De Moivre's Theorem 10.4

Introduction

In this Section we introduce De Moivre's theorem and examine some of its consequences. We shall see that one of its uses is in obtaining relationships between trigonometric functions of multiple angles (like $\sin 3x$, $\cos 7x$) and powers of trigonometric functions (like $\sin^2 x$, $\cos^4 x$). Another important use of De Moivre's theorem is in obtaining complex roots of polynomial equations. In this application we re-examine our definition of the argument arg(z) of a complex number.

Prerequisites

Before starting this Section you should ...

- be familiar with the polar form of a complex number
- be familiar with the Argand diagram
- be familiar with the trigonometric identity $\cos^2\theta + \sin^2\theta \equiv 1$
- know how to expand $(x+y)^n$ when n is a positive integer

Learning Outcomes

On completion you should be able to ...

- employ De Moivre's theorem in a number of applications
- fully define the argument arg(z) of a complex number
- obtain complex roots of complex numbers

1. De Moivre's theorem

We have seen, in Section 10.2 Key Point 7, that, in polar form, if $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and $w = t(\cos\phi + \mathrm{i}\sin\phi)$ then the product zw is:

$$zw = rt(\cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi))$$

In particular, if $r = 1, t = 1$ and $\theta = \phi$ (i.e. $z = w = \cos\theta + \mathrm{i}\sin\theta$), we obtain

$$(\cos\theta + \mathrm{i}\sin\theta)^2 = \cos 2\theta + \mathrm{i}\sin 2\theta$$

Multiplying each side of the above equation by $\cos\theta + \mathrm{i}\sin\theta$ gives

$$(\cos\theta + \mathrm{i}\sin\theta)^3 = (\cos 2\theta + \mathrm{i}\sin 2\theta)(\cos\theta + \mathrm{i}\sin\theta) = \cos 3\theta + \mathrm{i}\sin 3\theta$$

on adding the arguments of the terms in the product.
Similarly

$$(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta.$$

After completing p such products we have:

$$(\cos\theta + \mathrm{i}\sin\theta)^p = \cos p\theta + \mathrm{i}\sin p\theta$$

where p is a positive integer.
In fact this result can be shown to be true for those cases in which p is a negative integer and even when p is a rational number e.g. $p = \frac{1}{2}$.

Key Point 12

If p is a rational number:

$$(\cos\theta + \mathrm{i}\sin\theta)^p \equiv \cos p\theta + \mathrm{i}\sin p\theta$$

This result is known as **De Moivre's theorem**.

Recalling from Key Point 8 that $\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}$, De Moivre's theorem is simply a statement of the laws of indices:

$$(\mathrm{e}^{\mathrm{i}\theta})^p = \mathrm{e}^{\mathrm{i}p\theta}$$

2. De Moivre's theorem and root finding

In this subsection we ask if we can obtain fractional powers of complex numbers; for example what are the values of $8^{1/3}$ or $(-24)^{1/4}$ or even $(1+\mathrm{i})^{1/2}$?
More precisely, for these three examples, we are asking for those values of z which satisfy

$$z^3 - 8 = 0 \qquad \text{or} \qquad z^4 + 24 = 0 \qquad \text{or} \qquad z^2 - (1+\mathrm{i}) = 0$$

Each of these problems involve finding roots of a complex number.
To solve problems such as these we shall need to be more careful with our interpretation of arg(z) for a given complex number z.

Arg(z) revisited

By definition arg(z) is the angle made by the line representing z with the positive x-axis. See Figure 9(a). However, as the Figure 9(b) shows you can increase θ by 2π (or 360^0) and still obtain the *same* line in the xy plane. In general, as indicated in Figure 9(c) any integer multiple of 2π can be added to or subtracted from arg(z) without affecting the **Cartesian form** of the complex number.

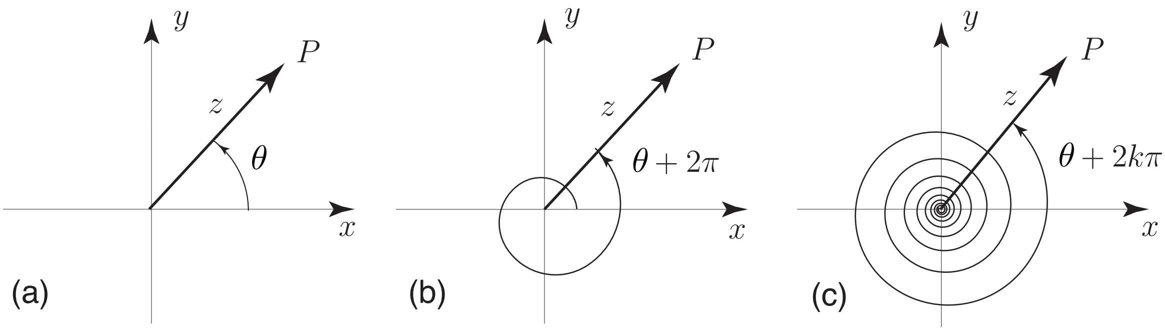

Figure 9

Key Point 13

arg(z) is unique only up to an integer multiple of 2π radians

For example:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}) \qquad \text{in polar form}$$

However, we could also write, equivalently:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2\pi))$$

or, in full generality:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2k\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2k\pi)) \qquad k = 0, \pm 1, \pm 2, \cdots$$

This last expression shows that in the polar form of a complex number the argument of z, arg(z), can assume infinitely many different values, each one differing by an integer multiple of 2π. This is nothing more than a consequence of the well-known properties of the trigonometric functions:

$$\cos(\theta + 2k\pi) \equiv \cos\theta, \qquad \sin(\theta + 2k\pi) \equiv \sin\theta \qquad \text{for } \textit{any} \text{ integer } k$$

We shall now show how we can use this more general interpretation of arg(z) in the process of finding roots.

Example 8

Find all the values of $8^{1/3}$.

Solution

Solving $z = 8^{1/3}$ for z is equivalent to solving the cubic equation $z^3 - 8 = 0$. We expect that there are three possible values of z satisfying this cubic equation. Thus, rearranging: $z^3 = 8$. Now write the right-hand side as a complex number in polar form:

$$z^3 = 8(\cos 0 + \mathsf{i} \sin 0)$$

(i.e. $r = |8| = 8$ and arg$(8) = 0$). However, if we now generalise our expression for the argument, by adding an arbitrary integer multiple of 2π, we obtain the modified expression:

$$z^3 = 8(\cos(2k\pi) + \mathsf{i}\sin(2k\pi)) \qquad k = 0, \pm 1, \pm 2, \cdots$$

Now take the cube root of both sides:

$$\begin{aligned} z &= \sqrt[3]{8}(\cos(2k\pi) + \mathsf{i}\sin(2k\pi))^{\frac{1}{3}} \\ &= \sqrt[3]{8}(\cos\frac{2k\pi}{3} + \mathsf{i}\sin\frac{2k\pi}{3}) \qquad \text{using De Moivre's theorem.} \end{aligned}$$

Now in this expression k can take any integer value or zero. The normal procedure is to take three consecutive values of k (say $k = 0, 1, 2$). Any other value of k chosen will lead to a root (a value of z) which repeats one of the three already determined.

$$\begin{aligned} \text{So if} \quad k &= 0 \qquad z_0 = 2(\cos 0 + \mathsf{i}\sin 0) = 2 \\ k &= 1 \qquad z_1 = 2(\cos\frac{2\pi}{3} + \mathsf{i}\sin\frac{2\pi}{3}) = -1 + \mathsf{i}\sqrt{3} \\ k &= 2 \qquad z_2 = 2(\cos\frac{4\pi}{3} + \mathsf{i}\sin\frac{4\pi}{3}) = -1 - \mathsf{i}\sqrt{3} \end{aligned}$$

These are the three (complex) values of $8^{\frac{1}{3}}$. The reader should verify, by direct multiplication, that $(-1 + \mathsf{i}\sqrt{3})^3 = 8$ and that $(-1 - \mathsf{i}\sqrt{3})^3 = 8$.

The reader may have noticed within this Example a subtle change in notation. When, for example, we write $8^{1/3}$ then we are expecting **three** possible values, as calculated above. However, when we write $\sqrt[3]{8}$ then we are only expecting **one** value: that delivered by your calculator.

Note the two complex roots are complex conjugates (since $z^3 - 8 = 0$ is a polynomial equation with real coefficients).

In Example 8 we have worked with the polar form. Precisely the same calculation can be carried through using the exponential form of a complex number. We take this opportunity to repeat this calculation but working exclusively in exponential form.
Thus

$$\begin{aligned} z^3 &= 8 \\ &= 8\mathrm{e}^{\mathrm{i}(0)} \qquad (\text{i.e. } r = |8| = 8 \quad \text{and} \quad \arg(8) = 0) \\ &= 8\mathrm{e}^{\mathrm{i}(2k\pi)} \qquad k = 0, \pm 1, \pm 2, \cdots \end{aligned}$$

therefore taking cube roots

$$\begin{aligned} z &= \sqrt[3]{8}\left[\mathrm{e}^{\mathrm{i}(2k\pi)}\right]^{\frac{1}{3}} \\ &= \sqrt[3]{8}\mathrm{e}^{\frac{\mathrm{i}2k\pi}{3}} \qquad \text{using De Moivre's theorem.} \end{aligned}$$

Again k can take any integer value or zero. Any three consecutive values will give the roots.

$$\begin{array}{llll} \text{So if} & k = 0 & & z_0 = 2\mathrm{e}^{\mathrm{i}0} = 2 \\ & k = 1 & & z_1 = 2\mathrm{e}^{\frac{\mathrm{i}2\pi}{3}} = -1 + \mathrm{i}\sqrt{3} \\ & k = 2 & & z_2 = 2\mathrm{e}^{\frac{\mathrm{i}4\pi}{3}} = -1 - \mathrm{i}\sqrt{3} \end{array}$$

These are the three (complex) values of $8^{\frac{1}{3}}$ obtained using the exponential form. Of course at the end of the calculation we have converted back to standard Cartesian form.

Task

Following the procedure outlined in Example 8 obtain the two complex values of $(1+\mathrm{i})^{1/2}$.

Begin by obtaining the polar form (using the general form of the argument) of $(1+\mathrm{i})$:

Your solution

Answer

You should obtain $1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2k\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2k\pi)) \quad k = 0, \pm 1, \pm 2, \cdots$.

Now take the square root and use De Moivre's theorem to complete the solution:

Your solution

Answer

You should obtain

$$z_1 = \sqrt[4]{2}(\cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}) = 1.099 + 0.455\mathrm{i}$$

$$z_2 = \sqrt[4]{2}(\cos(\frac{\pi}{8} + \pi) + \mathrm{i}\sin(\frac{\pi}{8} + \pi)) = -1.099 - 0.455\mathrm{i}$$

A good exercise would be to repeat the calculation using the exponential form.

Exercise

Find all those values of z which satisfy $z^4 + 1 = 0$. Write your values in standard Cartesian form.

Answer

$$z_0 = \frac{1}{\sqrt{2}} + \frac{\mathrm{i}}{\sqrt{2}} \qquad z_1 = -\frac{1}{\sqrt{2}} + \frac{\mathrm{i}}{\sqrt{2}} \qquad z_2 = -\frac{1}{\sqrt{2}} - \frac{\mathrm{i}}{\sqrt{2}} \qquad z_3 = \frac{1}{\sqrt{2}} - \frac{\mathrm{i}}{\sqrt{2}}$$