Contents 32

Numerical Initial Value Problems

Learning outcomes

In this Workbook you will learn about numerical methods for approximating solutions relating to a certain type of application area. Specifically you will see methods that approximate solutions to differential equations.

Initial Value Problems 32.1

Introduction

Many engineering applications describe the evolution of some process with time. In order to define such an application we require two distinct pieces of information: we need to know what the process is and also when or where the application started.

In this Section we begin with a discussion of some of these so-called **initial value problems**. Then we look at two numerical methods that can be used to approximate solutions of certain initial value problems. These two methods will serve as useful instances of a fairly general class of methods which we will describe in Section 32.2.

Prerequisites

Before starting this Section you should ...

- revise the trapezium method for approximating integrals in HELM 31.2
- review the material concerning approximations to derivatives in HELM 31.3

Learning Outcomes

On completion you should be able to ...

- recognise an initial value problem
- implement the Euler and trapezium method to approximate the solutions of certain initial value problems

1. Initial value problems

In HELM 19.4 we saw the following initial value problem which arises from Newton's law of cooling

$$\frac{d\theta}{dt} = -k(\theta - \theta_s), \qquad \theta(0) = \theta_0.$$

Here $\theta = \theta(t)$ is the temperature of some liquid at time t, θ_0 is the initial temperature at $t = 0$ and θ_s is the surrounding temperature. The constant of proportion k has units s^{-1} and depends on the properties of the liquid.

This initial value problem has two parts: the differential equation $\frac{d\theta}{dt} = -k(\theta - \theta_s)$, which models the physical process, and the initial condition $\theta(0) = \theta_0$.

Key Point 1

An initial value problem may be made up of two components

1. A mathematical model of the *process*, stated in the form of a differential equation.
2. An initial value, given at some value of the independent variable.

It should be noted that there are applications in which initial value problems do not model processes that are time dependent, but we will not dwell on this fact here.

The initial value problem above is such that we can write down an *exact* or **analytic** solution (it is $\theta(t) = \theta_s + (\theta_0 - \theta_s)e^{-kt}$) but there are many applications where it is impossible or undesirable to seek such a solution. The aim of this Section is to begin to describe numerical methods that can be used to find **approximate** solutions of initial value problems.

Rather than using the application-specific notation given above involving θ we will consider the following initial value problem in this Section. We seek $y = y(t)$ (or an approximation to it) that satisfies the differential equation

$$\frac{dy}{dt} = f(t, y), \qquad (t > 0)$$

and which is subject to the **initial condition**

$$y(0) = y_0,$$

a known quantity.

Some of the examples we will consider will be such that an analytic solution is readily available, and this fact can be used as a check on the accuracy of the numerical methods that follow.

2. Numerical solutions

We suppose that the initial value problem

$$\frac{dy}{dt} = f(t, y) \qquad\qquad y(0) = y_0$$

is such that we are unable (or unwilling) to seek a solution analytically (that is, by hand) and that we prefer to use a computer to approximate y instead. We begin by asking what we expect a numerical solution to look like.

Numerical solutions to initial value problems discussed in this Workbook will be in the form of a sequence of numbers approximating $y(t)$ at a sequence of values of t. The simplest methods choose the t-values to be equally spaced, and we will stick to these methods. We denote the common distance between consecutive t-values as h.

Key Point 2

A numerical approximation to the initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad\qquad y(0) = y_0$$

is a sequence of numbers y_0, y_1, y_2, y_3, $\ldots$.
The value y_0 will be exact, because it is defined by the initial condition.
For $n \geq 1$, y_n is the approximation to the exact value $y(t)$ at $t = nh$.

In Figure 1 the exact solution $y(t)$ is shown as a thick curve and approximations to $y(nh)$ are shown as crosses.

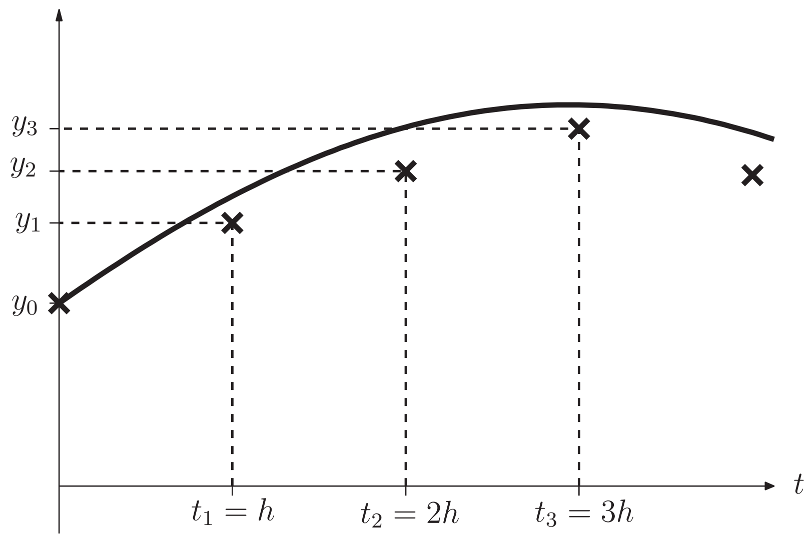

Figure 1

The general idea is to take the given initial condition y_0 and then use it together with what we know about the physical process (from the differential equation) to obtain an approximation y_1 to $y(h)$. We will have then carried out the first **time step**.

Then we use the differential equation to obtain y_2, an approximation to $y(2h)$. Thus the second time step is completed.

And so on, at the n^{th} time step we find y_n, an approximation to $y(nh)$.

Key Point 3

A **time step** is the procedure carried out to move a numerical approximation one increment forward in time.

The way in which we choose to "use the differential equation" will define a particular numerical method, and some ways are better than others. We begin by looking at the simplest method.

3. An explicit method

Guided by the fact that we only seek approximations to $y(t)$ at t-values that are a distance h apart we could use a **forward difference formula** to approximate the derivative in the differential equation. This leads to

$$\frac{y(t+h)-y(t)}{h} \approx f(t,y)$$

and we use this as the inspiration for the numerical method

$$y_{n+1} - y_n = hf(nh, y_n)$$

For clarity we denote $f(nh, y_n)$ as f_n. The procedure for implementing the method (called Euler's method - pronounced "Oil-er's method" - is summarised in the following Key Point.

Key Point 4

Euler's method for approximating the solution of

$$\frac{dy}{dt} = f(t,y), \qquad y(0) = y_0$$

is as follows. We choose a time step h, then

$$\begin{aligned} y(h) &\approx y_1 = y_0 + hf(0,y_0) \\ y(2h) &\approx y_2 = y_1 + hf(h,y_1) \\ y(3h) &\approx y_3 = y_2 + hf(2h,y_2) \\ y(4h) &\approx y_4 = y_3 + hf(3h,y_3) \\ &\vdots \end{aligned}$$

In general, $y(nh)$ is approximated by $\boxed{y_n = y_{n-1} + hf_{n-1}}$.

This is called an **explicit** method, but the reason why will be clearer in a page or two when we encounter an **implicit** method. First we look at an Example.

Example 1

Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = -1/(t+y)^2, \qquad y(0) = 0.9$$

Carry out two time steps of Euler's method with a step size of $h = 0.125$ so as to obtain approximations to $y(0.125)$ and $y(0.25)$.

Solution

In general, Euler's method may be written $y_{n+1} = y_n + hf_n$ and here $f(t,y) = -1/(t+y)^2$.
For the first time step we require $f_0 = f(0,y_0) = f(0,0.9) = -1.23457$ and therefore

$$y_1 = y_0 + hf_0 = 0.9 + 0.125 \times (-1.23457) = 0.745679$$

For the second time step we require $f_1 = f(h,y_1) = f(0.125, 0.745679) = -1.31912$ and therefore

$$y_2 = y_1 + hf_1 = 0.745679 + 0.125 \times (-1.31912) = 0.580789$$

We conclude that

$$y(0.125) \approx 0.745679 \qquad y(0.25) \approx 0.580789$$

where these approximations are given to 6 decimal places.

The simple, repetitive nature of this process makes it ideal for computational implementation, but this next exercise can be carried out by hand.

Task

Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = -y^2, \qquad y(0) = 0.5$$

Carry out two time steps of Euler's method with a step size of $h = 0.01$ so as to obtain approximations to $y(0.01)$ and $y(0.02)$.

Your solution

Answer

For the first time step we require $f_0 = f(0, y_0) = f(0, 0.5) = -(0.5)^2 = -0.25$ and therefore

$$y_1 = y_0 + hf_0 = 0.5 + 0.01 \times (-0.25) = 0.4975$$

For the second time step we require $f_1 = f(h, y_1) = f(0.01, 0.4975) = -(0.4975)^2 = -0.24751$ and therefore

$$y_2 = y_1 + hf_1 = 0.4975 + 0.01 \times (-0.24751) = 0.495025$$

We conclude that

$$y(0.01) \approx 0.497500 \qquad y(0.02) \approx 0.495025 \quad \text{to six decimal places.}$$

The following Task involves the so-called **logistic approximation** that may be used in modelling population dynamics.

Task

Given the logistic population dynamic model

$$\frac{dy}{dt} = 2y(1-y), \qquad y(0) = 1.2$$

carry out two time steps of Euler's method with a step size of $h = 0.125$ to obtain approximations to $y(0.125)$ and $y(0.25)$.

Your solution

Answer

For the first time step we require $f_0 = f(0, y_0) = f(0, 1.2) = 2 \times 1.2(1-1.2) = -0.48$ and therefore

$$\begin{aligned} y_1 &= y_0 + hf_0 \\ &= 1.2 + 0.125 \times (-0.48) \\ &= 1.14 \end{aligned}$$

For the second time step we require $f_1 = f(h, y_1) = f(0.125, 1.14) = -0.3192$ and therefore

$$\begin{aligned} y_2 &= y_1 + hf_1 \\ &= 1.14 + 0.125 \times (-0.3192) \\ &= 1.1001 \end{aligned}$$

We conclude that

$$\begin{aligned} y(0.125) &\approx 1.14 \\ y(0.25) &\approx 1.1001 \end{aligned}$$

Task

The following initial value problem models the population of the United Kingdom, suppose that

$$\frac{dP}{dt} = 2.5 \times 10^{-3}P, \qquad P(0) = 58.043$$

where P is the population in millions, t is measured in years and $t = 0$ corresponds to the year 1996.

(a) Show that Euler's method applied to this initial value problem leads to

$$P_n = (1 + 2.5 \times 10^{-3}h)^n \times 58.043$$

where P_n is the approximation to $P(nh)$.

(b) Use a time step of h equal to 6 months to approximate the predicted population for the year 2050.

Your solution

(a)

Answer

In general $P_{n+1} = P_n + hf_n$ where, in this case, $f(h, P_n) = 2.5 \times 10^{-3}P_n$ hence

$$P_{n+1} = P_n + 2.5 \times 10^{-3}hP_n \qquad \text{and so} \qquad P_{n+1} = (1 + 2.5 \times 10^{-3}h)P_n$$

But P_n will have come from the previous time step ($P_n = (1 + 2.5 \times 10^{-3}h)P_{n-1}$) and P_{n-1} will have come from the time step before that ($P_{n-1} = (1 + 2.5 \times 10^{-3}h)P_{n-2}$). Repeatedly applying this observation leads to

$$P_n = (1 + 2.5 \times 10^{-3}h)^n \times 58.043$$

since $P_0 = P(0) = 58.043$.

Your solution

(b)

Answer

For a time step of 6 months we take $h = \frac{1}{2}$ (in years) and we require 108 time steps to cover the 54 years from 1996 to 2050. Hence

UK population (in millions) in 2050 $\approx P(54) \approx P_{108} = (1 + 2.5 \times 10^{-3} \times \frac{1}{2})^{108} \times 58.043 = 66.427$

where this approximation is given to 3 decimal places.

Accuracy of Euler's method

There are two issues to consider when concerning ourselves with the accuracy of our results.

1. How accurately does the differential equation model the physical process?
2. How accurately does the numerical method approximate the solution of the differential equation?

Our aim here is to address only the second of these two questions.
Let us now consider an example with a known solution and consider just how accurate Euler's method is. Suppose that

$$\frac{dy}{dt} = y \qquad y(0) = 1.$$

We know that the solution to this problem is $y(t) = e^t$, and we now compare exact values with the values given by Euler's method. For the sake of argument, let us consider approximations to $y(t)$ at $t = 1$. The exact value is $y(1) = 2.718282$ to 6 decimal places. The following table shows results to 6 decimal places obtained on a spreadsheet program for a selection of choices of h.

h	Euler approximation to $y(1) = 2.718282$	Difference between exact value and Euler approximation
0.2	$y_5 = 2.488320$	0.229962
0.1	$y_{10} = 2.593742$	0.124539
0.05	$y_{20} = 2.653298$	0.064984
0.025	$y_{40} = 2.685064$	0.033218
0.0125	$y_{80} = 2.701485$	0.016797

Notice that the smaller h is, the more time steps we have to take to get to $t = 1$. In the table above each successive implementation of Euler's method halves h. Interestingly, the error halves (approximately) as h halves. This observation verifies something we will see in Section 32.2, that is that the error in Euler's method is (approximately) proportional to the step size h. This sort of behaviour is called **first-order**, and the reason for this name will become clear later.

Key Point 5

Euler's method is first order. In other words, the error it incurs is approximately proportional to h.

4. An implicit method

Another approach that can be used to address the initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(0) = y_0$$

is to consider integrating the differential equation

$$\frac{dy}{dt} = f(t, y)$$

from $t = nh$ to $t = nh + h$. This leads to

$$\Big[y(t)\Big]_{t=nh}^{t=nh+h} = \int_{nh}^{(n+1)h} f(t, y)\, dt$$

that is,

$$y(nh + h) - y(nh) = \int_{nh}^{(n+1)h} f(t, y)\, dt$$

and the problem now becomes one of approximating the integral on the right-hand side.

If we approximate the integral using the simple trapezium rule and replace the terms by their approximations we obtain the numerical method

$$y_{n+1} - y_n = \tfrac{1}{2}h\,(f_n + f_{n+1})$$

The procedure for time stepping with this method is much the same as that used for Euler's method, but with one difference. Let us imagine applying the method, we are given y_0 as the initial condition and now aim to find y_1 from

$$\begin{aligned} y_1 &= y_0 + \tfrac{h}{2}\,(f_0 + f_1) \\ &= y_0 + \tfrac{h}{2}\,\{f(0, y_0) + f(h, y_1)\} \end{aligned}$$

And here is the problem: the unknown y_1 appears on both sides of the equation. We cannot, in general, find an explicit expression for y_1 and for this reason the numerical method is called an **implicit** method.

In practice the particular form of f may allow us to find y_1 fairly simply, but in general we have to approximate y_1 for example by using the bisection method, or Newton-Raphson. (Another approach that can be used involves what is called a **predictor-corrector** method, in other words, a "guess and improve" method, and we will discuss this again later in this Workbook.)

And then, of course, we encounter the problem again in the second time step, when calculating y_2. And again for y_3 and so on. There is, in general, a genuine cost in implementing implicit methods, but they are popular because they have desirable properties, as we will see later in this Workbook.

Key Point 6

The trapezium method for approximating the solution of

$$\frac{dy}{dt} = f(t, y) \qquad\qquad y(0) = y_0$$

is as follows. We choose a time step h, then

$$\begin{aligned}
y(h) &\approx y_1 = y_0 + \tfrac{1}{2}\,h\Big(f(0, y_0) + f(h, y_1)\Big)\\
y(2h) &\approx y_2 = y_1 + \tfrac{1}{2}\,h\Big(f(h, y_1) + f(2h, y_2)\Big)\\
y(3h) &\approx y_3 = y_2 + \tfrac{1}{2}\,h\Big(f(2h, y_2) + f(3h, y_3)\Big)\\
y(4h) &\approx y_4 = y_3 + \tfrac{1}{2}\,h\Big(f(3h, y_3) + f(4h, y_4)\Big)\\
&\ \vdots
\end{aligned}$$

In general, $y(nh)$ is approximated by $\boxed{y_n = y_{n-1} + \tfrac{1}{2}\,h\Big(f_{n-1} + f_n\Big)}$

In Example 2 the implicit nature of the method is not a problem because y does not appear on the right-hand side of the differential equation. In other words, $f = f(t)$.

Example 2

Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = 1/(t+1), \qquad y(0) = 1$$

Carry out two time steps of the trapezium method with a step size of $h = 0.2$ so as to obtain approximations to $y(0.2)$ and $y(0.4)$.

Solution

For the first time step we require $f_0 = f(0) = 1$ and $f_1 = f(0.2) = 0.833333$ and therefore

$$y_1 = y_0 + \tfrac{1}{2}h(f_0 + f_1) = 1 + 0.1 \times 1.833333 = 1.183333$$

For the second time step we also require $f_2 = f(2h) = f(0.4) = 0.714286$ and therefore

$$y_2 = y_1 + \tfrac{1}{2}h(f_1 + f_2) = 1.183333 + 0.1 \times 1.547619 = 1.338095$$

We conclude that

$$y(0.1) \approx 1.183333 \qquad\qquad y(0.2) \approx 1.338095$$

Example 3 has f dependent on y, so the implicit nature of the trapezium method could be a problem. However in this case the way in which f depends on y is simple enough for us to be able to rearrange for an explicit expression for y_{n+1}.

Example 3

Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = 1/(t^2+1) - 2y, \qquad y(0) = 2$$

Carry out two time steps of the trapezium method with a step size of $h = 0.1$ so as to obtain approximations to $y(0.1)$ and $y(0.2)$.

Solution

The trapezium method is $y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1})$ and in this case y_{n+1} will appear on both sides because f depends on y. We have

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}\left\{\left(\frac{1}{t_n^2+1} - 2y_n\right) + \left(\frac{1}{t_{n+1}^2+1} - 2y_{n+1}\right)\right\} \\ &= y_n + \frac{h}{2}\left\{g(t_n) - 2y_n + g(t_{n+1}) - 2y_{n+1}\right\} \end{aligned}$$

where $g(t) \equiv \dfrac{1}{(t^2+1)}$ which is the part of f that depends on t. On rearranging to get all y_{n+1} terms on the left, we get

$$(1+h)y_{n+1} = y_n + \tfrac{1}{2}h\left\{g(t_n) - 2y_n + g(t_{n+1})\right\}$$

In this case $h = 0.1$.

For the first time step we require $g(0) = 1$ and $g(0.1) = 0.990099$ and therefore

$$1.1y_1 = 2 + 0.05\,(1 - 2 \times 2 + 0.990099)$$

Hence $y_1 = 1.726823$, to six decimal places.

For the second time step we also require $g(2h) = g(0.2) = 0.961538$ and therefore

$$1.1y_2 = 1.726823 + 0.05\,(0.990099 - 2 \times 1.726823 + 0.961538)$$

Hence $y_2 = 1.501566$. We conclude that $y(0.1) \approx 1.726823$ and $y(0.2) \approx 1.501566$ to 6 d.p.

Task

Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = t - y, \qquad y(0) = 2$$

Carry out two time steps of the trapezium method with a step size of $h = 0.125$ so as to obtain approximations to $y(0.125)$ and $y(0.25)$.

Your solution

Answer

The trapezium method is $y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1})$ and in this case y_{n+1} will appear on both sides because f depends on y. However, we can rearrange for y_{n+1} to give

$$1.0625 y_{n+1} = y_n + \tfrac{1}{2}h\left(g(t_n) - y_n + g(t_{n+1})\right)$$

where $g(t) = t$ is the part of f that depends on t.
For the first time step we require $g(0) = 0$ and $g(0.125) = 0.125$ and therefore

$$1.0625 y_1 = 2 + 0.0625\,(0 - 2 + 0.125)$$

Hence $y_1 = 1.772059$ to 6 d.p.
For the second time step we also require $g(2h) = g(0.25) = 0.25$ and therefore

$$1.0625 y_2 = 1.772059 + 0.0625\,(0.125 - 1.772059 + 0.25)$$

Hence $y_2 = 1.58564$, to 6 d.p.

Example 4

The current i in a simple circuit involving a resistor of resistance R and an inductance loop of inductance L with applied voltage E satisfies the differential equation

$$L\frac{di}{dt} + Ri = E$$

Consider the case where $L = 1$, $R = 100$ and $E = 1000$. Given that $i(0) = 0$ use a value of $h = 0.001$ in implementation of the trapezium method to approximate the current i at times $t = 0.001$ and $t = 0.002$.

Solution

The current i satisfies

$$\frac{di}{dt} = 1000 - 100i$$

and the trapezium approximation to this is

$$i_{n+1} - i_n = \frac{h}{2}(2000 - 100i_{n+1} - 100i_n)$$

Rearranging this for i_{n+1} gives

$$i_{n+1} = 0.904762i_n + 0.952381$$

It follows that

$$\begin{aligned} i(0.001) &\approx 0.904762 \times 0 + 0.952381 = 0.952381 \\ i(0.002) &\approx 0.904762 \times 0.952381 + 0.952381 = 1.814059 \end{aligned}$$

where these approximations are given to 6 decimal places.

Accuracy of the trapezium method

Let us now consider an example with a known solution and consider just how accurate the trapezium method is. Suppose that we look at the same test problem we considered when looking at Euler's method

$$\frac{dy}{dt} = y, \qquad y(0) = 1.$$

We know that the solution to this problem is $y(t) = e^t$, and we now compare exact values with the values given by the trapezium method. For the sake of argument, let us consider approximations to $y(t)$ at $t = 1$. The exact value is $y(1) = 2.718282$ to 6 decimal places. The following table shows results to 6 decimal places obtained on a spreadsheet program for a selection of choices of h.

h	Trapezium approximation to $y(1) = 2.718282$	Difference between exact value and trapezium approximation
0.2	$y_5 = 2.727413$	0.009131
0.1	$y_{10} = 2.720551$	0.002270
0.05	$y_{20} = 2.718848$	0.000567
0.025	$y_{40} = 2.718423$	0.000142
0.0125	$y_{80} = 2.718317$	0.000035

Notice that each time h is reduced by a factor of $\frac{1}{2}$, the error reduces by a factor of (approximately) $\frac{1}{4}$. This observation verifies something we will see in Section 32.2, that is that the error in the trapezium approximation is (approximately) proportional to h^2. This sort of behaviour is called **second-order**.

Key Point 7

The trapezium approximation is second order. In other words, the error it incurs is approximately proportional to h^2.

Exercises

1. Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = t + y \qquad y(0) = 3$$

Carry out two time steps of Euler's method with a step size of $h = 0.05$ so as to obtain approximations to $y(0.05)$ and $y(0.1)$.

2. Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = 1/(t^2 + 1) \qquad y(0) = 2$$

Carry out two time steps of the trapezium method with a step size of $h = 0.1$ so as to obtain approximations to $y(0.1)$ and $y(0.2)$.

3. Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = t^2 - y \qquad y(0) = 1.5$$

Carry out two time steps of the trapezium method with a step size of $h = 0.125$ so as to obtain approximations to $y(0.125)$ and $y(0.25)$.

4. The current i in a simple circuit involving a resistor of resistance R, an inductance loop of inductance L with applied voltage E satisfies the differential equation

$$L\frac{di}{dt} + Ri = E$$

Consider the case where $L = 1.5$, $R = 120$ and $E = 600$. Given that $i(0) = 0$ use a value of $h = 0.0025$ in implementation of the trapezium method to approximate the current i at times $t = 0.0025$ and $t = 0.005$.

Answers

1. For the first time step we require $f_0 = f(0, y_0) = f(0, 3) = 3$ and therefore

$$\begin{aligned} y_1 &= y_0 + hf_0 \\ &= 3 + 0.05 \times 3 \\ &= 3.15 \end{aligned}$$

For the second time step we require $f_1 = f(h, y_1) = f(0.05, 3.15) = 3.2$ and therefore

$$\begin{aligned} y_2 &= y_1 + hf_1 \\ &= 3.15 + 0.05 \times 3.2 \\ &= 3.31 \end{aligned}$$

We conclude that

$$\begin{aligned} y(0.05) &\approx 3.15 \\ y(0.1) &\approx 3.31 \end{aligned}$$

2. For the first time step we require $f_0 = f(0) = 1$ and $f_1 = f(0.1) = 0.990099$ and therefore

$$\begin{aligned} y_1 &= y_0 + \tfrac{1}{2}h(f_0 + f_1) \\ &= 2 + 0.05 \times 1.990099 \\ &= 2.099505 \end{aligned}$$

For the second time step we also require $f_2 = f(2h) = f(0.2) = 0.961538$ and therefore

$$\begin{aligned} y_2 &= y_1 + \tfrac{1}{2}h(f_1 + f_2) \\ &= 2.099505 + 0.05 \times 1.951637 \\ &= 2.197087 \end{aligned}$$

We conclude that

$$\begin{aligned} y(0.05) &\approx 2.099505 \\ y(0.1) &\approx 2.197087 \end{aligned}$$

to six decimal places.

Answers

3. The trapezium method is $y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1})$ and in this case y_{n+1} will appear on both sides because f depends on y. However, we can rearrange for y_{n+1} to give

$$1.0625y_{n+1} = y_n + \tfrac{1}{2}h\left\{g(t_n) - y_n + g(t_{n+1})\right\}$$

where $g(t) = t^2$ is the part of f that depends on t.
For the first time step we require $g(0) = 0$ and $g(0.125) = 0.015625$ and therefore

$$1.0625y_1 = 1.5 + 0.0625\,(0 - 1.5 + 0.015625)$$

Hence $y_1 = 1.324449$.
For the second time step we also require $g(2h) = g(0.25) = 0.0625$ and therefore

$$1.0625y_2 = 1.324449 + 0.0625\,(0.015625 - 1.324449 + 0.0625)$$

Hence $y_2 = 1.173227$.

4. Dividing through by $L = 1.5$ we find that the current i satisfies

$$\frac{di}{dt} = 400 - 80i$$

and the trapezium approximation to this is

$$i_{n+1} - i_n = \frac{h}{2}(800 - 80i_{n+1} - 80i_n)$$

Rearranging this for i_{n+1} gives

$$i_{n+1} = 0.818182i_n + 0.909091$$

It follows that

$$\begin{aligned} i(0.0025) &\approx 0.818182 \times 0 + 0.909091 = 0.909091 \\ i(0.005) &\approx 0.818182 \times 0.909091 + 0.909091 = 1.652893 \end{aligned}$$