LU Decomposition

30.3

Introduction

In this Section we consider another direct method for obtaining the solution of systems of equations in the form $AX = B$.

Prerequisites

Before starting this Section you should ...

- revise matrices and their use in systems of equations
- revise determinants

Learning Outcomes

On completion you should be able to ...

- find an LU decomposition of simple matrices and apply it to solve systems of equations
- determine when an LU decomposition is unavailable and when it is possible to circumvent the problem

1. LU decomposition

Suppose we have the system of equations

$$AX = B.$$

The motivation for an LU decomposition is based on the observation that systems of equations involving triangular coefficient matrices are easier to deal with. Indeed, the whole point of Gaussian elimination is to replace the coefficient matrix with one that is triangular. The LU decomposition is another approach designed to exploit triangular systems.
We suppose that we can write

$$A = LU$$

where L is a lower triangular matrix and U is an upper triangular matrix. Our aim is to find L and U and once we have done so we have found an LU decomposition of A.

Key Point 5

An LU decomposition of a matrix A is the product of a lower triangular matrix and an upper triangular matrix that is equal to A.

It turns out that we need only consider lower triangular matrices L that have 1s down the diagonal. Here is an example. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU \text{ where } L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}.$$

Multiplying out LU and setting the answer equal to A gives

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}.$$

Now we use this to find the entries in L and U. Fortunately this is not nearly as hard as it might at first seem. We begin by running along the top row to see that

$$\boxed{U_{11} = 1}, \quad \boxed{U_{12} = 2}, \quad \boxed{U_{13} = 4}.$$

Now consider the second row

$$L_{21}U_{11} = 3 \quad \therefore L_{21} \times 1 = 3 \quad \therefore \boxed{L_{21} = 3},$$

$$L_{21}U_{12} + U_{22} = 8 \quad \therefore 3 \times 2 + U_{22} = 8 \quad \therefore \boxed{U_{22} = 2},$$

$$L_{21}U_{13} + U_{23} = 14 \quad \therefore 3 \times 4 + U_{23} = 14 \quad \therefore \boxed{U_{23} = 2}.$$

Notice how, at each step, the equation being considered has only one unknown in it, and other quantities that we have already found. This pattern continues on the last row

$$L_{31}U_{11} = 2 \quad \therefore L_{31} \times 1 = 2 \quad \therefore \boxed{L_{31} = 2},$$

$$L_{31}U_{12} + L_{32}U_{22} = 6 \quad \therefore 2 \times 2 + L_{32} \times 2 = 6 \quad \therefore \boxed{L_{32} = 1},$$

$$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 13 \quad \therefore (2 \times 4) + (1 \times 2) + U_{33} = 13 \quad \therefore \boxed{U_{33} = 3}.$$

We have shown that

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

and this is an LU decomposition of A.

Task

Find an LU decomposition of $\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix}$.

Your solution

Answer

Let

$$\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} \end{bmatrix}$$

then, comparing the left and right hand sides row by row implies that $U_{11} = 3$, $U_{12} = 1$, $L_{21}U_{11} = -6$ which implies $L_{21} = -2$ and $L_{21}U_{12} + U_{22} = -4$ which implies that $U_{22} = -2$. Hence

$$\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$$

is an LU decomposition of $\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix}$.

Find an LU decomposition of $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$.

Your solution

Answer

Using material from the worked example in the notes we set

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix}$$

and comparing elements row by row we see that

$$\begin{array}{lll} U_{11}=3, & U_{12}=1, & U_{13}=6, \\ L_{21}=-2, & U_{22}=2, & U_{23}=-4 \\ L_{31}=0 & L_{32}=4 & U_{33}=-1 \end{array}$$

and it follows that

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

is an LU decomposition of the given matrix.

2. Using LU decomposition to solve systems of equations

Once a matrix A has been decomposed into lower and upper triangular parts it is possible to obtain the solution to $AX = B$ in a direct way. The procedure can be summarised as follows

- Given A, find L and U so that $A = LU$. Hence $LUX = B$.
- Let $Y = UX$ so that $LY = B$. Solve this triangular system for Y.
- Finally solve the triangular system $UX = Y$ for X.

The benefit of this approach is that we only ever need to solve triangular systems. The cost is that we have to solve two of them.

[Here we solve only small systems; a large system is presented in Engineering Example 1 on page 62.]

Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

Solution

- The first step is to calculate the LU decomposition of the coefficient matrix on the left-hand side. In this case that job has already been done since this is the matrix we considered earlier. We found that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

- The next step is to solve $LY = B$ for the vector $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. That is we consider

$$LY = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} = B$$

which can be solved by **forward substitution**. From the top equation we see that $y_1 = 3$. The middle equation states that $3y_1 + y_2 = 13$ and hence $y_2 = 4$. Finally the bottom line says that $2y_1 + y_2 + y_3 = 4$ from which we see that $y_3 = -6$.

Solution (contd.)

- Now that we have found Y we finish the procedure by solving $UX = Y$ for X. That is we solve

$$UX = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} = Y$$

by using **back substitution**. Starting with the bottom equation we see that $3x_3 = -6$ so clearly $x_3 = -2$. The middle equation implies that $2x_2 + 2x_3 = 4$ and it follows that $x_2 = 4$. The top equation states that $x_1 + 2x_2 + 4x_3 = 3$ and consequently $x_1 = 3$.

Therefore we have found that the solution to the system of simultaneous equations

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \qquad \text{is} \qquad X = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}.$$

Task

Use the LU decomposition you found earlier in the last Task (page 24) to solve

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix}.$$

Your solution

Answer

We found earlier that the coefficient matrix is equal to $LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$.

First we solve $LY = B$ for Y, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix}.$$

The top line implies that $y_1 = 0$. The middle line states that $-2y_1 + y_2 = 4$ and therefore $y_2 = 4$. The last line tells us that $4y_2 + y_3 = 17$ and therefore $y_3 = 1$.

Finally we solve $UX = Y$ for X, we have

$$\begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

The bottom line shows that $x_3 = -1$. The middle line then shows that $x_2 = 0$, and then the top line gives us that $x_1 = 2$. The required solution is $X = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

3. Do matrices always have an LU decomposition?

No. Sometimes it is impossible to write a matrix in the form "lower triangular" $\times$ "upper triangular".

Why not?

An invertible matrix A has an LU decomposition provided that all its **leading submatrices** have non-zero determinants. The k^{th} leading submatrix of A is denoted A_k and is the $k \times k$ matrix found by looking only at the top k rows and leftmost k columns. For example if

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

then the leading submatrices are

$$A_1 = 1, \qquad A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}.$$

The fact that this matrix A has an LU decomposition can be guaranteed in advance because none of these determinants is zero:

$$|A_1| = 1,$$
$$|A_2| = (1 \times 8) - (2 \times 3) = 2,$$
$$|A_3| = \begin{vmatrix} 8 & 14 \\ 6 & 13 \end{vmatrix} - 2\begin{vmatrix} 3 & 14 \\ 2 & 13 \end{vmatrix} + 4\begin{vmatrix} 3 & 8 \\ 2 & 6 \end{vmatrix} = 20 - (2 \times 11) + (4 \times 2) = 6$$

(where the 3×3 determinant was found by expanding along the top row).

Example 7

Show that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$ does not have an LU decomposition.

Solution

The second leading submatrix has determinant equal to

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (1 \times 4) - (2 \times 2) = 0$$

which means that an LU decomposition is not possible in this case.

Task

Which, if any, of these matrices have an LU decomposition?

(a) $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, (b) $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$, (c) $A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & 1 \\ 0 & 3 & -2 \end{bmatrix}$.

Your solution

(a)

Answer

$|A_1| = 3$ and $|A_2| = |A| = 3$. Neither of these is zero, so A **does** have an LU decomposition.

Your solution

(b)

Answer

$|A_1| = 0$ so A **does not** have an LU decomposition.

Your solution

(c)

Answer

$|A_1| = 1$, $|A_2| = 6 - 6 = 0$, so A **does not** have an LU decomposition.

Can we get around this problem?

Yes. It is always possible to re-order the rows of an **invertible** matrix so that all of the submatrices have non-zero determinants.

Example 8

Reorder the rows of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$ so that the reordered matrix has an LU decomposition.

Solution

Swapping the first and second rows does not help us since the second leading submatrix will still have a zero determinant. Let us swap the second and third rows and consider

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

the leading submatrices are

$$B_1 = 1, \quad B_2 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B_3 = B.$$

Now $|B_1| = 1$, $|B_2| = 3 \times 1 - 2 \times 1 = 1$ and (expanding along the first row)

$$|B_3| = 1(15-16) - 2(5-8) + 3(4-6) = -1 + 6 - 6 = -1.$$

All three of these determinants are non-zero and we conclude that B does have an LU decomposition.

Task

Reorder the rows of $A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & 1 \\ 0 & 3 & -2 \end{bmatrix}$ so that the reordered matrix has an LU decomposition.

Your solution

Answer

Let us swap the second and third rows and consider

$$B = \begin{bmatrix} 1 & -3 & 7 \\ 0 & 3 & -2 \\ -2 & 6 & 1 \end{bmatrix}$$

the leading submatrices are

$$B_1 = 1, \quad B_2 = \begin{bmatrix} 1 & -3 \\ 0 & 3 \end{bmatrix}, \quad B_3 = B$$

which have determinants 1, 3 and 45 respectively. All of these are non-zero and we conclude that B does indeed have an LU decomposition.

Exercises

1. Calculate LU decompositions for each of these matrices

(a) $A = \begin{bmatrix} 2 & 1 \\ -4 & -6 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 1 & -4 \\ 2 & 2 & -2 \\ 6 & 3 & -11 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 5 \\ 1 & 11 & 4 \end{bmatrix}$

2. Check each answer in Question 1, by multiplying out LU to show that the product equals A.

3. Using the answers obtained in Question 1, solve the following systems of equations.

(a) $\begin{bmatrix} 2 & 1 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 & -4 \\ 2 & 2 & -2 \\ 6 & 3 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 11 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 5 \\ 1 & 11 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

4. Consider $A = \begin{bmatrix} 1 & 6 & 2 \\ 2 & 12 & 5 \\ -1 & -3 & -1 \end{bmatrix}$

(a) Show that A does not have an LU decomposition.

(b) Re-order the rows of A and find an LU decomposition of the new matrix.

(c) Hence solve

$$\begin{aligned} x_1 + 6x_2 + 2x_3 &= 9 \\ 2x_1 + 12x_2 + 5x_3 &= -4 \\ -x_1 - 3x_2 - x_3 &= 17 \end{aligned}$$

Answers

1. (a) We let

$$\begin{bmatrix} 2 & 1 \\ -4 & -6 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} \end{bmatrix}.$$

Comparing the left-hand and right-hand sides row by row gives us that $U_{11} = 2$, $U_{12} = 1$, $L_{21}U_{11} = -4$ which implies that $L_{21} = -2$ and, finally, $L_{21}U_{12} + U_{22} = -6$ from which we see that $U_{22} = -4$. Hence

$$\begin{bmatrix} 2 & 1 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix}$$

is an LU decomposition of the given matrix.

(b) We let

$$\begin{bmatrix} 2 & 1 & -4 \\ 2 & 2 & -2 \\ 6 & 3 & -11 \end{bmatrix} = LU = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}.$$

Looking at the top row we see that $U_{11} = 2$, $U_{12} = 1$ and $U_{13} = -4$. Now, from the second row, $L_{21} = 1$, $U_{22} = 1$ and $U_{23} = 2$. The last three unknowns come from the bottom row: $L_{31} = 3$, $L_{32} = 0$ and $U_{33} = 1$. Hence

$$\begin{bmatrix} 2 & 1 & -4 \\ 2 & 2 & -2 \\ 6 & 3 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

is an LU decomposition of the given matrix.

(c) We let

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 5 \\ 1 & 11 & 4 \end{bmatrix} = LU = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}.$$

Looking at the top row we see that $U_{11} = 1$, $U_{12} = 3$ and $U_{13} = 2$. Now, from the second row, $L_{21} = 2$, $U_{22} = 2$ and $U_{23} = 1$. The last three unknowns come from the bottom row: $L_{31} = 1$, $L_{32} = 4$ and $U_{33} = -2$. Hence

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 5 \\ 1 & 11 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

is an LU decomposition of the given matrix.

2. Direct multiplication provides the necessary check.

Answers

3.

(a) We begin by solving

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Clearly $y_1 = 1$ and therefore $y_2 = 4$. The values y_1 and y_2 appear on the right-hand side of the second system we need to solve:

$$\begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

The second equation implies that $x_2 = -1$ and therefore, from the first equation, $x_1 = 1$.

(b) We begin by solving the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 11 \end{bmatrix}.$$

Starting with the top equation we see that $y_1 = 4$. The second equation then implies that $y_2 = -4$ and then, from the third equation, $y_3 = -1$. These values now appear on the right-hand side of the second system

$$\begin{bmatrix} 2 & 1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -1 \end{bmatrix}.$$

The bottom equation shows us that $x_3 = -1$. Moving up to the middle equation we obtain $x_2 = -2$. The top equation yields $x_1 = 1$.

(c) We begin by solving the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

Starting with the top equation we see that $y_1 = 2$. The second equation then implies that $y_2 = -1$ and then, from the third equation, $y_3 = 2$. These values now appear on the right-hand side of the second system

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$$

The bottom equation shows us that $x_3 = -1$. Moving up to the middle equation we obtain $x_2 = 0$. The top equation yields $x_1 = 4$.

Answers

4.

(a) The second leading submatrix has determinant $1 \times 12 - 6 \times 2 = 0$ and this implies that A has no LU decomposition.

(b) Swapping the second and third rows gives $\begin{bmatrix} 1 & 6 & 2 \\ -1 & -3 & -1 \\ 2 & 12 & 5 \end{bmatrix}$. We let

$$\begin{bmatrix} 1 & 6 & 2 \\ -1 & -3 & -1 \\ 2 & 12 & 5 \end{bmatrix} = LU = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}.$$

Looking at the top row we see that $U_{11} = 1$, $U_{12} = 6$ and $U_{13} = 2$. Now, from the second row, $L_{21} = -1$, $U_{22} = 3$ and $U_{23} = 1$. The last three unknowns come from the bottom row: $L_{31} = 2$, $L_{32} = 0$ and $U_{33} = 1$. Hence

$$\begin{bmatrix} 1 & 6 & 2 \\ -1 & -3 & -1 \\ 2 & 12 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is an LU decomposition of the given matrix.

(c) We begin by solving the system

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 17 \\ -4 \end{bmatrix}.$$

(Note that the second and third rows of the right-hand side vector have been swapped too.) Starting with the top equation we see that $y_1 = 9$. The second equation then implies that $y_2 = 26$ and then, from the third equation, $y_3 = -22$. These values now appear on the right-hand side of the second system

$$\begin{bmatrix} 1 & 6 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 26 \\ -22 \end{bmatrix}.$$

The bottom equation shows us that $x_3 = -22$. Moving up to the middle equation we obtain $x_2 = 16$. The top equation yields $x_1 = -43$.