$D^b(\mathbf{Intro})$

TOM BRIDGELAND

These are the lecture notes for the introductory school on derived categories in Warwick, September 2014. They cover some basic facts about derived categories of coherent sheaves on smooth projective varieties, assuming some kind of familiarity with the definition of a derived category. There are bound to be some mistakes that I haven't found yet: please feel free to let me know about them.

1. The derived category of an abelian category

In this section we summarize the most important properties of the derived category of an abelian category. We illustrate some of these by considering the duality functor for coherent sheaves on $\mathbb{A}^2$.

1.1. **Basics.** Let $\mathcal{A}$ be an abelian category, e.g. $\mathrm{Mod}(R)$ or $\mathrm{Coh}(X)$. Let $C(\mathcal{A})$ denote the category of cochain complexes in $\mathcal{A}$. A typical morphism $f^\bullet\colon M^\bullet \to N^\bullet$ in this category looks as follows

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M^{i-1} & \xrightarrow{d^{i-1}} & M^i & \xrightarrow{d^i} & M^{i+1} & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle f^{i-1}} & & \Big\downarrow{\scriptstyle f^i} & & \Big\downarrow{\scriptstyle f^{i+1}} & & \\
\cdots & \longrightarrow & N^{i-1} & \xrightarrow{d^{i-1}} & N^i & \xrightarrow{d^i} & N^{i+1} & \longrightarrow & \cdots
\end{array}$$

Recall that such a morphism $f^\bullet\colon M^\bullet \to N^\bullet$ is called a *quasi-isomorphism* if the induced maps on cohomology objects

$$H^i(f^\bullet)\colon H^i(M^\bullet) \to H^i(N^\bullet)$$

are all isomorphisms. The derived category $D(\mathcal{A})$ is obtained from $C(\mathcal{A})$ by formally inverting quasi-isomorphisms. Thus there is a localisation functor

$$Q\colon C(\mathcal{A}) \to D(\mathcal{A})$$

which is universal with the property that it takes quasi-isomorphisms to isomorphisms. The objects of $D(\mathcal{A})$ can be taken to be the same as the objects of $C(\mathcal{A})$.

Date: August 26, 2015.

It is immediate from the universal property that there are well-defined functors $H^i\colon D(\mathcal{A}) \to \mathcal{A}$ sending a complex to its cohomology objects. The bounded derived category is defined to be the full subcategory

$$D^b(\mathcal{A}) = \{E \in D(\mathcal{A}) : H^i(E) = 0 \text{ for } |i| \gg 0\} \subset D(\mathcal{A}).$$

There is an obvious functor $\mathcal{A} \to D(\mathcal{A})$ which sends an object $E \in \mathcal{A}$ to the corresponding trivial complex with E in position 0:

$$E \in \mathcal{A} \longmapsto (\cdots \longrightarrow 0 \longrightarrow E \longrightarrow 0 \longrightarrow \cdots) \in D(\mathcal{A}).$$

This functor is full and faithful. Its essential image is the full subcategory

$$\{E \in D(\mathcal{A}) : H^i(E) = 0 \text{ unless } i = 0\}.$$

Objects of this subcategory are said to be *concentrated in degree 0*. We shall always identify the category $\mathcal{A}$ with its image under this functor.

Two objects in $D(\mathcal{A})$ with the same cohomology objects need not be isomorphic (in much the same way as two modules with the same composition series need not be isomorphic). The extra information determining an object can be thought of as a 'cohomological glue' holding the cohomology objects together. If this glue vanishes then

$$E \cong \bigoplus_{i \in \mathbb{Z}} H^i(E)[-i]$$

$$\cong (\cdots \longrightarrow H^{i-1}(E) \xrightarrow{0} H^i(E) \xrightarrow{0} H^{i+1}(E) \longrightarrow \cdots).$$

Well-behaved functors between abelian categories $F\colon \mathcal{A} \to \mathcal{B}$ induce derived functors $\mathbf{F}\colon D(\mathcal{A}) \to D(\mathcal{B})$. The composite functors

$$A \in \mathcal{A} \mapsto H^i(\mathbf{F}(\mathcal{A})) \in \mathcal{A}$$

are the classical derived functors of F.

1.2. **Example: Duality for modules.** When $R = \mathbb{C}$ the dualizing functor

$$\mathbb{D}(M) = \mathrm{Hom}_R(M, R)$$

defines an anti-equivalence

$$\mathbb{D}\colon \mathrm{Mod}_{fg}(R) \longrightarrow \mathrm{Mod}_{fg}(R)$$

satsifying $\mathbb{D}^2 \cong \mathrm{id}$. What happens when R is a more interesting ring?

Consider the case $R = \mathbb{C}[x, y]$. Of course $\mathrm{Mod}_{fg}(R) = \mathrm{Coh}(\mathbb{A}^2_{\mathbb{C}})$. Defining a dualizing functor exactly as above we get an anti-equivalence

$$\mathbb{D}\colon \mathrm{Proj}_{fg}(R) \to \mathrm{Proj}_{fg}(R)$$

satisfying $\mathbb{D}^2 \cong \mathrm{id}$. (You may find it comforting to note that by the Quillen-Suslin theorem, any finitely-generated projective R-module is in fact free). But this functor is not an anti-equivalence on the full category $\mathrm{Mod}_{fg}(R)$ since, for example, if $M = R/(x)$ then

$$\mathbb{D}(M) = \mathrm{Hom}_R(R/(x), R) = (0).$$

To try to remedy this, let us consider also the classical derived functors

$$\mathbb{D}^i(M) = \mathrm{Ext}^i_R(M, R), \quad i \geqslant 0.$$

To compute these we replace $M = R/(x)$ by a free resolution

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$$

and apply $\mathbb{D}(-) = \mathrm{Hom}_R(-, R)$ to get

$$0 \longleftarrow R \xleftarrow{x} R \longleftarrow 0.$$

Taking cohomology gives

$$\mathbb{D}^i(M) = \begin{cases} M & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

so we have $\mathbb{D}^1(\mathbb{D}^1(M)) \cong M$.

Similarly, if we take the module $M = R/(x, y)$ then

$$\mathbb{D}^i(M) = \begin{cases} M & \text{if } i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and once again we have $\mathbb{D}^2(\mathbb{D}^2(M)) \cong M$.

But suppose now that we consider modules M fitting into a short exact sequence

$$0 \longrightarrow R/(x, y) \longrightarrow M \longrightarrow R/(x) \longrightarrow 0. \tag{1}$$

Then from the long exact sequence in Ext-groups

$$\mathbb{D}^i(M) = \begin{cases} R/(x) & i = 1, \\ R/(x, y) & i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

However M is not uniquely determined by the sequence (1), since

$$\mathrm{Ext}^1_R(R/(x), R/(x,y)) = \mathbb{C}.$$

We conclude that we cannot recover M from the objects $\mathbb{D}^i(M)$.

The solution (of course) is to consider the derived functor of $\mathbb{D}$, which defines an anti-equivalence

$$\underline{\mathbb{D}}\colon D^b \operatorname{Mod}_{fg}(R) \longrightarrow D^b \operatorname{Mod}_{fg}(R)$$

On the level of objects this means 'replace a complex by a quasi-isomorphic complex of projective modules and then dualize'. It is immediate that $\underline{\mathbb{D}}^2 \cong \mathrm{id}$ because we already know that duality works well for projective modules. If $M \in \operatorname{Mod}_{fg}(R)$ then we have

$$\mathbb{D}^i(M) = \mathrm{Ext}^i_R(M,R) = H^i(\underline{\mathbb{D}}(M)),$$

but as we saw above, these cohomology modules are not in general enough to determine the object $\underline{\mathbb{D}}(M)$, nor to recover the module M.

1.3. **Structure of $D(\mathcal{A})$.** The category $D(\mathcal{A})$ has two important structures which it is important to keep separate in one's mind.

(a) The category $D(\mathcal{A})$ is *triangulated*: it has a shift functors

$$[n]\colon D(\mathcal{A}) \to D(\mathcal{A}),$$

$$M^\bullet[n]^i = M^{i+n}, \qquad d^i_{M^\bullet[n]} = (-1)^n d^{i+n}_{M^\bullet},$$

and a collection of distinguished triangles

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & C \arrow[ul, dashed, "h"] & \end{tikzcd} \tag{2}$$

obtained from the mapping cone construction. Any such triangle is a sequence of maps

$$\cdots \longrightarrow C[-1] \xrightarrow{h[-1]} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \longrightarrow \cdots .$$

Distinguished triangles in a triangulated category play a very similar role to short exact sequences in an abelian category. All derived functors are triangulated: they commute with the shift functors and takes distinguished triangles to distinguished triangles.

Given objects $E, F \in D(\mathcal{A})$ we define

$$\mathrm{Hom}^i_{D(\mathcal{A})}(E,F) := \mathrm{Hom}_{D(\mathcal{A})}(E, F[i]).$$

If $E, F \in \mathcal{A}$ then these agree with the usual Ext-groups:

$$\mathrm{Ext}^i_{\mathcal{A}}(E,F) = \mathrm{Hom}_{D(\mathcal{A})}(E, F[i]).$$

It follows from the axioms of a triangulated category that if E is a fixed object and (2) is a distinguished triangle then there is a long exact sequences of abelian groups

(3)
$$\cdots \to \mathrm{Hom}^i_{D(\mathcal{A})}(E,A) \to \mathrm{Hom}^i_{D(\mathcal{A})}(E,B) \to \mathrm{Hom}^i_{D(\mathcal{A})}(E,C) \to$$
$$\to \mathrm{Hom}^{i+1}_{D(\mathcal{A})}(E,A) \to \mathrm{Hom}^{i+1}_{D(\mathcal{A})}(E,B) \to \cdots .$$

There is a similar long exact sequence involving Hom groups into E.

(b) The category $D(\mathcal{A})$ comes equipped with the *standard t-structure*. In particular, there is a full and faithful embedding $\mathcal{A} \hookrightarrow D(\mathcal{A})$ and cohomology functors $H^i\colon D(\mathcal{A}) \to \mathcal{A}$ as discussed above.

A short exact sequence

(4)
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{A}$ becomes a distinguished triangle of the form (2) in $D(\mathcal{A})$. The extra morphism $h \in \mathrm{Ext}^1_{\mathcal{A}}(C,A)$ is the extension-class defined by the sequence. Conversely, any distinguished triangle (2) induces a long exact sequence in cohomology objects

$$\cdots \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots .$$

Also important are the *truncation functors* $\tau_{\leqslant i}\colon D(\mathcal{A}) \to D(\mathcal{A})$ defined by

$$\tau_{\leqslant i}(M^\bullet) = \big(\cdots \to M^{i-1} \to \ker(d^i) \to 0 \to \cdots\big).$$

Note that

$$H^j(\tau_{\leqslant i}(M^\bullet)) = \begin{cases} H^j(M^\bullet) & j \leqslant i, \\ 0 & \text{otherwise.} \end{cases}$$

There is an obvious natural map of complexes $\tau_{\leqslant i-1}(M^\bullet) \to \tau_{\leqslant i}(M^\bullet)$ which induces isomorphisms in cohomology in degree $\leqslant i-1$. Taking the cone C on this map, and applying the long exact sequence in cohomology, we see that C is concentrated in degree i. We thus have distinguished triangles

$$\begin{tikzcd}[column sep=small] \tau_{\leqslant i-1}(M^\bullet) \arrow[rr] & & \tau_{\leqslant i}(M^\bullet) \arrow[dl] \\ & H^i(M^\bullet)[-i] \arrow[ul, dashed] & \end{tikzcd} \tag{5}$$

This is to be interpreted as saying that every object of $D(\mathcal{A})$ has a canonical 'filtration' whose 'factors' are shifts of objects of $\mathcal{A}$.

Note that genuinely derived functors do not preserve the standard t-structures. In fcat, a triangulated functor $D(\mathcal{A}) \to D(\mathcal{B})$ that preserves the standard t-structures induces an exact functor $\mathcal{A} \to \mathcal{B}$, and conversely, an exact functor $\mathcal{A} \to \mathcal{B}$ induces a functor $D(\mathcal{A}) \to D(\mathcal{B})$ in a trivial way. We often say that such functors are exact and hence 'do not need to be derived'.

1.4. **Grothendieck groups.** The Grothendieck group $K_0(D)$ of a triangulated category D is the free abelian group on isomorphism classes of objects modulo relations

$$[B] = [A] + [C]$$

for distinguished triangles

$$\begin{tikzcd}[column sep=small] A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, dashed] & \end{tikzcd}$$

It follows from the 'rotating triangle' axiom that $[E[n]] = (-1)^n[E]$.

Suppose that $D = D^b(\mathcal{A})$. The inclusion $\mathcal{A} \hookrightarrow D$ clearly induces a group homomorphism

$$I\colon K_0(\mathcal{A}) \to K_0(D).$$

It follows immediately from the existence of the filtration (5) that I is in fact an isomorphism, with inverse map P given by

$$P([E]) = \sum_{i\in\mathbb{Z}} [H^i(E)[i]] = \sum_{i\in\mathbb{Z}} (-1)^i [H^i(E)]$$

1.5. **Problems.**

1.5.1. *Two-step complexes.* Fix objects $A, B \in \mathcal{A}$ and consider objects $E \in D(\mathcal{A})$ such that

$$H^j(E) = \begin{cases} A & \text{if } j = -1 \\ B & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that any such object fits into a distinguished triangle

$$\begin{tikzcd} A[1] \arrow[rr] & & E \arrow[dl] \\ & B \arrow[ul, dashed, "\eta"] & \end{tikzcd}$$

Use this to give a complete classification of the isomorphism classes of such objects in terms of the group $\operatorname{Ext}^2_{\mathcal{A}}(B, A)$.

1.5.2. Consider the two-step complexes obtained by applying the functor $\mathbb{D}$ to the modules M which fit into a short exact sequence of the form (1). How is the extension class defining this short exact sequence reflected in the structure of $\mathbb{D}(M)$?

1.5.3. Let $\mathcal{A}$ be an abelian category of global dimension 1, i.e.

$$\operatorname{Ext}^p_{\mathcal{A}}(M, N) = 0 \text{ for all } p > 1 \text{ and all } M, N \in \mathcal{A}.$$

Prove that every $E \in D^b(\mathcal{A})$ satisfies $E \cong \bigoplus_{i \in \mathbb{Z}} H^i(E)[-i]$.

2. Derived categories of coherent sheaves

This lecture focuses on the derived category of coherent sheaves on a smooth projective variety. We introduce the basic abstract properties of this category and consider the example of the projective line.

2.1. **Basic properties.** Let X be a smooth complex projective variety of dimension d. We set $D(X) = D^b \operatorname{Coh}(X)$. Note that this is a $\mathbb{C}$-linear category: the Hom sets are all vector spaces over $\mathbb{C}$, and the composition maps are bilinear. From Section 1.4 we know that

$$K_0(D(X)) = K_0(\operatorname{Coh}(X)) = K_0(X)$$

is the usual Grothendieck group of X. Since X is smooth and projective this also coincides with the Grothendieck group of locally-free sheaves $K^0(X)$. This is a commutative ring, with multiplication induced by tensor product of vector bundles.

The category $D(X)$ has three very important properties

(a) **Finiteness**. $D(X)$ is of finite type: for all objects $E, F \in D(X)$

$$\dim_{\mathbb{C}} \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}^i_{D(X)}(E, F) < \infty.$$

This enables us to define

$$\chi(E,F)=\sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbb{C}}\operatorname{Hom}^i_{D(X)}(E,F).$$

Note that the long exact sequence (3) shows that this expression is additive: given a distinguished triangle (2) we have

$$\chi(E,B)=\chi(E,A)+\chi(E,C).$$

It follows that it descends to give a bilinear form

$$K_0(X)\times K_0(X)\to\mathbb{Z}$$

which is known as the *Euler form*.

(b) **Riemann-Roch.** The Chern character defines a ring homomorphism

$$\operatorname{ch}\colon K_0(X)\to H^*(X,\mathbb{Q})$$

$$\operatorname{ch}(E)=\big(c_0(E),c_1(E),\frac{1}{2}\,c_1(E)^2-c_2(E),\cdots\big).$$

The Riemann-Roch theorem states that for all $E,F\in D(X)$

$$\chi(E,F)=[\operatorname{ch}(E)^\vee\cdot\operatorname{ch}(F)\cdot\operatorname{td}(X)]_{2d}.$$

In this formula $\operatorname{ch}(E)^\vee$ denotes the sum $\sum_i(-1)^i\operatorname{ch}_i(E)$,

$$\operatorname{td}(X)=1+\frac{1}{2}\,c_1(X)+\frac{1}{12}\,(c_1(X)^2+c_2(X))+\cdots$$

is the Todd class of X, and $[\,\cdots\cdots]_{2d}$ means take the projection to the top degree component $H^{2d}(X,\mathbb{Q})=\mathbb{Q}$.

(c) **Serre duality.** There are functorial isomorphisms

$$\operatorname{Hom}^i_{D(X)}(E,F)\cong\operatorname{Hom}^{d-i}_{D(X)}(F,E\otimes\omega_X)^*$$

for all objects $E,F\in D(X)$. Here ω_X denotes the canonical line bundle of X, and $d=\dim_{\mathbb{C}}(X)$. If $E,F\in\operatorname{Coh}(X)$ this implies in particular that

$$\operatorname{Ext}^i_X(E,F)=0\text{ for }i>d.$$

Note that if X is Calabi-Yau (meaning that $\omega_X\cong\mathcal{O}_X$ is trivial) then the category $D(X)$ has the CY_d property:

$$\operatorname{Hom}^i_{D(X)}(E,F)\cong\operatorname{Hom}^{d-i}_{D(X)}(F,E)^*.$$

The Euler form $\chi(-,-)$ is then $(-1)^d$–symmetric.

Numerical Grothendieck group. Serre duality shows that the left- and right-kernels of the Euler form are the same: for a given class $\gamma \in K_0(X)$ we have

$$\chi(\alpha,\gamma) = 0 \quad \forall \alpha \in K_0(X) \iff \chi(\gamma,\beta) = 0 \quad \forall \beta \in K_0(X).$$

The numerical Grothendieck group is defined to be the quotient

$$\mathcal{N}(X) = K_0(X)/\ker \chi(-,-).$$

It is a finitely-generated free abelian group. Note that it is not clear that the Chern character descends to $\mathcal{N}(X)$ (this has to do with the standard conjectures), but this is certainly true for example when $\dim_{\mathbb{C}}(X) \leqslant 2$.

Serre functor. The functor $S_X \colon D(X) \to D(X)$ defined by

$$S_X(-) = (- \otimes \omega_X)[d]$$

is called the Serre functor. Serre duality may be trivially restated as the property that there are bifunctorial isomorphisms

$$\mathrm{Hom}_{D(X)}(E,F) \cong \mathrm{Hom}_{D(X)}(F, S_X(E))^*$$

for all objects $E, F \in D(X)$. It is easy to see using the Yoneda Lemma that this property determines S_X uniquely up to isomorphism of functors.

2.2. **Coherent sheaves.** Objects of $\mathrm{Coh}(X)$ can be thought of as 'vector bundles with varying fibres'. The fibre of $E \in \mathrm{Coh}(X)$ at a closed point $x \in X$ is

$$E_{(x)} = E_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}.$$

It is a simple consequence of Nakayama's Lemma that the subsets

$$S_i(E) = \{x \in X : \dim_{\mathbb{C}} E_{(x)} \geqslant i\} \subset X$$

are closed. Setting $V_i(E) = S_i(E) \setminus S_{i+1}(E)$ we get a stratification of X into disjoint, locally-closed subvarieties $V_i(E)$, such that each restriction $E|_{V_i(E)}$ is locally-free. In particular, given $E \in \mathrm{Coh}(X)$ the *support of* E is the closed subset

$$\mathrm{supp}(E) = S_0(E) \subset X$$

consisting of points where E has nonzero fibre. A sheaf E is *torsion-free* if $\mathrm{supp}(A) = X$ for all $A \subset E$. Note that a subsheaf of a torsion-free sheaf is automatically torsion-free.

To form non-stacky moduli spaces of coherent sheaves we must first restrict attention to a class of stable sheaves. There are several notions of stability, but for simplicity in what follows we will only consider μ-stability. To define this we must first fix a *polarization* of X: a class $\omega \in H^2(X, \mathbb{Z})$ which is the first Chern class of an ample line bundle. The *degree* of a sheaf E is then defined to be

$$d(E) = c_1(E) \cdot \omega^{d-1},$$

and the *slope* of a torsion-free sheaf is $\mu(E) = d(E)/r(E)$. A torsion-free sheaf is said to be *μ-semistable* if

$$0 \neq A \subsetneq E \implies \mu(A) \leqslant \mu(E).$$

Replacing the inequality with strict inequality gives the notion of *μ-stability*.

Theorem 2.1. (a) *Fix a Chern character v such that sheaves of this class have $r(E)$ and $d(E)$ coprime. Then there is a fine projective moduli scheme $\mathcal{M}_{X,\omega}(v)$ for μ-stable torsion-free sheaves of this class.*

(b) *Every torsion-free sheaf E has a unique Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

whose factors $F_i = E_i/E_{i-1}$ are μ-semistable with descending slopes:

$$\mu(F_1) > \mu(F_2) > \cdots > \mu(F_n).$$

(c) *If E and F are μ-semistable and $\mu(E) > \mu(F)$ then* $\mathrm{Hom}_X(E, F) = 0$.

(d) *If E and F are μ-stable of the same slope then any nonzero map $E \to F$ is an isomorphism.*

(e) *If E is μ-stable then* $\mathrm{End}_X(E) = \mathbb{C}$.

Proof. Part (a) comes from geometric invariant theory. The given assumptions ensure that for torsion-free sheaves of class v the notions of μ-stability and μ-semistability coincide, leading to a projective moduli space. They also ensure that this moduli space is fine. Part (b) is fairly easy. For (c) consider a nonzero map $f\colon E \to F$ and factor it via its image

$$0 \to K \hookrightarrow E \twoheadrightarrow I \hookrightarrow F \twoheadrightarrow Q \to 0.$$

Then $K = \ker(f)$ satisfies $\mu(K) < \mu(E)$, which by the additivity of rank and degree implies that $\mu(E) < \mu(I)$. On the other hand, $I = \mathrm{Im}(f)$ is a subsheaf of F and hence satisfies $\mu(I) \leqslant \mu(F)$. This implies that $\mu(E) < \mu(F)$, a contradiction. The same argument works for part (d). Part (e) then holds because $\mathrm{End}_X(E)$ is a finite-dimensional division algebra over $\mathbb{C}$. $\square$

2.3. **Derived category of $\mathbb{P}^1$.** By Exercise 1.5.3 every object in $D(\mathbb{P}^1)$ is a sum of its cohomology sheaves. Exercise 2.3.1 shows that any indecomposable sheaf is either a vector bundle or a fattened skyscraper. A well-known result (see Exercise 2.3.2) states that the only indecomposable vector bundles on $X = \mathbb{P}^1$ are the line bundles $\mathcal{O}(i)$ for $i \in \mathbb{Z}$.

We can represent the category $D(\mathbb{P}^1)$ graphically by drawing its Auslander-Reiten quiver: this has a vertex for each indecomposable object of $D(\mathbb{P}^1)$, and an arrow for each irreducible morphism (a morphism is called irreducible if it cannot be written as a composition $g \circ h$ with neither g nor h an isomorphism).

In fact the same category can be described in a different way. Consider the Kronecker quiver Q and the abelian category $\mathrm{Rep}(Q)$ of its finite-dimensional representations. It is easy enough to show that for all $n \geqslant 1$ there is a unique (up to isomorphism) indecomposable representation of Q of dimension vector $(n, n-1)$ and $(n-1, n)$, and a $\mathbb{P}^1$ worth of indecomposable representations of dimension vector (n, n). Categories of representations of quivers (without relations) always have global dimension 1, so Exercise 1.5.3 applies again, and we can draw the Auslander-Reiten quiver as before.

The pictures suggest that the categories $D(\mathbb{P}^1)$ and $D(Q)$ are equivalent. In fact, if we choose a basis for $\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}, \mathcal{O}(1)) \cong \mathbb{C}^2$ we can define a functor $F\colon \mathrm{Coh}(\mathbb{P}^1) \to \mathrm{Rep}(Q)$ by the rule

$$E \mapsto \big(\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(1), E) \Longrightarrow \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}, E)\big).$$

The associated derived functor is then an equivalence $D(\mathbb{P}^1) \to D(Q)$ which matches up the two pictures as in the diagram.

Note that the underived functor F is definitely not an equivalence since it kills the objects $\mathcal{O}(i)$ for $i < 0$. In fact, the abelian categories $\mathrm{Coh}(\mathbb{P}^1)$ and $\mathrm{Rep}(Q)$ are not equivalent: to see this note that the simple objects in $\mathrm{Coh}(\mathbb{P}^1)$ are the skyscraper sheaves $\mathcal{O}_x$ and the only finite-length objects are sheaves supported in dimension 0, whereas the category $\mathrm{Rep}(Q)$ is a finite-length category with only two simple objects $(0, 1)$ and $(1, 0)$ up to isomorphism.

We can use the above equivalence to identify the two derived categories and think of a single triangulated category D. But we then have two different abelian subcategories $\mathrm{Coh}(\mathbb{P}^1), \mathrm{Rep}(Q) \subset D$. There is an interesting auto-equivalence of D which corresponds to tensoring with $\mathcal{O}(1)$ in $D(\mathbb{P}^1)$. This auto-equivalence preserves the subcategory $\mathrm{Coh}(\mathbb{P}^1) \subset D$ but not $\mathrm{Rep}(Q) \subset$

D, illustrating the fact that the derived category of an abelian category can have extra symmetries not visible at the underived level.

Tilting objects. Let X be a smooth projective variety. An object $T \in D(X)$ is called a *tilting object* if

$$\operatorname{Ext}^i_X(T,T) = 0 \text{ unless } i = 0 \text{ and } \operatorname{Hom}^\bullet_X(T,E) = 0 \implies E \cong 0.$$

It follows that the (usually non-commutative) finite-dimensonal $\mathbb{C}$-algebra $A = \operatorname{End}_X(T)$ is of finite global dimension, and the derived functor

$$\mathbf{R}\operatorname{Hom}_X(T,-)\colon D^b(\operatorname{Coh}(X)) \to D^b(\operatorname{Mod}_{fg}(A))$$

is an equivalence. In the above example $T = \mathcal{O} \oplus \mathcal{O}(1)$, and A is the path algebra of Q. All known examples of tilting objects on smooth projective varieties are of the form $T = \bigoplus_i E_i$ for some exceptional collection $(E_1, \cdots, E_n)$. It is not known whether this is always the case.

Problems.

2.3.1. Let X be a curve. Prove that any indecomposable object $E \in \operatorname{Coh}(X)$ is either locally-free, or is of the form $\mathcal{O}_{nx}$ for some $x \in X$ and $n \geqslant 1$.

2.3.2. Prove that every indecomposable vector bundle on $X = \mathbb{P}^1$ is a line bundle as follows. First prove using the Harder-Narasimhan filtration and Serre duality that every indecomposable vector bundle is stable. Next use Serre duality to show that any stable vector bundle E is rigid, i.e. satisfies $\operatorname{Ext}^1_X(E,E) = 0$. Finally use Riemann-Roch to get the result.

2.3.3. Suppose that X is an elliptic curve and $E \in \operatorname{Coh}(X)$ is locally-free. Prove that

$$E \ \mu\text{-stable} \implies E \text{ indecomposable} \implies E \ \mu\text{-semistable}.$$

Conclude that if $\operatorname{ch}(E) = (r,d)$ with $\gcd(r,d) = 1$ then all three notions coincide.

2.3.4. Let $\mathcal{M}_X(2,1)$ be the moduli space of indecomposable vector bundles on an elliptic curve X of rank 2 and degree 1. Prove that $\mathcal{M}_X(2,1) \cong X$ by showing that every such bundle is an extension of line bundles of degrees 0 and 1 respectively.

3. Fourier-Mukai transforms

In this lecture we introduce integral functors and state the famous Bondal-Orlov theorem, which gives a criterion for when such a functor is an equivalence.

3.1. **Integral functors.** Let X, Y be smooth projective varitieties. For each $y \in Y$ we denote by $i_y \colon X \hookrightarrow Y \times X$ is the inclusion $x \mapsto (y, x)$. We can view an object $\mathcal{P} \in D(Y \times X)$ as defining a family of objects

$$\mathcal{P}_y = \mathbf{L}i_y^*(\mathcal{P}) \in D(X)$$

parameterised by $y \in Y$. Here $\mathbf{L}i_y^*$ denotes the left derived functor of the right exact functor i_y^*. Of course this is very familiar when $\mathcal{P}$ is a locally-free sheaf.

Lemma 3.1. *The objects $\mathcal{P}_y \in D(X)$ are all sheaves (i.e. they are all concentrated in degree 0), precisely if $\mathcal{P} \in D(Y \times X)$ is a sheaf, flat over Y.*

Proof. One implication is easy: if $\mathcal{P}$ is a Y-flat sheaf, then $\mathcal{P}_y$ is just the usual restricted sheaf $\mathcal{P}|_{\{y\}\times X}$. In the other direction let us assume that all the objects $\mathcal{P}_y \in D(X)$ are concentrated in degree 0. Let n be the maximum integer such that $H^n(\mathcal{P}) \neq 0$. Consider the triangle in $D(Y \times X)$

$$\tau_{\leqslant n-1}(\mathcal{P}) \longrightarrow \mathcal{P} \longrightarrow H^n(\mathcal{P})[-n].$$

Applying the derived functor $\mathbf{L}i_y^*(-)$ gives a triangle

$$\mathbf{L}i_y^*(\tau_{\leqslant n-1}(\mathcal{P})) \longrightarrow \mathcal{P}_y \longrightarrow \mathbf{L}i_y^*(H^n(\mathcal{P}))[-n] \tag{6}$$

in $D(X)$. Since $\mathbf{L}i_y^*$ is a left derived functor, it 'spreads things out to the left', so the first term is concentrated in degrees $\leqslant n-1$. Taking the long exact sequence in cohomology we get $H^n(\mathcal{P}_y) = i_y^*(H^n(\mathcal{P}))$ which by assumption on n is non-zero for some $y \in Y$. Since $\mathcal{P}_y$ is assumed to be concentrated in degree 0 we conclude that $n = 0$.

Taking cohomology of (6) again we see that $H^{-1}(\mathbf{L}i_y^*(H^0(\mathcal{P})) = 0$ for all $y \in Y$, which by the local criterion of flatness tells us that $H^0(\mathcal{P})$ is flat over Y. We now know that the last two terms in (6) are concentrated in degree 0. Since the first one is concentrated in degrees $\leqslant -1$ it must be zero. It follows that $\tau_{\leqslant -1}(\mathcal{P}) = 0$ which shows that $\mathcal{P}$ is concentrated in degree 0. $\square$

Remark 3.2. The same argument gives a local version of this statement: if some particular $\mathcal{P}_y$ is concentrated in degree 0, then for all $x \in X$, the stalk of $H^i(\mathcal{P})$ at (x, y) is zero for $i \neq 0$, and flat over $\mathcal{O}_{Y,y}$ for $i = 0$.

Define projection maps

$$Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$$

and consider the functor

$$\Phi^{\mathcal{P}}_{Y\to X}(-) = \mathbf{R}\pi_{X,*}(\mathcal{P}\otimes \pi_Y^*(-)).$$

Note that we do not need to derive π_Y^* because it is an exact functor (the projection map is flat). The $-\otimes -$ means the tensor product in $D(X)$ which is computed by first replacing the objects by quasi-isomorphic complexes of locally-free sheaves.

Lemma 3.3. *We have the relation*

$$\Phi^{\mathcal{P}}_{Y\to X}(\mathcal{O}_y) = \mathcal{P}_y.$$

Proof. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "i_y"] \arrow[d, "p"'] & X\times Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \\ \{y\} \arrow[r, "j_y"] & Y & \end{tikzcd}$$

First use base-change around the Cartesian square

$$\pi_Y^*(\mathcal{O}_y) = \pi_Y^*(j_{y,*}(\mathcal{O})) \cong i_{y,*}(p^*(\mathcal{O})) = i_{y,*}(\mathcal{O}_X).$$

Note that these functors are all exact. Now use the projection formula

$$\mathcal{P}\otimes i_{y,*}(\mathcal{O}_X) \cong i_{y,*}(\mathbf{L}i_y^*(\mathcal{P})\otimes \mathcal{O}_X) \cong i_{y,*}(\mathcal{P}_y).$$

Finally, using the fact that $\pi_X\circ i_y \cong \mathrm{id}_X$, we get

$$\Phi^{\mathcal{P}}_{Y\to X}(\mathcal{O}_y) = \mathbf{R}\pi_{X,*}(i_{y,*}(\mathcal{P}_y)) \cong \mathcal{P}_y.$$

which completes the proof. □

A functor $\Phi\colon D(Y)\to D(X)$ isomorphic to one of the form $\Phi^{\mathcal{P}}_{Y\to X}$ is called an *integral functor*. Such functors are very important due to

Theorem 3.4 (Orlov)**.** *If X and Y are smooth projective varieties then any triangulated equivalence $\Phi\colon D(Y)\to D(X)$ is an integral functor.* □

3.2. **The Bondal-Orlov theorem.** The following very useful result allows us to write down many examples of varieties with equivalent derived categories.

Theorem 3.5 (Bondal, Orlov)**.** *Let X and Y be smooth projective varieties. An integral functor $\Phi\colon D(Y)\to D(X)$ is an equivalence if and only if*

(a) $\mathrm{Hom}^i_{D(X)}(\Phi(\mathcal{O}_{y_1}),\Phi(\mathcal{O}_{y_2}))=0$ *unless* $y_1=y_2$ *and* $0\leqslant i\leqslant \dim(Y)$,

(b) $\mathrm{Hom}_{D(X)}(\Phi(\mathcal{O}_y),\Phi(\mathcal{O}_y))=\mathbb{C}$,

(c) $\Phi(\mathcal{O}_y) \otimes \omega_X \cong \Phi(\mathcal{O}_y)$.

One can easily check that the conditions of Theorem 3.5 are necessary. Indeed, using a Koszul resolution, one can compute

$$\operatorname{Ext}^i_Y(\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) = \begin{cases} \bigwedge^i \mathbb{C}^d, & \text{if } y_1 = y_2 \\ 0 & \text{otherwise.} \end{cases}$$

If Φ is an equivalence commuting with the shift functors then it preserves the $\operatorname{Hom}^i(-,-)$ spaces so (a) and (b) must hold. For (c) note that an equivalence must intertwine the Serre functors on $D(Y)$ and $D(X)$ since these are uniquely defined by categorical conditions. Since $\mathcal{O}_y \otimes \omega_X \cong \mathcal{O}_y$ it follows that the objects $\Phi(\mathcal{O}_y)$ must also be invariant under $-\otimes \omega_X$ up to shift. It follows that they are in fact invaraint under $-\otimes \omega_X$ and that moreover X and Y must have the same dimension.

Example 3.6. Let X be an abelian variety, and let $Y = \operatorname{Pic}^0(X)$ be the dual abelian variety. By definition Y parameterizes line bundles L on X with $c_1(L) = 0$. There is a universal object $\mathcal{P}$ on $Y \times X$ called the Poincaré line bundle. The resulting functor $\Phi^{\mathcal{P}}_{Y\to X}$ is called the Fourier-Mukai transform; it was the first non-trivial example of an equivalence betwen derived categories of coherent sheaves.

The conditions (b) and (c) of Theorem 3.5 are immediate in this example. To check (a) one needs to know a non-trivial fact, namely that if $L \in \operatorname{Pic}^0(X)$ is non-trivial then $H^i(X, L) = 0$ for all i. In the dimension one case when X is an elliptic curve this is easy: $H^0(X, L) = 0$ because any nonzero section $\mathcal{O}_X \to L$ would have to be an isomorphism, and Serre duality then implies that also $H^1(X, L) = 0$.

Example 3.7. Take an isomorphism of smooth projective varieties $f\colon Y \to X$, a line bundle $L \in \operatorname{Pic}(Y)$ and an integer $n \in \mathbb{Z}$. Then the functor

$$F(-) = f_*(L \otimes -)[n].$$

is a Fourier-Mukai equivalence $F\colon D(Y) \to D(X)$. Functors of this form are called standard equivalences.

The following result gives a useful characterisation of standard equivalences.

Lemma 3.8. *Suppose $\Phi\colon D(Y) \to D(X)$ is a triangulated equivalence. Then Φ is a standard equivalence precisely if for every point $y \in Y$ the object $\Phi(\mathcal{O}_y) \in D(X)$ is a shift of a skyscraper sheaf.*

Proof. One implication is easily checked so let us assume that Φ takes skyscrapers to shifts of skyscrapers. We can write $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$ for some object $\mathcal{P}$. By assumption, the object $\mathcal{P}_y = \mathbf{L}i_y^*(\mathcal{P})$ is concentrated in some degree for each $y \in Y$. By Remark 3.2 it follows that $\mathcal{P}$ is concentrated in a fixed degree in a neighbourhood of each point, and since Y is connected it follows that this degree is constant, and by composing Φ with a shift we can therefore assume that $\mathcal{P}$ is a sheaf, flat over Y.

Now, by Exercise 3.4.4, X is a fine moduli space for skyscraper sheaves on X, so there is a morphism $f\colon Y \to X$ and a line bundle $L \in \mathrm{Pic}(Y)$ such that

$$\mathcal{P} \cong (f \times \mathrm{id}_X)^*(\mathcal{O}_\Delta) \otimes \pi_Y^*(L) = \mathcal{O}_{\Gamma(f)} \otimes \pi_Y^*(L)$$

where $\Gamma_f \subset Y \times X$ is the graph of f. It follows easily that $\Phi(-) \cong f_*(L \otimes -)$. Since Φ is an equivalence it follows that f is an isomorphism. $\square$

3.3. **Auto-equivalences.** As well as looking for varieties with equivalent derived categories, it is interesting to study self-equivalences of derived categories of coherent sheaves. We denote by $\mathrm{Aut}\, D(X)$ the group of $\mathbb{C}$-linear, triangulated auto-equivalences of the category $D(X)$, these being considered up to isomorphism of functors.

The *standard auto-equivalences* define a subgroup

$$\mathrm{Aut}_{\mathrm{stand}}\, D(X) = \mathbb{Z} \times \mathrm{Aut}(X) \ltimes \mathrm{Pic}(X) \subset \mathrm{Aut}\, D(X).$$

The following result shows that in many interesting cases these are all auto-equivalences.

Lemma 3.9. *Suppose $\omega_X^{\pm 1}$ is ample and $\Phi\colon D(Y) \to D(X)$ is a triangulated equivalence. Then $Y \cong X$ and Φ is a standard equivalence.*

Proof. Set $\mathcal{P}_y = \Phi(\mathcal{O}_y)$. By condition (c) of the Theorem we have $\mathcal{P}_y \otimes \omega_X = \mathcal{P}_y$. This implies the same for each $H^i(\mathcal{P}_y)$. But by the condition, the only sheaves invariant under $- \otimes \omega_X$ are zero-dimensional. Since $\mathrm{End}_X(\mathcal{P}_y) = \mathbb{C}$ it is indecomposable so we conclude that each $H^i(\mathcal{P}_y)$ is supported at the same point $x \in X$. Note that any two such sheaves $E, F \in \mathrm{Coh}(X)$ there are nonzero maps $E \to F$, because any such sheaf has a filtration with factors $\mathcal{O}_x$. Now consider the spectral sequence

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}_X^p(H^i(\mathcal{P}_y), H^{i+q}(\mathcal{P}_y)) \implies \mathrm{Ext}_X^{p+q}(\mathcal{P}_y, \mathcal{P}_y). \tag{7}$$

Since Φ is an equivalence this gives zero unless $0 \leqslant p + q \leqslant d$. If w is the maximum integer such that there is an $i \in \mathbb{Z}$ with $H^i(E)$ and $H^{i+w}(E)$ both nonzero, then we get a nontrivial term $E^{0,-w}$-term in this spectral sequence which survives to ∞. Thus $w = 0$ and $\mathcal{P}_y$ is concentrated in a fixed degree. Since the class of $\Phi(\mathcal{O}_y)$ in $\mathcal{N}(X)$ must be primitive it follows that $\mathcal{P}_y$ is a shift of a skyscraper. □

3.4. **Problems.**

3.4.1. *Adjoints of integral functors.* Using standard adjunctions from algebraic geometry calculate the left and right adjoints to the functor $\Phi^{\mathcal{P}}_{Y\to X}$. Use your answer to give another proof that smooth projective varieties with equivalent derived categories have the same dimension.

3.4.2. Suppose that $\mathcal{P}$ is a sheaf on $Y \times X$, flat over Y, and set $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$. Using the cohomology and base-change theorem, show that for any sheaf $E \in \mathrm{Coh}(Y)$ and any ample line bundle L, the image $\Phi(E \otimes L^n)$ is a locally-free sheaf for $n \gg 0$.

3.4.3. Prove that there is a well-defined functor

$$\mathrm{FM}\colon D(Y \times X) \longrightarrow \mathrm{Fun}(D(Y), D(X)), \quad \mathcal{P} \mapsto \Phi^{\mathcal{P}}_{Y\to X}.$$

Show that this functor is not in general faithful, as follows. Take $Y = X$ an elliptic curve and prove that any morphism of functors $\mathrm{id} \to [2]$ is zero. On the other hand show using Serre duality that $\mathrm{Ext}^2_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \mathbb{C}$.

3.4.4. *Moduli of skyscrapers.* Prove that the moduli space of skyscraper sheaves on a smooth variety X is the variety X itself, and that the universal object can be taken to be the structure sheaf of the diagonal in $X \times X$.

3.4.5. *Integral transforms preserve families.* Let S be an arbitrary variety, and Y, X smooth, projective varieties as usual. An object $E \in D(S \times Y)$ is said to be S-perfect if the derived restrictions $E_s = E|_{\{s\}\times Y}$ all have bounded cohomology objects and hence live in $D(Y)$. Suppose that $\Phi\colon D(Y) \to D(X)$ is an integral functor. By using a relative integral functor defined by the projections $S \times Y \leftarrow S \times Y \times X \to S \times X$, prove that if $E \in D(S \times Y)$ is S-perfect then there is an S-perfect object $F \in D(S \times X)$ such that $F_s = \Phi(E_s)$ for all $s \in S$.

4. Calabi-Yau examples

This lecture is devoted to working out some of the general theory considered above in the case of low-dimensional Calabi-Yau varieties, namely elliptic curves and K3 surfaces.

4.1. **Elliptic curves.** In this section we shall prove

Theorem 4.1. *Let X be a smooth projective curve of genus 1. Then $D(Y) \cong D(X)$ implies that $Y \cong X$, and moreover there is a short exact sequence*

$$1 \longrightarrow \operatorname{Aut}(X) \ltimes \operatorname{Pic}^0(X) \times \mathbb{Z} \longrightarrow \operatorname{Aut} D(X) \longrightarrow \mathrm{SL}(2,\mathbb{Z}) \longrightarrow 1. \tag{8}$$

Proof. The Chern character map descends to the numerical Grothendieck group and gives an isomorphism

$$\mathrm{ch}\colon \mathcal{N}(X) \to \mathbb{Z} \oplus \mathbb{Z}, \quad [E] \mapsto (r(E), d(E)).$$

Riemann-Roch shows that the Euler form is

$$\chi(E,F) = r(E)\,d(F) - r(F)\,d(E).$$

Any triangulated auto-equivalence of $D(X)$ induces an automorphism of $\mathcal{N}(X)$ preserving the Euler form, so we get a group homomorphism

$$\varpi\colon \operatorname{Aut} D(X) \to \mathrm{SL}(2,\mathbb{Z}).$$

Our first aim is to show that this map is surjective.

The dual abelian variety $Y = \operatorname{Pic}^0(X)$ is non-canonically isomorphic to X, by mapping $x \mapsto \mathcal{O}_X(x - x_0)$ for some base-point $x_0 \in X$. The original Fourier-Mukai transform therefore gives an auto-equivalence $\Phi \in \operatorname{Aut} D(X)$. This satisfies $\Phi(\mathcal{O}_y) = \mathcal{P}_y$. By Exercise 3.4.1, the inverse is given by $\Phi^{\mathcal{P}^*}_{X\to Y}[1]$ and so $\Phi(\mathcal{P}^*_y) = \mathcal{O}_y[1]$. We conclude that

$$\varpi(\Phi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Tensoring with a degree 1 line bundle L gives another auto-equivalence, which clearly satisfies

$$\varpi(- \otimes L) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Since these two matrices generate $\mathrm{SL}(2,\mathbb{Z})$ the map ϖ is indeed surjective.

Consider an auto-equivalence Φ lying in the kernel of ϖ. It must take $\mathcal{O}_x$ to an indecomposable object of class $(0,1)$. Up to shift such an object is a sheaf. But then it must be a skyscraper. So any such autoequivalence is standard. Conversely a standard autoequivalence acts trivially on $\mathcal{N}(X)$ precisely if the line bundle L has degree 0 and the shift is even. This gives the above short exact sequence.

Finally we prove the first part of the statement. Suppose $\Phi\colon D(Y) \to D(X)$ is an equivalence. Then Φ induces an isomorphism of numerical Grothendieck groups; in particular Φ takes skyscrapers to indecomposable objects having some primitive class $(a,b) \in \mathcal{N}(X)$. Composing with an element of $\operatorname{Aut} D(X)$ we can assume that $(a,b) = (0,1)$. But any indecomposable object of this class is a shift of a skyscraper. Thus Φ is standard, and in particular, $Y \cong X$. $\square$

4.2. **K3 surfaces.** Recall that a K3 surface is a smooth surface which is Calabi-Yau ($\omega_X = \mathcal{O}_X$) and satisfies $H^1(X, \mathcal{O}_X) = 0$. It is an important fact that all such surfaces are deformation equivalent as complex manifold and hence have the same cohomology groups. In particular $H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{\oplus 22}$. The Hodge decomposition takes the form

$$H^2(X,\mathbb{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X).$$

where $H^{2,0}(X) = H^0(\omega_X) = \mathbb{C}$. The key point is that the isomorphism class of a K3 surface is completely determined by the position of the line $H^{2,0}(X)$ in the complexification of the lattice $H^2(X,\mathbb{Z})$. This is called the Torelli theorem:

Theorem 4.2. *Two K3 surfaces are isomorphic if they are Hodge isometric, i.e. if there is an isomorphism*

$$\phi\colon H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$$

such that ϕ preserves the intersection form and

$$(\phi \otimes \mathbb{C})\,(H^{2,0}(X_1)) = H^{2,0}(X_2).$$

It is useful to introduce a minor variant of the Chern character called the *Mukai vector*

$$v\colon K(X) \to H^*(X,\mathbb{Z})$$

$$v(E) = \operatorname{ch}(E) \cdot \sqrt{\operatorname{td}(X)} = (r(E), \operatorname{ch}_1(E), \operatorname{ch}_2(E) + r(E)).$$

We put a symmetric form on

$$H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^{22} \oplus \mathbb{Z}$$

by setting

$$\langle (r_1, d_1, s_1), (r_2, d_2, s_2) \rangle = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

The Riemann-Roch theorem now takes the simple form

$$\chi(E,F) = -\langle v(E), v(F) \rangle.$$

The map v descends to the numerical Grothendieck group, and allows us to identify $\mathcal{N}(X)$ with the image of this map.

Theorem 4.3. *Fix $v = (r, D, s) \in \mathcal{N}(X)$ with $r > 0$. Suppose there is a polarization ω such that $\gcd(r, D \cdot \omega) = 1$. Then the moduli space $\mathcal{M}_{X,\omega}(v)$ is a non-empty, smooth, complex symplectic, projective variety of dimension $2 + \langle v, v\rangle$.*

Proof. The non-emptiness statement is tricky: one has to consider a deformation to an elliptic K3. For the rest, recall that the tangent space to the moduli space of sheaves at a point E is given by $\mathrm{Ext}^1_X(E,E)$. In our case Riemann-Roch gives

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_X(E,E) - \dim_{\mathbb{C}} \mathrm{Ext}^0_X(E,E) - \dim_{\mathbb{C}} \mathrm{Ext}^2_X(E,E) = \langle v, v\rangle.$$

Since E is stable, $\mathrm{End}_X(E) = \mathbb{C}$ and Serre duality gives $\mathrm{Ext}^2_X(E,E) \cong \mathrm{Hom}_X(E,E)^*$. Thus the tangent space to $\mathcal{M}_{X,\omega}(v)$ has constant dimension and hence the space is smooth. The symplectic form given by the Serre duality pairing $\mathrm{Ext}^1_X(E,E) \cong \mathrm{Ext}^1_X(E,E)^*$. $\square$

These (and deformations of them) are basically the only known examples of compact complex symplectic manifolds. Suppose now that $(v, v) = 0$ and ω can be chosen as in the statement of the Theorem. Then $Y = \mathcal{M}_{X,\omega}(v)$ is a smooth projective surface.

Lemma 4.4. *The surface Y is a K3 surface and the functor $\Phi\colon D(Y) \to D(X)$ defined by the universal object $\mathcal{P}$ on $Y \times X$ is an equivalence.*

Proof. By the Bondal-Orlov theorem, to check that Φ is an equivalence we just have to check that if $y_1 \neq y_2$ are distinct points of Y then $\mathrm{Ext}^i_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0$ for all i. There are no maps in degree 0 since these are distinct stable sheaves of the same slope. Serre duality then shows that there are no maps in degree 2. Since $\chi(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = -\langle v, v\rangle = 0$ this is enough.

Since any equivalence commutes with Serre functors it is clear that $\omega_Y \cong \mathcal{O}_Y$ (this also follows from the fact that Y is complex symplectic). To show that Y is a K3 surface we must check that $H^1(Y, \mathcal{O}_Y) = 0$. If we take a sufficiently ample line bundle L on X then

$$\mathrm{Hom}^i_Y(\Phi^{-1}(L^*), \mathcal{O}_y) = \mathrm{Hom}^i_X(L^*, \mathcal{P}_y) = H^i(X, \mathcal{P}_y \otimes L)$$

which is concentrated in degree 0. It follows that $\Phi^{-1}(L^*) = M$ is a vector bundle on Y. Now L (and hence also M) is rigid:

$$\mathrm{Ext}^1_Y(M,M) = \mathrm{Ext}^1_X(L,L) = H^1(X, \mathcal{O}_X) = 0.$$

But for any vector bundle on Y the obvious map $\mathcal{O}_Y \to \mathcal{H}om_{\mathcal{O}_Y}(M,M)$ is split by the trace map, which implies that $H^1(Y,\mathcal{O}_Y)$ is a summand of $\mathrm{Ext}^1_Y(M,M)$. □

We have now proved one of the implications in the following derived Torelli result.

Theorem 4.5 (Mukai, Orlov)**.** *Let X, Y be K3 surfaces. Then the following are equivalent*

(a) *There is a $\mathbb{C}$-linear triangulated equivalence $D(Y) \cong D(X)$,*
(b) *$Y \cong \mathcal{M}_{X,\omega}(v)$ is a fine moduli space of μ-stable vector bundles on X,*
(c) *There is a Hodge isometry*

$$H^*(Y,\mathbb{Z}) \cong H^*(X,\mathbb{Z})$$

This means an isomorphism of groups which preserves the form $\langle -,-\rangle$ and whose complexification takes $H^{0,2}(Y) \subset H^*(Y,\mathbb{Z})$ to $H^{0,2}(X) \subset H^*(X,\mathbb{Z})$.

Proof. Above, we proved (b) $\implies$ (a). To get (a) $\implies$ (c) one must show that a Foruier-Mukai equivalence induces an isomorphism on the full cohomology groups (not just the numerical Grothendieck groups, which are the algebraic part). This can be done by hand in a slightly ad hoc way; a better approach would be via periodic cyclic homology, but this seems difficult.

The implication (c) $\implies$ (a) goes as follows. Let $p = (0,0,1)$ denote the Mukai vector of a skyscraper sheaf. Given an isomorphism $\psi\colon H^*(Y,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ of the required type, put $v = \psi(p)$. Then v is algebraic and integral, and hence defines a primitive class in $\mathcal{N}(X)$. With a bit of jiggery-pokery involving known auto-equivalences of $D(X)$ we can even assume that $v = (r,D,s)$ is such that $r > 0$ and there exists a polarization ω with $\gcd(r, D\cdot\omega) = 1$. (If we use Gieseker stability instead of slope stability we don't need this last condition). Let $Z = \mathcal{M}_{X,\omega}(v)$ be the resulting fine moduli space, and $\Phi\colon D(Z) \to D(X)$ the corresponding equivalence. Let $\phi\colon H^*(Z,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ be the induced Hodge isometry; by definition it takes p to v. Now $\psi^{-1}\circ\phi$ is a Hodge isometry $H^*(Z,\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ which preserves the class p. But since $p^{\perp}/\mathbb{Z}\cdot p = H^2(X,\mathbb{Z})$ this then induces a Hodge isometry $H^2(Z,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$. The usual Torelli theorem then implies $Y \cong Z$ and we are done. □

4.3. **Problems.**

4.3.1. *Moduli of bundles on an elliptic curve.* Let $\mathcal{M}_X(r,d)$ denote the moduli space of indecomposable vector bundles on an elliptic curve X of rank r and degree d. Prove that $\mathcal{M}_X(r,d) \cong X$. (If you are being careful about moduli spaces you might need Exercise 3.4.5).

4.3.2. *Auto-equivalences of an abelian surface.* Let X be an abelian surface. This is a smooth projective surface with $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = \mathbb{C}^2$.

(a) Prove that any nonzero object $E \in \operatorname{Coh}(X)$ satisfies $\dim_{\mathbb{C}} \operatorname{Ext}^1_X(E,E) \geqslant 2$. (When E is a vector bundle use the argument from the proof of Lemma 4.4, in general use the Fourier-Mukai trandsform and Exercise 3.4.2).

(b) Use the spectral sequence (7) to show that if $\Phi\colon D(Y) \to D(X)$ is an equivalence then it takes skyscraper sheaves to shifts of sheaves.

(c) Show that any auto-equivalence which acts trivially on $\mathcal{N}(X)$ is standard and hence determine the kernel of the map $\operatorname{Aut} D(X) \to \operatorname{Aut} \mathcal{N}(X)$.

4.3.3. *Reflection functor.* Let X be a K3 surface and let $\mathcal{I}_\Delta \in \operatorname{Coh}(X \times X)$ denote the ideal sheaf of the diagonal $\Delta \subset X \times X$.

(a) Prove that $\Phi^{\mathcal{I}_\Delta}_{X \to X}$ defines an auto-equivalence $\Phi \in \operatorname{Aut} D(X)$.

(b) Prove that for any $E \in D(X)$ there is a distinguished triangle

$$\bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{D(X)}(\mathcal{O}_X, E) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow E \longrightarrow \Phi(E).$$

(c) Let $\phi \in \operatorname{Aut} K_0(X)$ be the effect of $D(X)$ on the Grothendieck group, and set $q = [\mathcal{O}_X]$. Show that

$$\phi(v) = v - \chi(q,v)q.$$

Conclude that $\Phi^2 \in \operatorname{Aut} D(X)$ is a non-standard auto-equivalence which acts trivially on $K_0(X)$.