Mathematical Explorations

47.2

Introduction

This Section revisits some of the mathematics already introduced in other Workbooks but explores several alternative ways of looking at it, with the intention of widening the range of techniques or approaches which the student can call on when dealing with mathematical problems. Familiar functions such as the tangent function of trigonometry are used to illustrate the methods which are introduced, with an application to optics being used to motivate the study of the mathematics involved. Several links between apparently different pieces of mathematics are introduced to encourage the student to appreciate the value of looking for underlying structures which will permit 'knowledge transfer' between what at first sight appear to be topics from different Workbooks.

Prerequisites

Before starting this Section you should . . .

- be familiar with the concept of the Maclaurin series and the method of comparing coefficients of powers of x for power series
- know and understand the definitions of the trigonometric functions
- know and be able to use the theorem of Pythagoras
- know the basic theory of geometric series

Learning Outcomes

On completion you should be able to . . .

- appreciate the value of moving between different approaches to a problem e.g. from a trigonometric diagram to algebra or from a complex number calculation to a matrix calculation
- use a power series approach to problems in order to supplement or replace alternative calculations based on traditional algebra
- have some appreciation of the idea of an isomorphism in mathematics and how this can be useful in extending the range of applicability of a mathematical technique

Introduction to the exploratory mathematical examples

The $\tan(x)$ function is usually regarded as the most "difficult" of the three standard trigonometric functions. Examples 1 to 4 show that it has interesting properties and that it has uses in coordinate geometry and in optics. Example 5 deals with approximating a parabola by an arc of a circle. Example 6 explores efficient ways to find Maclaurin series. Example 7 demonstrates a surprising link between complex arithmetic and matrix algebra. Examples 8 and 9 provide unusual approaches to the exponential function, and finally Example 10 derives the link between the inverse hyperbolic tangent and natural logarithms.

Example 1

Maclaurin series of $\tan x$

Derive the first four non-zero terms in the Maclaurin series of $\tan(x)$ by using the fact that $\tan(x)$ has the derivative $1 + \tan^2(x)$.

Solution

First we note that the series has only odd powers of x in it because $\tan(x)$ is an odd function, with the property $f(-x) = -f(x)$. Thus we have the Maclaurin series

$$\tan(x) = A_1x + A_3x^3 + A_5x^5 + \dots$$

Using the known result $T' = 1 + T^2$ we must have

$$\begin{aligned} A_1 + 3A_3x^2 + 5A_5x^4 + 7A_7x^6 \dots &= 1 + (A_1x + A_3x^3 + A_5x^5 \dots)^2 \\ &= 1 + A_1^2x^2 + 2A_1A_3x^4 + (A_3^2 + 2A_1A_5)x^6 + \dots \end{aligned}$$

Comparing the coefficients of x^N on both sides gives the results

$(N = 0)$ $A_1 = 1$

$(N = 2)$ $3A_3 = A_1^2$, so that $A_3 = 1/3$

$(N = 4)$ $5A_5 = 2A_1A_3 = 2/3$, so that $A_5 = 2/15$

$(N = 6)$ $7A_7 = A_3^2 + 2A_1A_5 = 1/9 + 4/15 = 17/45$, so that $A_7 = 17/315$

We have thus found the expansion up to the x^7 term.

A careful study of the structure of the equations used above shows us how to obtain a general equation which describes the process. The general recurrence relation for the calculation is

$$(N + 1)A_{N+1} = \sum_{J=1}^{N-1} A_JA_{N-J}$$

This can easily be programmed for a computer to generate many terms of the series.

Example 2

Derivatives of $\tan x$

When higher derivatives are being used in the differential calculus the symbol $F^{(n)}$ is often used to denote the n^{th} derivative of a function F. For the function $T = \tan(x)$ we know that $T^{(1)} = 1+T^2$. Use the chain rule to work out the derivatives of $\tan(x)$ up to $T^{(5)}$ and show how this can give the first three terms in the Maclaurin series for $\tan(x)$.

Solution

By the chain rule we have

$$T^{(n+1)} = \frac{d}{dx}(T^{(n)}) = \frac{d}{dT}(T^{(n)}) \times \frac{dT}{dx} = (1+T^2)\frac{d}{dT}(T^{(n)})$$

Thus given the result

$$T^{(1)} = 1 + T^2$$

we find that

$$T^{(2)} = (1+T^2)\frac{d}{dT}(T^{(1)}) = (1+T^2)2T = 2T + 2T^3$$

$$T^{(3)} = (1+T^2)(2+6T^2) = 2 + 8T^2 + 6T^4$$

$$T^{(4)} = (1+T^2)(16T+24T^3) = 16T + 40T^3 + 24T^5$$

$$T^{(5)} = (1+T^2)(16+120T^2+120T^4) = 16 + 136T^2 + 240T^4 + 120T^6$$

To find the Maclaurin series by the textbook method we need to find the n^{th} derivative of $\tan(x)$ at $x = 0$ and then divide it by n factorial $(n!)$. $T = 0$ at $x = 0$ and so the required n^{th} derivative is just the constant term in the $T^{(n)}$ calculated above. This term is zero for the even n and leads to the correct first three terms of the $\tan(x)$ series when the terms with odd n are taken. If we set down the general equation relating derivatives of T to powers of T,

$$T^{(n)} = \sum A(n,m)T^m$$

and insert this in the chain rule equation used in the calculations above, then we arrive at a recurrence relation by comparing the T^m coefficients on both sides:

$$A(n+1,\ m) = (m+1)\ A(n,m+1) + (m-1)\ A(n,m-1)$$

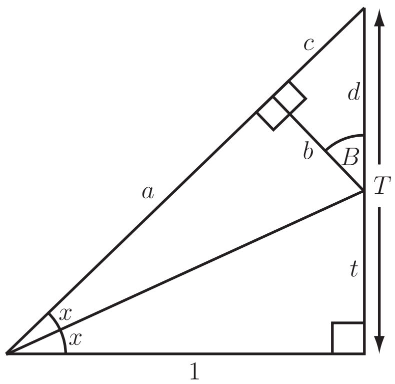
which can be used in a computer calculation or to set up a tabular method of calculating the results above, rather like a more complicated version of Pascal's triangle. Here are the first few rows:

$n \setminus m$	0	1	2	3	4	5
1	1	0	1	0	0	0
2	0	2	0	2	0	0
3	2	0	8	0	6	0
4	0	16	0	40	0	24

Example 3

A geometric derivation of the equation for tan($2x$)

To study the reflections from a parabolic mirror (Example 4) we need to know how to find $\tan(2x)$ from $\tan(x)$. This Example presents a simple way to derive the required formula, while giving some practice at algebraic manipulation and simple geometry. Study the following diagram, in which t denotes $\tan(x)$ and T denotes $\tan(2x)$. Find the lengths a, b, c and the angle B on the diagram and thence derive an equation for T by studying the small triangle with the sides $b,\ c$ and d. You will need to do some algebra to work the equation into a form where it gives T in terms of t. When you have obtained the equation, use it to include the length d in your calculation and so find an equation which gives $\cos(2x)$ in terms of t. (We are in fact using a geometrical approach to find what are called "the tan half angle formulae" in trigonometry.)



Solution

By studying the two similar triangles we see that $a = 1$ and $b = t$. Using the fact that the angles in a plane triangle add up to $180°$, we find that the angle B is just $2x$, which means that $T = \tan(2x)$ must equal c/b. To find c we have to note that $a + c$ is the hypotenuse of a triangle in which the other two sides have the lengths 1 and T. We can thus use the theorem of Pythagoras to find $a + c$. Putting all these facts together leads to an equation:

$$T = c/b = \left(\sqrt{1 + T^2} - 1\right)/t$$

This leads to $Tt + 1 = \sqrt{1 + T^2}$. Squaring gives $T^2t^2 + 2Tt + 1 = 1 + T^2$

Cancelling the 1 on both sides and dividing by the non-zero number T gives, after re-arranging terms

$$T = 2t/(1 - t^2), \quad \text{so that} \quad \tan 2x = \frac{2\tan x}{(1 - \tan^2 x)}.$$

If we now look at the length d we find that it equals $T - t$, so that $\cos(2x) = b/d = t/(T - t)$

Inserting the result for T in this equation and tidying up the fractions which appear leads to the results

$$\cos(2x) = (1 - t^2)/(1 + t^2), \quad \sin(2x) = 2t/(1 + t^2)$$

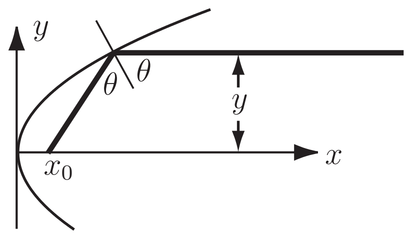
where the result for $\sin(2x)$ follows since $\sin(2x) = \tan(2x)\cos(2x)$.

Example 4

An application of the tan($2x$) equation to optics

One of the useful applications of the tangent function arises in coordinate geometry. If a line has the equation $y = mx + c$ then the angle θ which the line makes with the x-axis satisfies $\tan(\theta) = m$. In dealing with optical reflections we know that the angle of incidence equals the angle of reflection. The angle between the incident and reflected light beams is thus exactly twice the angle of incidence. This problem exploits that fact to make use of the double angle equation derived in Example 3.

Consider a parabola which is described by the equation $y = A\sqrt{x}$. Suppose that a beam of light travelling horizontally from right to left hits the parabola at a height y above the x-axis and that it undergoes a mirror reflection from the parabola. Find the x coordinate at which the light beam crosses the x-axis and show that it is independent of the value of y. The diagram below shows the details.



Solution

The gradient at the point (x, y) on the parabola is $\dfrac{dy}{dx} = \dfrac{1}{2}Ax^{-1/2}$. This is more conveniently written as $(1/2)A^2/y$. The normal to the parabola has the gradient $-2y/A^2$, since two lines at right-angles have the property that the product of their gradients is -1. This result means that the normal makes an angle θ with the horizontal such that $\tan(\theta) = 2y/A^2$. The reflected beam makes an angle with the horizontal which is exactly twice this angle. From the result for $\tan(2x)$ in terms of $\tan(x)$ (see Example 3) we thus know that the reflected beam has a gradient m such that

$$m = \tan(\pi - 2\theta) = -\tan(2\theta) = -2(2y/A^2)/(1 - 4y^2/A^4).$$

This looks like a complicated result but it turns out that the end result is quite simple. To find the x value x_0 at which the beam will cross the axis we note that the beam starts off on the parabola at height y and thus has the initial x coordinate $x = y^2/A^2$. In falling a distance y it will travel a further horizontal distance y/m, where m is the gradient of the line along which it travels. Combining all these facts leads to a result for the x value at which the beam crosses the axis:

$$x_0 = y^2/A^2 - y/m = y^2/A^2 + y(1 - 4y^2/A^4)A^2/(4y)$$

Working out the terms shows that the y^2/A^2 is cancelled out, leaving the simple result $x_0 = A^2/4$. The calculation above was carried out in the plane i.e. in two dimensions. A three dimensional parabolic mirror would be constructed by rotating the parabola around the x axis. Our result shows that all rays arriving parallel to the axis will arrive at the focal point at $x = A^2/4$. If a light source is placed at the focus then a strong parallel beam of light will emerge from the mirror (as in a searchlight).

Example 5

A calculation involving a circle

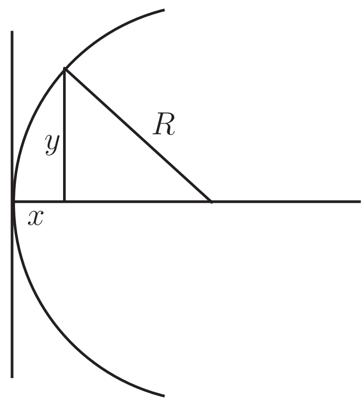
Suppose that the parabolic mirror surface $y = A\sqrt{x}$ of Example 4 is too difficult to make and so is replaced by an arc of a circle. (The full three dimensional parabolic mirror is thus replaced by a portion of a sphere.)

If the circle has radius R, show that it will simulate the parabolic mirror fairly well for light beams striking the mirror close to the x axis and find the x coordinate of the focal point in terms of the radius R.

HINT: Do not consider the reflection of a light beam; try to write the equation of the arc of the circle in the same form as that of the parabola and thus read off the value of $A^2/4$ directly.

Solution

A diagram will clarify the geometry of the situation:



Using the theorem of Pythagoras we see that

$$R^2 = y^2 + (R - x)^2 = R^2 + y^2 + x^2 - 2Rx$$

so that

$$y^2 = x(2R - x)$$

For light beams near to the axis, with x very small, the arc of the circle will be well represented by the equation $y = \sqrt{2Rx}$.

This is identical with the equation of the parabola if we set $A = \sqrt{2R}$.

Thus the focal point of the simulating spherical surface is at the distance $x = A^2/4 = 2R/4 = R/2$ from the centre of the mirror at $x = 0$.

Example 6

Maclaurin series and geometric series

In HELM 16.5 it is shown how to use the geometric series for $1/(1+x)$ to find the Maclaurin series for $\ln(1+x)$ by using the fact that the derivative of $\ln(1+x)$ is $1/(1+x)$. This particular example can be taken further and the same technique can be applied to the $\tan$ function, as illustrated by this problem.

Use the geometric series approach to do the following:

(1) Find the Maclaurin series for $\ln\{(1+x)/(1-x)\}$ and use it to estimate the numerical value of $\ln(2)$.

(2) Find the Maclaurin series for $\tan^{-1}(x)$ and thus obtain an infinite sum of terms which give $\pi/4$.

Solution

(1) We have $\dfrac{d}{dx}\{\ln(1+x)\} = 1/(1+x) = 1 - x + x^2 - x^3 + x^4 \ldots$

Integrating gives

$$\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 \ldots$$

However, changing sign gives

$$\ln(1-x) = -x - x^2/2 - x^3/3 - x^4/4 \ldots$$

(Note that this makes sense; if a number is less than 1 we expect to get a negative logarithm, since e to a negative index is needed to produce a number less than 1.)

We know that

$$\ln\{(1+x)/(1-x)\} = \ln(1+x) - \ln(1-x).$$

Subtracting the two series in accord with this result gives

$$\ln\{(1+x)/(1-x)\} = 2(x + x^3/3 + x^5/5 \ldots)$$

Note that this series has only odd powers in it, because the function being expanded is odd (like $\tan(x)$). Changing x to $-x$ changes the $\ln$ of a number to the $\ln$ of its reciprocal; this simply changes the sign of the logarithm.

Suppose that we want to find $\ln(Y)$ for some number Y. A little algebra shows that the value of x to use in the series above is given by the formula $x = (Y-1)/(Y+1)$ [work it out!]. To find $\ln(2)$ we thus need to use the small number $x = (2-1)/(2+1) = 1/3$ in the series. Adding the first five terms gives the approximate value 0.693145 for $\ln(2)$, whereas the correct value to 6 d.p. is 0.693147. Using more terms would give greater accuracy.

Solution

(2) We know that $\tan^{-1}(x)$ has derivative $1/(1+x^2)$ and so can set

$$\frac{d}{dx}(\tan^{-1}(x)) = 1/(1+x^2) = 1 - x^2 + x^4 - x^6 + x^8 \ldots$$

Integrating gives

$$\tan^{-1}(x) = x - x^3/3 + x^5/5 - x^7/7 \ldots$$

$\pi/4$ is the angle in the central range which has a tangent equal to 1. Thus we set $x = 1$ in the series to obtain

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 \ldots$$

The successive numerical sums in this alternating series straddle the exact value of $\pi/4$ and thus give upper and lower bounds to it. This series is not very good for estimating π, however, since it converges very slowly. Two more effective series (which arise in the theory of Fourier series) are the following:

$\pi^2/6 =$ sum of terms $1/n^2$ $\quad (n = 1, 2, 3, \ldots)$, i.e. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$\pi^4/90 =$ sum of terms $1/n^4$ $\quad (n = 1, 2, 3, \ldots)$, i.e. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Example 7

A link between matrices and complex numbers

Consider the family of 2×2 matrices of the form

$$Z(x, y) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

Show that the $Z(x, y)$ matrices multiply together exactly in the same way as complex numbers, if we take $Z(x, y)$ to represent the complex number $x + \mathrm{i}y$. Show also that complex number division can be carried out using this matrix model, with the matrix inverse playing the role of the inverse of its associated complex number.

Solution

The product $Z(a, b)Z(c, d)$ as given by the standard row-column matrix multiplication rule is quickly found to be

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{pmatrix}$$

The results in the product matrix are exactly the real and imaginary parts of the complex number product $(a+\mathrm{i}b)(c+\mathrm{i}d)$. The rule for the inverse of a 2×2 matrix tells us that the inverse of $Z(x, y)$ is given by

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} = \frac{1}{(x^2 + y^2)} \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

Again, this fits to the inverse of the complex number $(x + \mathrm{i}y)$, which is $(x - \mathrm{i}y)/(x^2 + y^2)$. Thus complex number division can be carried out by multiplying $Z(a, b)$ by the inverse of $Z(c, d)$ to yield correctly the real and imaginary parts of the complex number $(a + \mathrm{i}b)/(c + \mathrm{i}d)$. Complex number multiplication is known to be commutative and we can easily verify that the family of matrices $Z(x, y)$ also commute with one another under matrix multiplication. We have what is called by mathematicians an **isomorphism** (identity of form) between the set of complex numbers $x + \mathrm{i}y$ and the set of 2×2 matrices $Z(x, y)$ with real components x and y. You may check for yourself that complex number addition and subtraction are also correctly described by this matrix model. Note that since we have found an infinite family of matrices which commute with one another under multiplication the statement $AB \neq BA$ which sometimes appears in textbooks on matrices is not correct, strictly speaking, since it appears to be a *universal* statement that matrices *never* commute under multiplication!

Example 8

An approximation for the exponential function

Find an expression for exp(x) by comparing the Maclaurin series for exp(x) with the binomial expansion for $(1+\dfrac{x}{N})^N$.

Solution

If we compare the exponential series for $\exp(x)$ (often written e^x) with the binomial series for $(1+x/N)^N$, where N is a large integer, then we find that the terms of the binomial series become closer and closer to the terms in $\exp(x)$ as N increases. Suppose, for example, that we wish to work out $e=\exp(1)$. The J^{th} term in the exponential series would be just $1/J!$. The J^{th} term in the binomial expansion of $(1+1/N)$ would be $(1/N)^J\ N!/[J!(N-J)!]$ as discussed in HELM 16 on Sequences and Series. This binomial term can be written more usefully in the form

$$\frac{N\times(N-1)\times(N-2)\times\ldots(N+1-J)}{N\times N\times N\times\ldots N}\times(1/J!)$$

For any finite J it is clear that the fraction multiplying $(1/J!)$ becomes closer and closer to 1 as N is increased. Thus the binomial expansion fits more and more closely to the exponential expansion. This leads to an equation involving a limit:

$$\exp(x)=\lim_{N\to\infty}\left\{\left(1+\frac{x}{N}\right)^N\right\}$$

Note

This equation is not only a celebrated one in classical mathematics; it is actually applied in some areas of scientific research to estimate the exponential of x for difficult cases in which x is not just a number but is a matrix or an operator. Example 9 deals only with the case in which x is a number but it introduces an interesting discovery which has recently appeared in the literature of mathematical education:

'A modification to the e limit', Tony Robin, Mathematical Gazette Vol 88; 2004, pp 279-281.

Example 9

An improved approximation for the exponential function

The function $F(k,x) = (1 + x/N)^{kx}$ tends to 1 as the value of N increases for fixed values of k and x. Thus if we modify the classical equation for $\exp(x)$ given above by multiplying the term $(1 + x/N)^N$ by $F(k,x)$ we shall *still* obtain $\exp(x)$ in the limit $N \to \infty$. We thus have an infinite family of approximations, with the classical one corresponding to the case $k = 0$. Use a calculator or computer to estimate $e = \exp(1)$ by using the N values $8, 16, 32, \ldots$ (a doubling sequence) and compare the results obtained by using $k = 0$ and $k = 1/2$ in the multiplying factor $F(k,x)$.

Solution

Calculating with a simple 8 digit calculator and rounding the results to 6 significant digits (because of the rounding error in taking high powers of a number) we obtain the following results:

N	$k = 0$	$k = 1/2$
8	2.56578	2.72143
16	2.63793	2.71911
32	2.67699	2.71849
64	2.69734	2.71833
128	2.70772	2.71828

Thus the $k = 1/2$ approximations reach the correct result much more quickly than the classical $k = 0$ approximations do.

Example 10

An example of an inverse hyperbolic function

Find an expression for the inverse $\tanh$ of x, $\tanh^{-1}(x)$, in terms of the $\ln$ (natural logarithm) function by two different methods:

1. by using algebra to solve the equation $x = \tanh(y)$ for y.
2. by using an approach via geometric series, starting from the fact that the derivative of $\tanh(x)$ is $1 - \tanh^2(x)$.

Solution

(1) Recalling that $\tanh(y) = \sinh(y)/\cosh(y)$ and using the temporary symbol Y for e^y, we have the starting equation

$$x = \tanh(y) = (Y - Y^{-1})/(Y + Y^{-1}) = (Y^2 - 1)/(Y^2 + 1)$$

from which we find that $xY^2 + x = Y^2 - 1$ and thus that

$$Y^2 = e^{2y} = (1 + x)/(1 - x), \quad \text{so that} \quad y = (1/2)\ln\{(1 + x)/(1 - x)\}$$

which expresses the inverse $\tanh$ function, $y = \tanh^{-1}(x)$, in terms of natural logs.

(2) By using an approach which is the same as that used for the inverse tangent function in trigonometry, we find that the inverse $\tanh$ function has the derivative $1/(1 - x^2)$. Using the geometric series approach explained previously we can set

$$\frac{d}{dx}(\tanh^{-1}(x)) = 1/(1 - x^2) = 1 + x^2 + x^4 + \ldots$$

so that by integrating we find

$$\tanh^{-1}(x) = x + x^3/3 + x^5/5 + \ldots$$

Comparison with the result of Example 8 shows that this series is exactly half of the series for the function $\ln\{(1 + x)/(1 - x)\}$, which leads us to the same result as that obtained by algebra in the first part of the solution. Arguments based on the use of infinite series for functions are often used in simple calculus but have the limitation that, strictly speaking, they can only be regarded as trustworthy if the series involved actually converge for the range of x values for which the functions are to be used. The geometric series appearing above are only convergent if x is between 1 and -1 (for the case of real x). Thus a series approach is sometimes only a "shortcut" way of getting a result which can be obtained more rigorously by an alternative method. That is the case for the example treated in this problem. However, if the series involved converges for all x (however large) then a series approach is fully legitimate and there are several such pieces of mathematics involving the series for the exponential and the trigonometric functions.