frac{1}{2} \pi a^2\end{aligned}$$

So $$\iint_S \underline{F}(\underline{r}) \cdot \underline{dS} = \pi a^2 + \frac{1}{2} \pi a^2 = \frac{3}{2} \pi a^2$$

Engineering Example 4

The current continuity equation

Introduction

When an electric current flows at a constant rate through a conductor, then the **current continuity equation** states that

$$\oint_S \underline{J} \cdot \underline{dS} = 0$$

where $\underline{J}$ is the current density (or current flow per unit area) and S is a closed surface. The equation is an expression of the fact that, under these conditions, the current flow into a closed volume equals the flow out.

Problem in words

A person is standing nearby when lightning strikes the ground. Find the potential difference between the feet of that person.

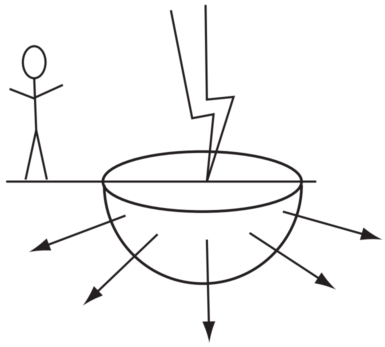

Figure 13: Lightning: a current dissipating into the ground

Mathematical statement of problem

The current from the lightning dissipates radially (see Fig 13).

(a) Find a relationship between the current I and current density J at a distance r from the strike by integrating the current density over the hemisphere $I = \displaystyle\int_S \underline{J} \cdot \underline{dS}$

(b) Find the field $\underline{E}$ from the equation $\quad E = \dfrac{\rho I}{2\pi^2 r}$ where $E = |\underline{E}|$ and I is the current.

(c) Find V from the integral $\displaystyle\int_{R_1}^{R_2} \underline{E} \cdot \underline{dr}$

Mathematical analysis

Imagine a hemisphere of radius r level with the surface of the ground so that the point of lightning strike is at its centre. By symmetry, the pattern of current flow from the point of strike will be uniform radial lines, and the magnitude of $\underline{J}$ will be a constant, i.e. over the curved surface of the hemisphere $\underline{J} = J\underline{\hat{r}}$.

Since the amount of current entering the hemisphere is I, then it follows that the current leaving must be the same i.e.

$$\begin{aligned} I &= \int_{S_c} \underline{J} \cdot \underline{dS} \qquad \text{(where } S_c \text{ is the curved surface of the hemisphere)} \\ &= \int_{S_c} (J\underline{\hat{r}}) \cdot (dS\ \underline{\hat{r}}) \\ &= J \int_{S_c} dS \\ &= 2\pi r^2 J \qquad [= \text{surface area } (2\pi r^2) \times \text{flux } (J)] \end{aligned}$$

since the surface area of a sphere is $4\pi r^2$. Therefore

$$J = \frac{I}{2\pi r^2}$$

Note that if the current density $\underline{J}$ is uniformly radial over the curved surface, then so must be the electric field $\underline{E}$, i.e. $\underline{E} = E\underline{\hat{r}}$. Using Ohm's law

$$\underline{J} = \sigma \underline{E} \quad \text{or} \quad E = \rho J$$

where σ = conductivity $= 1/\rho$.

Hence $\quad E = \dfrac{\rho I}{2\pi r^2}$

The potential difference between two points at radii R_1 and R_2 from the lightning strike is found by integrating $\underline{E}$ between them, so that

$$\begin{aligned}
V &= \int_{R_1}^{R_2} \underline{E}\cdot\underline{dr} \\
&= \int_{R_1}^{R_2} E\ dr \\
&= \frac{\rho I}{2\pi}\int_{R_1}^{R_2} \frac{dr}{r^2} \\
&= \frac{\rho I}{2\pi}\left[\frac{-1}{r}\right]_{R_1}^{R_2} \\
&= \frac{\rho I}{2\pi}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad \left(= \frac{\rho I}{2\pi}\left(\frac{R_2-R_1}{R_1 R_2}\right)\right)
\end{aligned}$$

Interpretation

Suppose the lightning strength is a current $I = 10,000$ A, that the person is 12 m away with feet 0.35 m apart, and that the resistivity of the ground is 80 Ω m. Clearly, the worst case (i.e. maximum voltage) would occur when the difference between R_1 and R_2 is greatest, i.e. R_1=12 m and R_2=12.35 m which would be the case if both feet were on the same radial line. The voltage produced between the person's feet under these circumstances is

$$\begin{aligned}
V &= \frac{\rho I}{2\pi}\left[\frac{1}{R_1}-\frac{1}{R_2}\right] \\
&= \frac{80\times 10000}{2\pi}\left[\frac{1}{12}-\frac{1}{12.35}\right] \\
&\approx 300 \text{ V}
\end{aligned}$$

Task

For $\underline{F} = (x^2+y^2)\underline{i} + (x^2+z^2)\underline{j} + 2xz\underline{k}$ and S the square bounded by $(1,0,1)$, $(1,0,-1)$, $(-1,0,-1)$ and $(-1,0,1)$ find the integral $\displaystyle\int_S \underline{F}\cdot\underline{dS}$

Your solution

Answer

$\underline{dS} = dxdz\underline{j} \qquad \displaystyle\int_{-1}^{1}\int_{-1}^{1}(x^2+z^2)\ dxdz = \frac{8}{3}$

Task

For $\underline{F} = (x^2 + y^2)\underline{i} + (x^2 + z^2)\underline{j} + 2xz\underline{k}$ and S being the rectangle bounded by $(1, 0, 1)$, $(1, 0, -1)$, $(-1, 0, -1)$ and $(-1, 0, 1)$ (i.e. the same $\underline{F}$ and S as in the previous Task), find the integral $\displaystyle\int_S \underline{F} \times \underline{dS}$

Your solution

Answer

$$\left\{\int_{-1}^{1}\int_{-1}^{1}(-2xz)\underline{i} + \int_{-1}^{1}\int_{-1}^{1}(x^2 + 0)\underline{k}\right\} dxdz = \frac{4}{3}\underline{k}$$

Exercises

1. Evaluate the integral $\displaystyle\iint_S \underline{\nabla}\phi \cdot \underline{dS}$ for $\phi = x^2 z \sin y$ and S being the rectangle bounded by $(0, 0, 0)$, $(1, 0, 1)$, $(1, \pi, 1)$ and $(0, \pi, 0)$.

2. Evaluate the integral $\displaystyle\iint_S (\underline{\nabla} \times \underline{F}) \times \underline{dS}$ where $\underline{F} = xe^y\underline{i} + ze^y\underline{j}$ and S represents the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

3. Using spherical polar coordinates (r, θ, ϕ), evaluate the integral $\displaystyle\iint_S \underline{F} \cdot \underline{dS}$ where $\underline{F} = r\cos\theta\hat{\underline{r}}$ and S is the curved surface of the top half of the sphere $r = a$.

Answers 1. $-\dfrac{2}{3}$, 2. $(e - 1)\underline{j}$, 3. πa^3

2. Volume integrals involving vectors

Integrating a scalar function of a vector over a volume involves essentially the same procedure as in HELM 27.3. In 3D cartesian coordinates the volume element dV is $dxdydz$. The scalar function may be the divergence of a vector function.

Example 29

Integrate $\underline{\nabla} \cdot \underline{F}$ over the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$ where $\underline{F}$ is the vector function $x^2 y \underline{i} + (x - z)\underline{j} + 2xz^2 \underline{k}$.

Solution

$\underline{\nabla} \cdot \underline{F} = \dfrac{\partial}{\partial x}(x^2 y) + \dfrac{\partial}{\partial y}(x - z) + \dfrac{\partial}{\partial z}(2xz^2) = 2xy + 4xz$

The integral is

$$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(2xy + 4xz)dzdydx = \int_{x=0}^{1}\int_{y=0}^{1}\left[2xyz + 2xz^2\right]_0^1 dydx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}(2xy + 2x)\,dydx = \int_{x=0}^{1}\left[xy^2 + 2xy\right]_0^1 dx$$

$$= \int_{x=0}^{1} 3xdx = \left[\frac{3}{2}x^2\right]_0^1 = \frac{3}{2}$$

Key Point 6

The volume integral of a scalar function (including the divergence of a vector) is a scalar.

Task

Using spherical polar coordinates (r, θ, ϕ) and the vector field $\underline{F} = r^2\,\hat{\underline{r}} + r^2 \sin\theta\,\hat{\underline{\theta}}$, evaluate the integral $\displaystyle\iiint_V \underline{\nabla} \cdot \underline{F}\,dV$ over the sphere given by $0 \leq r \leq a$.

Your solution

Answer

$\underline{\nabla} \cdot \underline{F} = 4r + 2r\cos\theta,\qquad \displaystyle\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\{(4r + 2r\cos\theta)r^2\sin\theta\}\,d\phi d\theta dr = 4\pi a^4$

The $r^2 \sin\theta$ term comes from the Jacobian for the transformation from spherical to cartesian coordinates (see HELM 27.4 and HELM 28.3).

Exercises

1. Evaluate $\iiint_V \underline{\nabla} \cdot \underline{F} dV$ when $\underline{F}$ is the vector field $yz\underline{i} + xy\underline{j}$ and V is the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$

2. For the vector field $\underline{F} = (x^2y + \sin z)\underline{i} + (xy^2 + e^z)\underline{j} + (z^2 + x^y)\underline{k}$, find the integral $\iiint_V \underline{\nabla} \cdot \underline{F} dV$ where V is the volume inside the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Answers 1. $\underline{\nabla} \cdot \underline{F} = x$, $\frac{1}{2}$ 2. $\frac{7}{60}$

Integrating a vector function over a volume integral is similar, but less common. Care should be taken with the various components. It may help to think in terms of a separate volume integral for each component. The vector function may be of the form $\underline{\nabla} f$ or $\underline{\nabla} \times \underline{F}$.

Example 30

Integrate the function $\underline{F} = x^2\underline{i} + 2\underline{j}$ over the prism given by $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq (1 - x)$. (See Figure 14.)

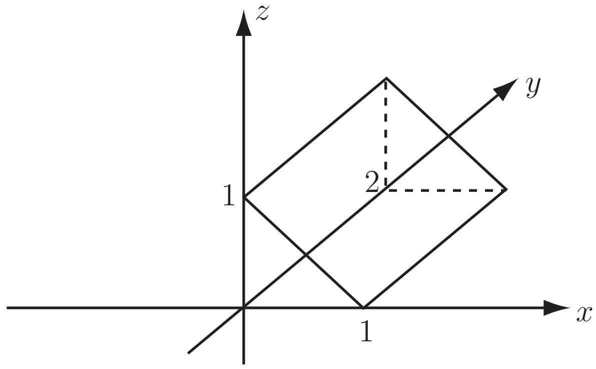

Figure 14: The prism bounded by $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq (1 - x)$

Solution

The integral is

$$\int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{1-x} (x^2\underline{i} + 2\underline{j}) dz dy dx = \int_{x=0}^{1} \int_{y=0}^{2} \left[x^2 z\underline{i} + 2z\underline{j} \right]_{z=0}^{1-x} dy dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{2} \left\{ x^2(1-x)\underline{i} + 2(1-x)\underline{j} \right\} dy dx = \int_{x=0}^{1} \int_{y=0}^{2} \left\{ (x^2 - x^3)\underline{i} + (2 - 2x)\underline{j} \right\} dy dx$$

$$= \int_{x=0}^{1} \left\{ (2x^2 - 2x^3)\underline{i} + (4 - 4x)\underline{j} \right\} dx = \left[(\frac{2}{3}x^3 - \frac{1}{2}x^4)\underline{i} + (4x - 2x^2)\underline{j} \right]_0^1$$

$$= \frac{1}{6}\underline{i} + 2\underline{j}$$

Example 31

For $\underline{F} = x^2y\underline{i} + y^2\underline{j}$ evaluate $\iiint_V (\underline{\nabla} \times \underline{F})dV$ where V is the volume under the plane $z = x + y + 2$ (and above $z = 0$) for $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

Solution

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & y^2 & 0 \end{vmatrix} = -x^2\underline{k}$$

so

$$\begin{aligned} \iiint_V (\underline{\nabla} \times \underline{F})dV &= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=0}^{x+y+2} (-x^2)\underline{k}\ dzdydx \\ &= \int_{x=-1}^{1}\int_{y=-1}^{1} \Big[(-x^2)z\underline{k}\Big]_{z=0}^{x+y+2} dydx \\ &= \int_{x=-1}^{1}\int_{y=-1}^{1} \left[-x^3 - x^2y - 2x^2\right] dydx\ \underline{k} \\ &= \int_{x=-1}^{1} \left[-x^3y - \frac{1}{2}x^2y^2 - 2x^2y\right]_{y=-1}^{1} dx\ \underline{k} \\ &= \int_{x=-1}^{1} \left(-2x^3 - 0 - 4x^2\right) dx\ \underline{k} = \left[-\frac{1}{2}x^4 - \frac{4}{3}x^3\right]_{-1}^{1} \underline{k} = -\frac{8}{3}\underline{k} \end{aligned}$$

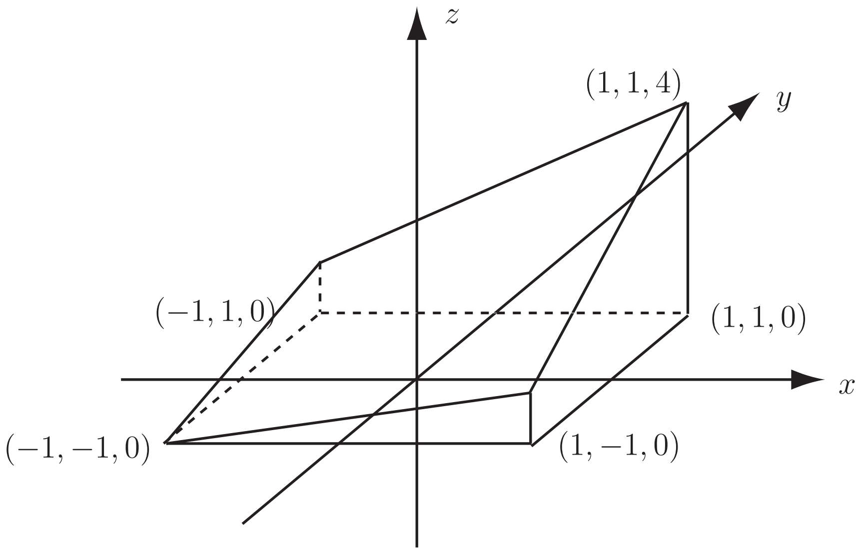

Figure 15: The plane defined by $z = x + y + z$, for $z > 0$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$

Key Point 7

The volume integral of a vector function (including the gradient of a scalar or the curl of a vector) is a vector.

Task

Evaluate the integral $\int_V \underline{F} dV$ for the case where $\underline{F} = x\underline{i} + y^2\underline{j} + z\underline{k}$ and V is the cube $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

Your solution

Answer

$$\int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=-1}^{1} (x\underline{i} + y^2\underline{j} + z\underline{k}) dz dy dx = \frac{8}{3}\underline{j}$$

Exercises

1. For $f = x^2 + yz$, and V the volume bounded by $y = 0$, $x + y = 1$ and $-x + y = 1$ for $-1 \leq z \leq 1$, find the integral $\iiint_V (\underline{\nabla} f) dV$.

2. Evaluate the integral $\int_V (\underline{\nabla} \times \underline{F}) dV$ for the case where $\underline{F} = xz\underline{i} + (x^3 + y^3)\underline{j} - 4y\underline{k}$ and V is the cube $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

Answers

1. $\iiint_V (2x\underline{i} + z\underline{j} + y\underline{k}) dV = \frac{2}{3}\underline{k}$,

2. $\iiint_V (-4\underline{i} + x\underline{j} + 3x^2\underline{k}) dV = -32\underline{i} + 8\underline{k}$