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Minimum Wage Spillover Effects and Social Welfare in a Model of Stochastic Job Matching *

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Abstract

In this paper, I carry out a welfare analysis of the minimum wage in the framework of a Diamond-Mortensen-Pissarides model with stochastic job matching. I explore the role of the minimum wage in a labour market with trading externalities and present the necessary and sufficient condition for a minimum wage hike to be efficiency-enhancing. In this context, I characterise minimum wage spillover effects and demonstrate that there is a direct link between the welfare effects and spillover effects of a minimum wage. This theoretical finding suggests that the welfare impact of minimum wage changes can be inferred from the empirical observation of spillover effects on the wage distribution.

JEL Classification: J08, J64, H21, H23
Keywords: minimum wage, wage distribution, social welfare, policy evaluation

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1 Introduction

Recent empirical studies have challenged the predictions of the simple textbook model about the effects of the minimum wage on labour market outcomes: following the influential work of Card and Krueger (1995), a large body of empirical work has documented that moderate increases in the level of the wage floor decrease wage inequality and have little or no adverse effect on employment (for example, see Cengiz et al. (2019) for the USA, Butcher et al. (2012) for the UK, and Dustmann et al. (2021) for Germany).[1] Although this evidence is suggestive of an overall positive effect of the minimum wage on the welfare of labour market participants, the evaluation of the minimum wage as a policy instrument necessitates a complete characterisation of its impact on the distribution of wages, unemployment, and social welfare.

In this paper, I present a tractable general equilibrium search and matching model of the labour market and use it to analyse the effects of minimum wage changes on equilibrium labour market outcomes, to describe the channels through which such changes impact social welfare, and to derive specific conditions for a welfare improving minimum wage. Within a Diamond (1982), Mortensen (1982), Pissarides (1985) (henceforth DMP) model with match-specific productivity,[2] I demonstrate that the minimum wage can be welfare improving when workers' bargaining power is low, and more specifically lower than the elasticity of matching with respect to unemployment, implying an underlying inefficiency in the allocation of match surplus between workers and firms, as in Hosios (1990). In this environment, a social planner considering whether to intervene by imposing a minimum wage faces a trade-off: a minimum wage increases the average share of match surplus captured by workers, which improves welfare, but at the same time it prevents some low-productivity matches from being consummated, which reduces welfare. I study this trade-off and present the necessary and sufficient conditions for a minimum wage hike to be welfare enhancing.

My theoretical setting is closely related to the environment developed by Flinn (2006) to empirically assess the welfare impact of minimum wage changes. The empirical findings of Flinn (2006) suggest that the workers' share of the surplus is different from their corresponding elasticity of the matching function, which is an indication of underlying inefficiency, à la Hosios (1990). Contrary to Hosios (1990), Flinn (2006) considers the bargaining power of workers to be a primitive parameter and treats the minimum wage as a policy variable. Flinn (2006) argues that the social planner can alleviate the existing inefficiency in the labour market and improve welfare through increases in the minimum wage, but does not provide analytical conditions under which minimum wage changes are welfare enhancing.

My work builds on and extends the analysis of Flinn (2006). Under the assumption that the minimum wage is the only policy instrument available to the social planner, I

[1] There are notable counter examples of empirical studies that report negative employment effects of the minimum wage: see Neumark and Wascher (2008), Jardim et al. (2022), and Clemens and Wither (2019). However, the weight of evidence suggests that minimum wage changes have little or no impact on employment.

[2] I assume continuous time, a stationary environment, job search, Nash bargaining, a constant returns to scale matching technology, a fixed mass of workers, and an endogenously determined population of firms.

show how social welfare varies in response to minimum wage changes, derive a Hosios-type condition for efficiency in a labour market constrained by a wage floor, and demonstrate that the minimum wage has positive spillover effects on the wage distribution if and only if it improves social welfare.

The key intuition for my results is as follows. The minimum wage truncates the wage distribution by eliminating a range of otherwise acceptable wages, and by placing a constraint on the set of legally acceptable wages. This constraint implies that many firm-worker pairs with different match-specific productivities will pay the minimum wage, thereby generating a spike/ mass point in the wage distribution.[3] These firm-worker pairs that pay the minimum wage divide the match surplus in a way that "violates" the solution to the Nash-bargaining problem: workers in matches that pay the minimum wage receive a higher share of the surplus than what their bargaining power would suggest. Therefore, in the presence of a wage floor, employed workers capture, on average, a share of the surplus that exceeds their bargaining power, leading to an increase in their "effective" bargaining power. I formalise the concept of the "effective" bargaining power and present the labour market equilibrium in a way that encompasses the case without a wage floor (the unconstrained case) and the case with a binding wage floor (the constrained case). Within this setting, I specify the necessary and sufficient conditions for a minimum wage induced increase in the "effective" bargaining power of workers to be efficiency enhancing. I also demonstrate that for incremental changes in the minimum wage the efficiency condition depends only on the relationship between workers' bargaining power and the elasticity of the matching function, which are assumed to be primitive parameters; hence, I derive a Hosios-type condition for efficiency in market with a binding minimum wage.

In addition to the welfare implications of changes in the minimum wage, I analyse minimum wage effects further up the wage distribution, that is, minimum wage spillover effects.[4] The introduction or the uprating of a binding minimum wage implies an increase or a decrease in the value of search, which in turn affects accepted wages. I derive the condition for the existence of spillovers and show that there is a direct relationship between changes in social welfare and spillovers: an increase in the minimum wage has positive spillover effects on the wage distribution if and only if it improves social welfare. An important implication of this result is that one can infer the welfare impact of the minimum wage by comparing the wage distribution before and after a minimum wage change.

Besides Flinn (2006), many studies have argued that minimum wage increases can be welfare improving in imperfect labour markets exhibiting inefficiency in the distribution of match surplus: Manning (2003, 2004) and Engbom and Moser (2021) demonstrate this in a Burdett and Mortensen (1998) setting, Hungerbühler and Lehmann (2009) show this in a Pissarides (2000) model with redistributive taxation, while Rocheteau and Tasci (2008)

[3]The observation of a spike in the wage distribution at the level of the minimum wage is a stylised fact reported in a number of studies: see Card and Krueger (1995) and Flinn (2006) for the USA, Butcher et al. (2012) for the UK, Engbom and Moser (2021) for Brazil.

[4]The empirical relevance of spillover effects has been highlighted by Card and Krueger (1995), Lee (1999), Flinn (2006), Autor et al. (2016), and Cengiz et al. (2019) for the USA, Butcher et al. (2012) for the UK, Engbom and Moser (2021) for Brazil, Dustmann et al. (2021) for Germany.

study the effects of minimum wages on labour market outcomes and social welfare in the context of many alternative search and matching environments and show that it can have positive and negative welfare effects. While the mechanism through which the minimum wage improves welfare is different across these studies, the common idea is that the minimum wage corrects an underlying inefficiency caused by the excessively low share of the surplus captured by workers. My contribution to this literature is twofold: first, I derive an analytical condition for the minimum wage to be efficiency enhancing, and a Hosios-type condition for efficiency under a binding minimum wage; second, I present a testable prediction that relates minimum wage welfare effects to the empirical observation of minimum wage spillover effects.

The welfare impact of the minimum wage has also been analysed in markets suffering from different types of inefficiency. For example, Acemoglu (2001) builds a DMP model with an endogenous distribution of jobs: in this environment, rent-sharing implies that workers capture part of the return on capital investment made by firms, so firms open too few capital-intensive/ high-wage jobs and too many low-wage jobs. Acemoglu (2001) shows that a minimum wage improves the composition of jobs by alleviating this hold-up problem. A similar hold-up problem arises in the model developed by Bauducco and Janiak (2018), who build a large-firm search and matching model and present a comprehensive analysis of the strategic interactions between workers, the firm, and capital; they show that moderate increases in the minimum wage have positive welfare effects by pushing capital investment closer to its efficient level. In a wage posting model with coordination frictions, Gautier and Moraga-Gonzalez (2018) demonstrate that a minimum wage can have a welfare enhancing effect by reducing excessive search intensity. In a directed search model where firms enjoy market power, Galenianos et al. (2011) show that a minimum wage exacerbates existing inefficiencies, and thus decreases welfare, by increasing the misallocation of workers to low productivity firms. There are also studies examining how the minimum wage affects welfare in competitive labour markets. Lee and Saez (2012) argue that, despite higher unemployment, the minimum wage may be desirable in a competitive market if the government values redistribution and if low productivity jobs are eliminated. Deltas (2007) considers a competitive labour market where worker effort and working hours are endogenous and shows that the minimum wage decreases worker welfare even when employment increases: workers work so much harder that the extra compensation is not worth the extra effort.

The framework of Flinn (2006) has been extended to study the impact of minimum wages on labour market outcomes in studies adopting a positive focus. Gorry (2013) evaluates the effects of minimum wages on the pattern of unemployment over the life-cycle in a search and matching model with experience accumulation; he shows that high minimum wages have a pronounced disemployment effect on young/inexperienced workers, who are unable to accumulate job experience and face worse labour market outcomes later in their working lives. In a search and matching model with heterogeneous occupations and workers, Liu (2022) demonstrates that minimum wages reduce the occupational mobility of young/ low-ability workers and increase job mismatch. My work contributes to the literature by adopting a normative approach: I solve the problem of a social planner setting a minimum wage in a labour market with search frictions.

This paper is organized as follows. In Section 2, I describe the behaviour of agents and characterise the equilibrium allocations in the presence of an exogenously given wage floor. In Section 3, I treat the minimum wage as a policy instrument and discuss the implications of changes in its level for social welfare, accepted wages, and unemployment. Section 4 concludes with a discussion of the implications of my findings for empirical research on the minimum wage. Technical details, derivations, and proofs are gathered in Appendices A, B, C, and D. Appendix E discusses the properties of the labour market equilibrium with and without a minimum wage using numerical illustrations. The discussion in the main body of the paper (and Appendices A-E) assumes that agents' discount rate is zero; Appendix F presents the model and the key results under the assumption that the discount rate is positive.

2 The Model

2.1 The Environment

The model is set up in continuous time and assumes a stationary labour market environment. The labour market is populated by a mass of workers normalized to 1. The mass of firms in the market is endogenous: firms enter (exit) the market if the cost of posting a vacancy is smaller (greater) than the expected value of a filled job position. Each firm can employ one worker.

The technology that brings together unemployed workers (u) and firms with unfilled vacancies (v) is characterised by a constant returns to scale matching function: $M(u,v)$. Labour market tightness is denoted $\theta = \frac{v}{u}$. A firm with an open vacancy meets a job searcher at rate $\lambda_e = \frac{M}{v} = q(\theta)$, where $q(\theta)$ is decreasing in θ; symmetrically, an unemployed worker meets a vacancy at rate $\lambda_w = \frac{M}{u} = \theta q(\theta)$.

For the sake of exposition, I also assume that M is Cobb-Douglas, which implies that the elasticity of matching with respect to unemployment, $\eta = \frac{\partial \ln M}{\partial \ln u}$, is constant and independent of market tightness.[5] The constant returns to scale matching technology also implies that $\eta \in (0,1)$.

When a firm meets a potential employee, they both observe the productive value of the match, z, which is drawn randomly from a predetermined distribution, $G(z)$; G is assumed to be continuous with infimum $\underline{z}$ and supremum $\overline{z}$. The realized match value is divided between the firm and the worker using the generalised Nash bargaining framework.

Given that my ultimate goal is the characterisation of labour market outcomes in the steady state, I assume that agents have no preference for the present, that is, agents' common discount rate, r, goes to zero. This assumption is without loss of generality: it

[5]The assumption that M is Cobb-Douglas facilitates the presentation of the model's predictions as η becomes a primitive parameter. The analysis can be generalised to account for any constant returns to scale matching technology by expressing $\eta(\theta)$.

simplifies the analysis and allows me to directly compare steady-state solutions without considering the dynamic version of the same model.[6] Consummated matches are destroyed at an exogenously given rate $\delta \in (0,1)$, in which case workers join the unemployment pool receiving a constant instantaneous payoff $b > 0$. Firms advertise their open vacancies incurring an instantaneous cost $c > 0$. In the labour market setting considered in this paper, employed individuals do not receive alternative offers of employment, that is, there is no on the job search.

2.2 Individual Behaviour under a Binding Minimum Wage

In this part, I adjust the standard DMP model with match-specific productivity to account for the existence of a wage floor. My framework extends Flinn (2006) by formalising the concept of the "effective" bargaining power, which allows me to define the labour market equilibrium in a way that encompasses the unconstrained and the constrained case. This facilitates the comparison of the two types of equilibria, and the characterisation of the welfare impact of the minimum wage.

To fix ideas, I present the theoretical framework assuming that the minimum wage is an exogenous feature of the labour market environment, rather than a policy variable set by the social planner. This assumption is relaxed in Section 3, where I examine how a varying minimum wage affects steady state outcomes and present the solution to the problem of a social planner, whose only policy instrument is the minimum wage.

2.2.1 Decisions of Workers and Firms

Consider a worker and a firm, whose match has productivity value z; let $w(z)$ denote the wage paid to the worker and $z - w(z)$ the profit flow of the firm. A binding minimum wage, m, places a constraint on acceptable wages and match productivities. Assuming full compliance, acceptable matches are those that pay wages at least as large as the minimum wage, $w(z) \geq m$. From the point of view of the firm, acceptable matches are those with productivity value weakly greater than the minimum wage, $z \geq m$, otherwise the firm incurs a negative profit flow. This implies that, in the presence of a legislated wage floor, a random worker-firm contact results in an acceptable match if their joint productivity value is $z \geq m$, which happens with probability $1 - G(m)$.

Let $w^e = E(w(z)|z \geq m)$ and $z^e = E(z|z \geq m)$ respectively denote the mean wage and mean productivity of acceptable matches. The steady-state flow values of unmatched

[6] In Appendix F, I present the model with $r > 0$, and demonstrate that this assumption does not change the key results presented in Section 3.

workers and firms, denoted U_m and V_m, are:[7]

$$U_m = b + \lambda_w (1 - G(m)) \frac{w^e - U_m}{\delta}, \tag{1}$$

$$V_m = -c + \lambda_e (1 - G(m)) \frac{z^e - w^e}{\delta}. \tag{2}$$

U_m represents the average income of an unemployed worker and is equal to the sum of a flow term, b, plus an expected "capital gain" due to a change in employment status. Similarly, V_m represents the average rent of a vacant job and is equal to the expected profit, if the job is filled, minus the cost of posting the vacancy. In equilibrium, the free entry and exit of firms in the labour market drive rents from vacant jobs to zero ($V_m = 0$).

The firm-worker pair bargain over the division of the flow surplus, which is given by the match value minus the corresponding disagreement values of the worker and the firm. The disagreement value of a worker is equal to the flow value of unemployment, U_m, while the disagreement value of a firm is equal to the flow value of holding a vacancy, which is equal to zero in equilibrium due to the free entry condition. For any match with value $z \geq m$, the flow surplus, S, is given by:

$$S(z) = [w(z) - U_m] + [z - w(z)] = z - U_m.$$

Setting the minimum wage equal to the reservation value of job seekers would give the the model without a binding minimum wage. Let U and z^* respectively denote the value of unemployment and reservation productivity in the unconstrained case. For $m \leq U_m = U$, the value of unemployment is equal to the minimum acceptable wage and the minimum acceptable match value: $z^* = U = w(z^*)$, see Appendix A. Under a binding minimum wage, the minimum acceptable productivity, denoted z_m^*, is equal to the legislated wage floor, m, which exceeds the reservation value of a job seeker, U_m: $z_m^* = m > U_m$. The minimum wage eliminates a range of wages that would otherwise be acceptable: workers are willing to accept any match paying at least U_m, but the lowest feasible wage in the labour market is the minimum, m. Therefore, U_m corresponds to the "implicit" reservation wage of workers.

As in Flinn (2006), wages are determined by Nash bargaining with the additional constraint that wages must be weakly greater than m:

$$w(z) = \max\{\beta z + (1 - \beta) U_m, m\}, \text{ for } z \geq m,$$

where β is the bargaining power of workers. For some productivity value, denoted $\hat{z}$, the Nash-bargained wage is equal to the minimum wage:

$$\hat{z} = \frac{m - (1 - \beta) U_m}{\beta}. \tag{3}$$

[7] The steady state flow values of unattached workers and firms are given by the limit of the corresponding asset values multiplied by the discount rate as the discount rate tends to zero; the derivation is similar to the unconstrained case, presented in Appendix A.

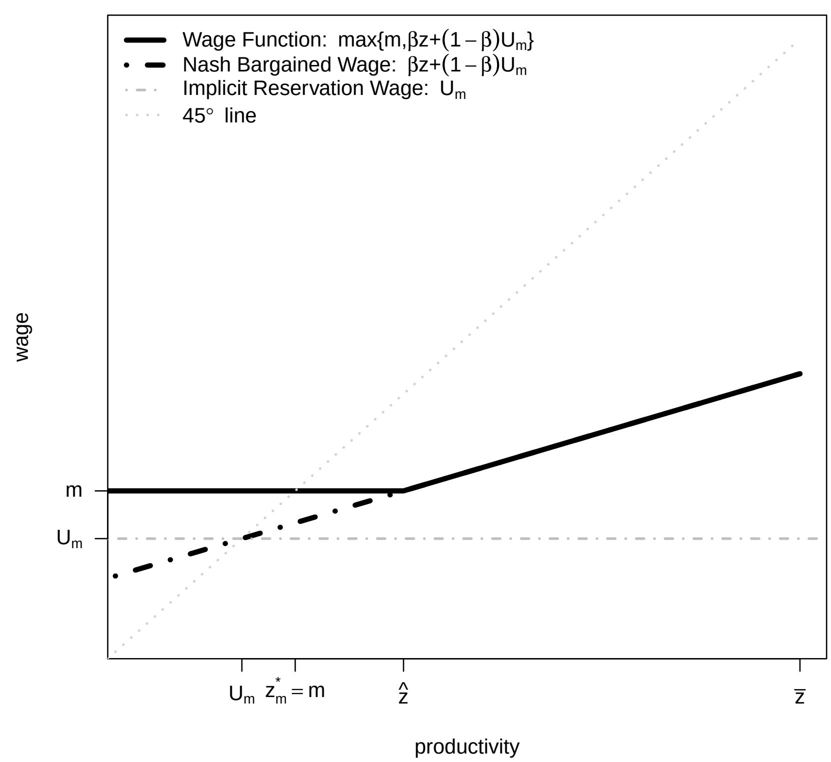
All matches with productivity below $\hat{z}$ pay the minimum wage, while matches further up the productivity distribution pay wages that exceed the minimum (this analysis is relevant only if $\hat{z} \geq m$, otherwise, the minimum wage is not binding and the interactions of agents are given by the unconstrained model, see Appendix A).

The intuition for this result is simple. Any match with productivity $z \in [m, \hat{z})$ would be acceptable from the firm's point of view, but is not feasible because the Nash-bargained wage is below the legislated minimum. The match would become feasible, if the firm paid the worker a wage equal to m; in this case, the firm would make non-negative profits. Therefore, feasible matches pay wages given by:

$$w(z) = \begin{cases} \beta z + (1-\beta) U_m, \text{ for } z \in [\hat{z}, \overline{z}], \\ m, \text{ for } z \in [m, \hat{z}). \end{cases} \tag{4}$$

Figure 1 illustrates the wage function under a binding minimum wage, m. The 45-degree

Figure 1: The Wage Function With and Without a Binding Minimum Wage



line gives the value of the output produced by the firm-worker pair: any match paying a wage above the 45-degree line would have negative flow surplus, so it would not be consummated. Matches with $z \in [\hat{z}, \overline{z}]$ pay wages given by Nash-bargaining, so workers in these matches receive their "implicit" reservation wage, U_m, plus a share β of the match specific flow surplus, S, while firms receive their reservation profit flow, $V = 0$, plus a share $(1-\beta)$

of S. Matches with $z \in [m, \hat{z})$ have Nash-bargained wages below the minimum wage; to comply with the minimum wage, these matches pay m; therefore, workers in matches with $z \in [m, \hat{z})$ capture a share of S that is strictly greater than their bargaining power:

$$\frac{m - U_m}{z - U_m} > \beta, \text{ for } z \in [m, \hat{z}).$$

Flinn (2006) describes this as an increase in the "effective" bargaining power of workers caused by the minimum wage. To formalise Flinn's idea, I present a definition of the "effective" bargaining power of workers, which I then use to examine the implications of the minimum wage for equilibrium outcomes.

Definition 1 *Define the "effective" bargaining power of workers, ε, to be the average share of the match surplus received by employed workers*

$$\varepsilon = \frac{w^e - U_m}{z^e - U_m}, \tag{5}$$

where z^e and w^e denote the average match specific productivity and the average accepted wage under a binding wage floor: $z^e = E(z|z \geq m)$ and $w^e = E(w(z)|z \geq m)$.

Unlike the bargaining power, β, which is a primitive parameter of the model, the "effective" bargaining power, ε, is endogenously determined in the constrained decentralised equilibrium and can be expressed as a function of β, m, and U_m: $\varepsilon(\beta, m, U_m)$. In Appendix C.1, I demonstrate that changes in the minimum wage have a direct and an indirect impact on ε. A minimum wage hike increases the average wage, which implies a direct positive effect on ε. A minimum wage hike also has an indirect negative effect on ε arising due to changes in the value of search. The total effect of the minimum wage on the "effective" bargaining power of workers depends on U_m, which is endogenously determined.

The above Definition suggests that the upper bound of ε is 1, while its lower bound is β: a non-binding minimum wage, $m_0 \leq U_m = U$, implies that $\varepsilon(\beta, m_0, U) = \beta$. A binding minimum wage, $m > U_m$, raises the constrained average wage above the unconstrained average wage, which implies that $\varepsilon(\beta, m, U_m) > \beta$. Therefore, the existence of a binding minimum wage raises the "effective" bargaining power of workers above β.

2.3 Equilibrium under a Binding Minimum Wage

2.3.1 The Decentralised Equilibrium Allocation

At stationary equilibrium a mass of workers u are unemployed. At every point in time, the measure of unemployed workers finding a job amounts to $\lambda_w [1 - G(m)] u$, and the measure of job destructions is $\delta(1 - u)$. Equating these flows gives equilibrium unemployment/

unemployment rate:[8]

$$u = \frac{\delta}{\delta + \lambda_w \left[1 - G\left(m\right)\right]}. \tag{6}$$

The effect of the minimum wage on equilibrium unemployment is ambiguous. The ambiguity arises due to the impact of the minimum wage on market tightness, which may be lower or higher than in the unconstrained case, leading to a lower or higher contact rate, λ_w, and higher or lower equilibrium unemployment. In addition, a binding minimum wage truncates the distribution of acceptable match specific productivities above the level of the unconstrained reservation productivity, $m = z_m^* > z^* = U$, leading to fewer acceptable matches and higher equilibrium unemployment. A formal analysis of the effect of the minimum wage on equilibrium unemployment is examined in Section 3.

Using expression (5) to substitute the average wage, w^e, into equations (1) and (2), taking into account that in equilibrium $V_m = 0$, leads to two conditions characterising the equilibrium behaviour of workers and firms searching for a match:

$$U_m = b + \frac{\varepsilon}{(1-\varepsilon)} c\theta, \tag{7}$$

$$\frac{\delta c}{q\left(\theta\right)\left[1 - G\left(m\right)\right]} = \left(1-\varepsilon\right)\left(z^e - b\right) - \varepsilon c\theta. \tag{8}$$

In equilibrium, the value of unemployment and job creation depend on market tightness, the minimum wage, the "effective" bargaining power, and the primitive parameters of the model.

Combining equilibrium conditions (7) and (8) with the definition of the "effective" bargaining power, condition (5), it is possible to compute market tightness, the value of unemployed workers, and the "effective" bargaining power. These values can then be substituted into equation (6), to calculate equilibrium unemployment and vacancies. The following definition summarises this analysis:

Definition 2 *A labour market equilibrium consists of the quadruplet* $(\varepsilon^*, u^*, \theta^*, U_m^*)$*, which satisfies equations (5), (6), (7), and (8).*

The distribution of wages in the constrained equilibrium, denoted $F(w)$, is a mapping from $G\left(z\right)$, conditional on the match value (and wage) being on the acceptable range. The observed wage distribution exhibits a mass point, i.e. a spike, at the minimum wage; this happens because a range of match specific productivity values, $z \in [m, \hat{z})$, lead to matches paying the minimum wage. The density function of observed wages in the constrained steady state is

$$f\left(w; m\right) = \begin{cases} \frac{\beta^{-1} g(x(w;m))}{1-G(m)} & \text{for } w > m, \\ \frac{G(\hat{z})-G(m)}{1-G(m)} & \text{for } w = m, \\ 0 & \text{for } w < m, \end{cases} \tag{9}$$

[8] The population of workers is normalized to 1, so there is no need to distinguish between unemployment (the measure of unemployed workers) and the unemployment rate (the proportion of unattached workers relative to the aggregate worker population). Henceforth, I use the term unemployment.

where $x(w; m)$ is the inverse function of $w(z)$ for any m, obtained by solving (4) for z.

2.3.2 Characterising the Equilibrium

Introducing the "effective" bargaining power as an additional endogenous variable, and equation (5) as an additional equilibrium condition, I can characterise the constrained equilibrium allocation based on equilibrium conditions that are algebraically similar to the unconstrained equilibrium equations with the notable difference that the bargaining power, β, is now replaced by the "effective" bargaining power, ε.[9] The decentralised equilibrium allocation, as given by Definition 2, encompasses both the constrained and the unconstrained case: if the minimum wage is not binding, then $\varepsilon = \beta$ and the constrained equilibrium conditions, i.e. equations (5), (6), (7), and (8), are reduced to the corresponding decentralised equilibrium conditions of the DMP model. This facilitates the characterisation of the equilibrium allocation and the welfare analysis presented in the following Section.

A notable difference between the constrained and unconstrained cases is that under a binding minimum wage the flow value of unemployment, U_m, is no longer the reservation productivity/ wage, but rather an "implicit" reservation wage that affects wage determination. The implications of this difference for efficiency in the decentralised equilibrium allocation are explored in Section 2.4.

2.4 Welfare and Efficiency

The benchmark for efficiency in the DMP model is the solution to the problem of a social planner, who chooses the strategies of agents to maximise aggregate output net of search costs subject to the same matching constraints faced by individual agents, see Hosios (1990) and Pissarides (2000). In the framework I consider, individual agents make two types of decisions, job creation and job acceptance, which lead to trading externalities. Within a similar framework but in the absence of a minimum wage, Hosios (1990) derives the necessary and sufficient condition for efficiency in the unconstrained equilibrium allocation: the workers' bargaining share is equal to the elasticity of matching with respect to unemployment, $\beta = \eta$. The intuition is that efficient job creation and job acceptance require that the share of the surplus captured by individual agents is equal to their marginal contribution to the creation of match opportunities.

To analyse the implications of a binding minimum wage for efficiency in the equilibrium allocation, consider a social planner constrained by an exogenous minimum wage.[10] Let SW_m denote social welfare, i.e. aggregate output net of search costs. The planner's problem

[9]For completeness, the unconstrained case is presented in Appendix A.

[10]In this section the minimum wage is not a choice (policy) variable. The implications of a varying minimum wage for the efficiency of the labour market equilibrium allocation and the derivation of a "welfare maximising" minimum wage level are presented and discussed in Section 3.

is:

$$\max_{\theta, z_m^*, u} \{SW_m = ub + (1-u) z^e - vc\}, \tag{10}$$

$$\text{subject to}$$
$$u = \frac{\delta}{\delta + \lambda_w [1 - G(z_m^*)]}$$
$$z_m^* \geq m$$

Solving this problem (see Appendix B) yields the following conditions:

$$(1-\eta)(z^e - b) = \eta c\theta + \frac{\delta c}{q(\theta)[1 - G(z_m^*)]}, \tag{11}$$

$$\text{and } z_m^* = m. \tag{12}$$

The efficient allocation with a binding minimum wage is characterised by (θ^*, z_m^*) that jointly satisfy the above conditions for any given level of the minimum wage, m.

When the social planner maximises social welfare, his objective is to achieve optimal job creation and optimal job acceptance. A binding minimum wage places a constraint on the set of acceptable reservation productivity values that can be chosen by the social planner and leads to an equilibrium allocation with inefficiently low job acceptance and inefficiently high job creation. To demonstrate this, compare the efficient allocation and the decentralised equilibrium allocation: equations (11) and (8) are identical if the "effective" bargaining power is equal to the elasticity of matching with respect to unemployment: $\varepsilon = \eta$. This condition would guarantee efficient job creation in the decentralised equilibrium. From condition (7), which holds in the decentralised equilibrium, if $\varepsilon = \eta$, the value of unemployment would be:

$$U_m^{opt} = b + \frac{\eta}{(1-\eta)} c\theta, \tag{13}$$

Efficient job acceptance is achieved if the reservation productivity is equal to the worker's reservation value, that is, $z_m^* = U_m^{opt}$. Taking into account condition (12), it has to be the case that $z_m^* = U_m^{opt} = m$, but this would render the minimum wage non-binding. A binding minimum wage has to be greater than the value of search: $z_m^* = U_m^{opt} = m > U_m$, which implies that $U^{opt} > U_m$ or equivalently (from conditions (7) and (13)) that $\eta > \varepsilon$. This suggests that the solution to the social planner's problem under a binding minimum wage is a corner solution.

To interpret this result, consider how a binding minimum wage affects worker-firm matches: a minimum wage truncates the match-specific productivity distribution and eliminates a range of productivity values that would be feasible from the point of view of workers and firms. The reservation productivity is equal to the binding minimum wage, which has to be greater than the value of unemployment. This wedge between the reservation productivity and the value of unemployment implies that job acceptance is inefficiently low under a binding minimum wage. Inefficiently low job acceptance implies that the value of unemployment is below its optimal level, or equivalently implies that the average share of the surplus captured by workers is below their contribution to the creation of match opportunities, i.e. $\varepsilon < \eta$. Therefore, under a binding minimum wage firms receive, on average, a higher share of the surplus than in the optimal allocation, $(1-\varepsilon) > (1-\eta)$, which leads to inefficiently high job creation.

Consider now a modified version of the social planner's problem: the planner maximises social welfare with respect to unemployment, market tightness, and reservation productivity (as in (10)), but also with respect to the minimum wage. If the planner were allowed to determine the level of unemployment, market tightness and reservation productivity, he would be able to achieve the efficient allocation without a minimum wage, so the optimal minimum wage would be a non-binding minimum wage: the efficient allocation without m dominates the efficient allocation with m.

The above analysis characterises the welfare costs of a minimum wage: it describes why a binding minimum wage prevents the social planner from achieving the efficient equilibrium allocation (as in Hosios (1990)). In Section 3, I consider an inefficient labour market and compare the potential welfare benefits with the welfare costs of a minimum wage.

3 The Effects of Minimum Wage Changes: Welfare, Wages, Unemployment

In Section 2, I described the predictions of the model treating m as an exogenous constraint. I now consider m to be the policy variable chosen by the social planner, and examine the implications of a varying minimum wage for the welfare of labour market participants, accepted wages, and unemployment.

3.1 Welfare and Efficiency

In an efficient labour market the minimum wage has a distortionary effect, so consider the interesting case of a labour market where $\beta \neq \eta$: search externalities are not taken into account by individual agents leading to a suboptimal decentralised equilibrium allocation. The social planner could intervene and restore efficiency in this market. Suppose the social planner has no control over the division of the surplus between the matched firm-worker, and cannot determine the equilibrium values of unemployment, market tightness, or reservation productivity. The only way the planner can intervene in this market is by varying the level of the minimum wage. Under what conditions would such an intervention increase welfare?

The objective of the social planner is the maximisation of social welfare, as given by (10), with respect to the minimum wage, i.e. the policy variable under his control. The social planner's decisions impact the behaviour of agents in the labour market, whose reactions have to be factored into the planner's maximisation problem: in other words, the planner sets the minimum wage; given the minimum wage, the behaviour of firms and workers in the market leads to a decentralised equilibrium allocation; given the decentralised equilibrium allocation the planner computes Social Welfare. The planner's objective is to set the minimum wage at a level such that the ensuing decentralised equilibrium allocation achieves the highest possible Social Welfare given the constraints faced by the planner and the agents. In Appendix B, I demonstrate that, in equilibrium, the planner's welfare criterion is equal to the equilibrium value of unemployment, U_m.[11]

To determine the conditions for a welfare improving minimum wage, I conduct a comparative statics analysis: I examine how a change in m affects equilibrium conditions (5), (6), (7), and (8), and solve for $\frac{dU_m}{dm}$. The calculations are presented in Appendix C and summarised in Lemma 1, which is rewritten here for ease of exposition:

$$\frac{dU_m}{dm} \overset{sgn}{=} \left[G\left(\hat{z}\right) - G\left(m\right)\right]\left(\eta - \varepsilon\right) - \eta\left[1 - \varepsilon\right]\left(m - U_m\right) g\left(m\right). \tag{14}$$

To interpret this result, consider the possible channels through which an increase in the wage floor affects the value of search: there is a positive effect caused by the higher "effective" bargaining power of workers, a negative effect due to lower job creation (caused by the fall in the surplus of firms), and one additional negative effect due to lower job acceptance (implied by the higher cutoff productivity level). The first term on the right-hand side of equation (14) shows that the positive effect of the higher match-surplus, captured by the mass of workers receiving the minimum $\left[G\left(\hat{z}\right) - G\left(m\right)\right]$, dominates the corresponding negative effect on job-creation if the marginal contribution of workers to the creation of match opportunities exceeds their average or "effective" share of the surplus: i.e. $\eta > \varepsilon$. The second term on the right-hand side of (14) corresponds to the negative effect of the minimum wage rise on job acceptance: the incrementally higher wage floor eliminates $g\left(m\right)$ matches with surplus $\left(m - U_m\right)$. Condition (14) gives the slope of the welfare criterion in the direction of increasing m: a positive sign suggests that welfare is increasing in m, so there is a new higher minimum wage that increases welfare; this change in m can be marginal or discrete.

Using the above relationship, I determine the conditions under which the minimum wage has a positive or a negative effect on the flow value of unemployment. Simple inspection of equation (14) gives a sufficient condition for the minimum wage to be welfare decreasing. This result is summarized in Proposition 1:

Proposition 1 *If the minimum wage binds and the workers' "effective" bargaining power* (ε) *is weakly greater than the elasticity of the matching function with respect to unemployment* (η)*, any increase in the minimum wage results in a decrease in the flow value of unemployment* (U_m) *and in social welfare* (SW_m)*.*

Proof See Appendix D. ■

Proposition 1 is only a sufficient (but not necessary) condition for changes in the minimum wage to decrease welfare. In this sense, the minimum wage can have a negative

[11] In a setting without a minimum wage, Pissarides (2000) demonstrates that social welfare maximisation is achieved if workers and firms divide the match surplus in a way that maximises the value of unemployment. In addition, the value of unemployed search is used as a welfare criterion in environments with a minimum wage, see Flinn (2006) and Flinn (2011).

welfare effect even in cases where the workers' "effective" bargaining power is lower than the elasticity of the matching function with respect to unemployment. Moreover, this result suggests that there is scope for the minimum wage to increase welfare only if $\eta > \varepsilon$. This necessary condition for positive welfare effects is summarized in Corollary 1:

Corollary 1 *A minimum wage increase can only improve social welfare if the workers' "effective" bargaining power (ε) is lower than the elasticity of the matching function with respect to unemployment (η).*

3.1.1 A Hosios-type Condition for Constrained Efficiency

Let us now examine the necessary and sufficient condition for a welfare increasing minimum wage. According to equation (14), an important determinant of the minimum wage effect is the mass of matches paying the minimum wage $(G(\hat{z}) - G(m))$, that is, the spike at the level of the legislated wage floor.[12] It is evident that the magnitude of the spike is determined by the shape of the match specific productivity distribution as well as the gap between $\hat{z}$ and m. For expositional purposes, I present the following simplification:

Definition 3 *For any binding minimum wage, m, there is at least one productivity level x_m, which corresponds to the average match specific productivity density, $g(x_m)$, on the interval $[m, \hat{z}]$. Clearly, $g(x_m)$ satisfies*

$$(G(\hat{z}) - G(m)) = \int_m^{\hat{z}} g(z)\, dz = (\hat{z} - m)\, g(x_m) = \frac{(1-\beta)}{\beta}(m - U_m)\, g(x_m).^{13} \tag{15}$$

Assume that $\frac{dU_m}{dm} \geq 0$ and that the necessary condition for positive welfare effects is satisfied $\eta > \varepsilon$. Rewriting expression (14) using Definition 3, we have:

$$\begin{aligned}\frac{dU_m}{dm} &\overset{sgn}{=} \frac{(1-\beta)}{\beta}(\eta - \varepsilon)\, g(x_m) - \eta(1-\varepsilon)\, g(m) = \\ &= \eta\left[\frac{(1-\beta)}{\beta} g(x_m) - g(m)\right] - \varepsilon\left[\frac{(1-\beta)}{\beta} g(x_m) - \eta g(m)\right] \geq 0. \end{aligned} \tag{16}$$

Inequality (16) leads to a second necessary condition for a welfare enhancing minimum wage:

Proposition 2 *A minimum wage increase can only improve social welfare if*

$$\frac{(1-\beta)}{\beta} > \frac{g(m)}{g(x_m)}. \tag{17}$$

Proof See Appendix D. ■

The intuition for this necessary condition is that the matches eliminated due to the higher minimum wage imply a loss of worker surplus, $g(m) \times \beta$, which has to be lower than the

[12] Formally, the spike is given by the mass of match-values paying the minimum wage conditional on them being on the acceptable range, that is, $\frac{G(\hat{z})-G(m)}{1-G(m)}$.

[13] Note that x_m is unique if g(z) is monotonic for $z \in [m, \hat{z}]$.

extra surplus captured by the average worker earning the new minimum, $g(x_m) \times (1-\beta)$. This condition is not particularly restrictive: for example, it is satisfied if the bargaining power of workers, β, is sufficiently low and if the shape of the match-specific productivity distribution where the minimum wage "bites" is such that $g(m)$ is not much greater than $g(x_m)$. This condition is always satisfied if the bargaining power of workers is less than the bargaining power of firms, $\beta < \frac{1}{2}$, and the minimum wage "bites" at a part of the match-specific productivity distribution where $g(x_m) \geq g(m)$, e.g. a match-specific productivity density that does not decrease between m and $\hat{z}$.

Solving inequality (16) for the workers' "effective" bargaining power (ε), yields the necessary and sufficient condition for positive welfare effects, presented in Proposition 3:

Proposition 3 *A binding minimum wage, m, increases U_m and social welfare if and only if the following condition holds*

$$\varepsilon(m) \leq \frac{\eta\left[g(x_m) - \beta g(x_m) - \beta g(m)\right]}{g(x_m) - \beta g(x_m) - \beta\eta g(m)} \leq \eta, \tag{18}$$

where $\beta < \varepsilon(m)$.

Proof See Appendix D. ■

Proposition 3 shows that the necessary and sufficient condition for a welfare improving minimum wage depends only on the policy variable (the minimum wage), the primitive parameters of the model, and the parameters of the match-specific productivity distribution. Exploiting the continuity of $G(z)$, it is possible to simplify this condition:

Corollary 2 *There is a binding minimum wage, m, that increases the value of search and social welfare if and only if*

$$\beta < \frac{\frac{g(x_m)}{g(m)} \times \eta}{\frac{g(x_m)}{g(m)} + \eta}. \tag{19}$$

There is a marginally binding minimum wage, m, that increases the value of search and social welfare if and only if

$$\beta < \frac{\eta}{1+\eta}. \tag{20}$$

Proof Assume that the existing minimum wage, m_0, just binds: $m_0 \approx U_m$, which means that $\varepsilon(m_0) \approx \beta$. From condition (18), a higher minimum wage $m > m_0$ (with $\varepsilon(m) > \beta$) increases the value of search, $\frac{dU_m}{dm} > 0$, if

$$\beta \approx \varepsilon(m_0) < \varepsilon(m) \leq \frac{\eta\left[g(x_m) - \beta g(x_m) - \beta g(m)\right]}{g(x_m) - \beta g(x_m) - \beta\eta g(m)},$$

which implies $\beta < \frac{\eta[g(x_m)-\beta g(x_m)-\beta g(m)]}{g(x_m)-\beta g(x_m)-\beta\eta g(m)}$, and

$$\frac{\beta\eta}{\eta-\beta} < \frac{g(x_m)}{g(m)}. \tag{21}$$

Rearranging (21) yields (19).

In the case of a marginally higher minimum wage, $m > m_0$, the continuity of $G(z)$ and Definition 3 imply that $\lim_{m\to m_0} \frac{g(x_m)}{g(m)} = 1$, which if substituted into (21) gives $\frac{\beta\eta}{\eta-\beta} < 1$, a rearranged version of condition (20).

If condition (19) holds, there is an $\varepsilon(m) > \beta$ corresponding to a binding minimum wage, $m > m_0$, such that condition (18) is satisfied. Condition (18) guarantees that this minimum wage is welfare improving. Condition (20) is the limiting version of (19) for $m \to U_m$. ■

This result implies that the social planner can assess the welfare impact of a minimum wage imposition by considering the relationship between the bargaining power parameter (β) and the elasticity of the matching function with respect to unemployment (η), with limited information (condition (19)) or no information (condition (20)) about the match specific productivity distribution. It is worth emphasising that condition (19) can be used to evaluate discrete changes in the minimum wage: if this condition holds, the slope of the welfare criterion in the direction of increasing m is positive, so there is a new binding m that increases social welfare. Condition (20) holds in the limit, i.e. when m is marginally binding, so it can only be used to evaluate marginal changes in the minimum wage.

3.1.2 Optimal Minimum Wage

If condition (20) holds, then the value of search is increasing in the minimum wage at m_0, where m_0 satisfies $\varepsilon(m_0) = \beta$. Proposition 1 suggests that the value of search is decreasing at m_1, where m_1 satisfies $\varepsilon(m_1) = \eta$. The value of search is continuous, so there exists at least one value of the minimum wage, $m^* \in (m_0, m_1)$, such that $\frac{dU_m}{dm}|_{m=m^*} = 0$. Therefore, there is at least one $m^* \in (m_0, m_1)$ at which the value of search and social welfare are (locally) maximised. From the above analysis, m^* satisfies

$$\varepsilon(m^*) = \frac{\eta\left[g(x_m) - \beta g(x_m) - \beta g(m^*)\right]}{g(x_m) - \beta g(x_m) - \beta\eta g(m^*)}. \tag{22}$$

Although this is only a necessary condition for an optimal minimum wage, as m^* may correspond to a local rather than a global maximum, a large number of numerical experiments conducted for a wide range of parameter values yielded a unique welfare maximising m^* satisfying (22): some indicative experiments are presented in Appendix E.

3.1.3 Discussion

The discussion on the efficiency of the constrained labour market equilibrium in Section 2.4 facilitates the interpretation of Proposition 3, Corollary 2, and Section 3.1.2.

When the contribution of workers to the creation of match opportunities (η) exceeds the share of the surplus they capture (β), firms make high profits and create an inefficiently high number of vacancies; at the same time, the value of unemployment is inefficiently low leading to high job acceptance. Introducing a binding minimum wage in this labour market would correct in part this inefficiency by adjusting job acceptance and job creation to levels that are closer to the efficient labour market equilibrium. A binding minimum wage has two effects on job acceptance: first, it raises the average share of the surplus captured by workers, i.e. the “effective” bargaining power of workers, thus increasing their value of unemployment; second, it imposes a restriction on acceptable match productivities and eliminates a mass of previously feasible matches. Both effects lead to a decrease in job acceptance. In addition, a binding minimum wage reduces the average share of the surplus captured by firms causing a fall in job creation and in the contact rate.

A binding minimum wage may reduce inefficiency in the labour market, but cannot completely eliminate it. Proposition 3 and condition (22) state that increasing the minimum wage reduces the underlying inefficiency in the labour market and increases social welfare until the average share of the match surplus captured by workers reaches some upper bound ($\varepsilon(m^*) = \overline{\varepsilon} = \frac{\eta[g(x_m)-\beta g(x_m)-\beta g(m)]}{g(x_m)-\beta g(x_m)-\beta\eta g(m)}$), which corresponds to the constrained social optimum. In the constrained optimum, workers' average share of the surplus is smaller than their contribution to the creation of match opportunities: $\varepsilon(m^*) = \overline{\varepsilon} < \eta$. The wedge between $\varepsilon(m^*)$ and η guarantees that in the constrained optimum firms create more vacancies than in the unconstrained optimum. Moreover, in the constrained optimum there is a wedge between the cutoff productivity level and the value of unemployment ($z_m^* = m^* > U_m^*$), implying that job acceptance is inefficiently low: the binding minimum wage truncates the productivity distribution at a level that exceeds the cutoff productivity of the unconstrained optimum. Therefore, the constrained optimum is characterised by inefficiently high job creation and inefficiently low job acceptance relative to the unconstrained optimum.

The welfare improving effect of the minimum wage depends on the values of the bargaining power parameter and the elasticity of matching with respect to unemployment as well as the shape of the match productivity distribution. As demonstrated in Corollary 2, the welfare effect of a marginally binding minimum wage can be determined with no knowledge of the match productivity distribution: it only depends on the relationship between β and η. Condition (20) indicates that a marginally binding minimum wage can only have a welfare improving effect if workers capture a share of the surplus that is lower than their contribution to the creation of match opportunities and also lower than the share of the surplus captured by firms: i.e. $\beta < \eta$ and $\beta < \frac{1}{2}$. Intuitively, the loss of surplus due to the elimination of matches earning the old minimum is outweighed by the extra surplus captured by the average worker earning the new marginally higher minimum. In the limit, the distribution of match productivities plays no role because the mass of matches eliminated is equal to the mass of matches benefiting from the marginally higher minimum.

The shape of the underlying productivity distribution has important implications for the magnitude of the welfare benefits and losses of a minimum wage hike. For example, suppose the productivity density is everywhere decreasing; introducing a minimum wage in such a setting would imply that the welfare loss due to the elimination of a higher mass of matches would have to be outweighed by a much higher extra surplus captured on average by the mass of workers earning the new minimum. This would only happen if the Nash-bargained worker surplus is lower than firm surplus, i.e. $\beta < \frac{1}{2}$. By contrast, if the productivity density is increasing, a new minimum wage would force into unemployment a mass of matches smaller than the mass of matches benefiting from the new higher surplus. Therefore, if the productivity density is increasing, a minimum wage hike may improve welfare even if workers capture a bigger share of the surplus than firms, i.e. $\beta > \frac{1}{2}$.

In Appendix E, I consider a number of parametric examples, and discuss the implications of the shape of the match productivity distribution, the values of the primitive parameters, and the "bite" of the minimum wage for overall social welfare.

3.2 Minimum Wage Effects on Accepted Wages

In this model, the imposition of a binding minimum wage or the uprating of an already binding wage floor may have three effects on employed workers. First, some matches become unprofitable and are terminated −the standard disemployment effect. Second, some workers receiving a wage below the current minimum have their pay increased to the level of the mandated wage floor −the compliance effect. Finally, the minimum wage change may result in the renegotiation of wages strictly above the current minimum −the spillover effect.

Wage determination under two binding minimum wages implies that:

Proposition 4 *A minimum wage increase has positive (negative) spillover effects on accepted wages if and only if the value of search under the new minimum is higher (lower) than the value of search under the initial minimum.*

Proof See Appendix D. ■

In equilibrium, social welfare is equal to the value of unemployment, which implies:

Corollary 3 *A minimum wage increase has positive (negative) spillovers if and only if it increases (decreases) social welfare.*

This result suggests that it is possible to infer the welfare impact of minimum wage changes from the empirical observation of spillover effects on the wage distribution.[14]

The assumption that all agents are ex ante homogeneous implies that the value of search is the same for all workers. This means that any change in the minimum wage would have a uniform absolute effect and a monotonically diminishing relative effect on the wages of all workers across the wage distribution. This is consistent with the empirical literature on the spillover effects of the minimum wage: this literature considers relative changes in accepted wages and reports a pattern of spillovers that is monotonically diminishing, see Lee (1999), Manning (2003), Stewart (2012a, 2012b), Butcher et al. (2012), Autor et al. (2016), Cengiz et al. (2019), Dustmann et al. (2021), and Engbom and Moser (2021).

[14] The results in Proposition 4 and Corollary 3 are derived in a general equilibrium framework with endogenous contact rates. Flinn (2002) presents a similar result that links spillover effects to changes in the value of unemployment, but in a partial equilibrium setting, assuming constant and exogenously given contact rates.

3.3 Minimum Wage Effects on Unemployment

This model predicts that introducing or uprating a binding minimum wage affect equilibrium unemployment in two ways: first, it increases the cutoff productivity level implying that fewer firm-worker contacts are consummated; second, the minimum wage change affects the contact rate, which is endogenously determined and can be greater or lower than the initial equilibrium contact rate. Hence, the overall effect of a minimum wage change on equilibrium unemployment is ambiguous. Considering the impact of a higher minimum wage on the value of search and market tightness leads to the following result:

Proposition 5 *If an increase in the minimum wage raises the value of search, it will also raise equilibrium unemployment.*

Proof See Appendix D. ■ An immediate implication of this result is the following necessary condition for a decrease in equilibrium unemployment:

Corollary 4 *Equilibrium unemployment can only decrease in response to a minimum wage hike if the value of search decreases.*

These results should not come as a surprise. The mass of workers in the labour market is fixed, so changes in unemployment arise due to the entry and exit of firms. A higher value of search reflects a higher expected wage, lower firm profitability, and fewer firms in the labour market, which in turn implies higher unemployment and lower employment. [15]

4 Conclusion

In a general equilibrium search and matching model with Nash bargaining and match-specific productivity, I conduct a formal welfare analysis of the minimum wage and derive testable results. I demonstrate that in a labour market with trading externalities, where the share of the surplus captured by workers is less than their marginal contribution to the creation of match opportunities, a minimum wage hike can raise the "effective" bargaining power of workers, and thus, mitigate the inefficiency of the decentralised equilibrium allocation. My analysis draws on the work of Flinn (2006), who alludes to this minimum wage effect without providing an explicit derivation. Building on Flinn (2006), I describe the adjustment margins in response to a minimum wage change, I analyse the channels through which the minimum wage can lead to efficiency gains, derive the necessary and sufficient condition under which a minimum wage increase is welfare enhancing, and calculate the welfare maximising minimum wage. I specify a Hosios-type condition for constrained efficiency by showing that the welfare impact of the minimum wage depends on the relationship between the Nash bargaining power parameter and the elasticity of the matching function.

[15] Many studies explore mechanisms through which a minimum wage can be welfare improving despite its adverse employment effects. For example, see Hungerbühler and Lehmann (2009), Swinnerton (1996), and Lee and Saez (2012). Engbom and Moser (2021) lend empirical support to my result: they show that the Brazilian minimum wage caused a notable decrease in wage inequality and significant spillover effects reaching up to the 80th percentile of the wage distribution, with modest disemployment effects.

Moreover, I extend this framework to describe the phenomenon of minimum wage spillover effects. Variation in the level of the minimum wage changes the value of unemployed workers. Given that wages are determined through Nash bargaining, if a minimum wage hike alters the value of search, it also affects accepted wages further up the wage distribution, thus causing spillover effects. This analysis relates to my discussion on efficiency as social welfare and the value of search change in the same direction in response to a minimum wage increase. Combining these findings, I demonstrate that the minimum wage improves social welfare if and only if it has positive spillover effects.

My main conclusion is that spillover effects characterise the impact of the minimum wage on the welfare of labour market participants, so the welfare effects of minimum wage changes can be inferred by the observation of spillovers on the wage distribution. In this way, I establish theoretically the view that minimum wage welfare effects can be empirically assessed by the comparison of pre- and post-change wage distributions. This is an important result as it provides a testable prediction that calls for further empirical research.

There are a number of issues that have not been considered in my highly stylised environment and are worth pointing out as potential avenues for future research. First, an obvious way to extend the current framework would be to endogenise the decision of individuals to participate in the labour force and actively search for a match, as in Flinn (2006) and Flinn (2011). In such a setting, minimum wage hikes would affect labour force participation through their impact on the value of unemployment: a higher value of unemployment would attract more participants in the labour market, increase the unemployment rate, and have an ambiguous effect on the measure of employed individuals.[16] Second, the current framework could be extended with endogenous job separations and on-the-job search: Flinn and Mabbli (2008) and Flinn and Mullins (2021) show that such extensions have important implications for the division of the match surplus between the firm and the worker, and thus, affect the characterisation of the welfare impact of the minimum wage.

[16]In that case, analysing the impact of the minimum wage on the welfare of labour market participants would be more complicated: one would have to account for the endogenous population of labour market participants, which depends on but also affects the value of unemployment (through market tightness).

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Appendices

- Appendix A presents the standard DMP model with stochastic job matching, and also includes the details of the constained model.
- In Appendix B, I solve the planner's problem presented in Section 2.4, and I demonstrate that, in equilibrium, the social planner's objective function is equal to the value of unemployed workers.
- In Appendix C, I consider the effect of the minimum wage on the equilibrium values of the endogenous variables of the model presented in Section 2.
- Appendix D includes proofs of Propositions.
- Appendix E presents a numerical illustration of minimum wage effects on equilibrium outcomes under different parameterisations.
- Appendix F presents the model with discounting: $r > 0$.

A Individual Behaviour with and without a Minimum Wage

A.1 Individual Behaviour with a Minimum Wage

In this part, I outline the basic features of the theoretical setting without a minimum wage $-$the unconstrained case. Although this is a standard DMP model with stochastic job matching, its presentation places the exposition of the constrained case in context, and facilitates the discussion of the efficiency gains/losses caused by a wage floor (see Sections 2 and 3).

A.1.1 Decisions of Individual Workers and Firms

Consider a worker and a firm whose match has productivity value z; let $w(z)$ denote the wage paid to the worker and $z - w(z)$ the profit flow of the firm. Not all matches between unemployed workers and vacant firms are mutually beneficial. There is some common reservation productivity value, denoted z^*, below which neither the firm nor the worker will want to trade. A random worker-firm contact results in an acceptable match if their joint productivity value is $z \geq z^*$, which happens with probability $1 - G(z^*)$.

The steady-state flow value of unmatched workers and firms, denoted U and V, respectively, are given by:[17]

$$V = -c + \lambda_e (1 - G(z^*)) E\left(\frac{z - w(z) - V}{\delta} | z \geq z^*\right), \tag{A6}$$

$$U = b + \lambda_w (1 - G(z^*)) E\left(\frac{w(z) - U}{\delta} | z \geq z^*\right). \tag{A7}$$

[17]Let Y_f and V_f denote the present discounted value (PDV) of a firm with a filled and an unfilled job, respectively. Using the discount rate, r, the PDV of filled and unfilled jobs satisfy, respectively:

$$Y_f - V_f = \frac{(z - w(z)) - rV_f}{r + \delta}, \tag{A1}$$

$$rV_f = -c + \lambda_e (1 - G(z^*)) E(Y_f - V_f | z \geq z^*). \tag{A2}$$

Substituting (A1) into (A2) gives

$$rV_f = -c + \lambda_e (1 - G(z^*)) E\left(\frac{z - w(z) - rV_f}{r + \delta} | z \geq z^*\right). \tag{A3}$$

Defining the steady state flow value of a vacant job to be $V = rV_f$ and taking the limit of (A3) as r goes to zero leads to (A6).
Similarly, denote Y_w and V_w the present discounted value of an employed and an unemployed worker, respectively. Using the discount rate, r, the expected PDV of employed and unemployed workers satisfy, respectively:

$$Y_w - V_w = \frac{w(z) - rV_w}{r + \delta}, \tag{A4}$$

$$rV_w = b + \lambda_w (1 - G(z^*)) E((Y_w - V_w) | z \geq z^*). \tag{A5}$$

Substituting (A4) into (A5) gives

$$rV_w = b + \lambda_w (1 - G(z^*)) E\left(\frac{w(z) - rV_w}{r + \delta} | z \geq z^*\right).$$

Defining the steady state flow value of an unemployed worker to be $U = rV_w$ and taking the limit of the above expression as r tends to zero yields (A7).

U represents the average income of an unemployed worker and is equal to the sum of a flow term, b, plus an expected "capital gain" due to a change in employment status. Similarly, V represents the average rent of a vacant job and is equal to the expected profit, if the job is filled, minus the cost of posting the vacancy. In equilibrium, the free entry and exit of firms in the labour market drive rents from vacant jobs to zero. This equilibrium condition for the supply of vacant jobs ($V = 0$) implies that

$$E\left(\frac{z - w\left(z\right)}{\delta}|z \geq z^*\right) = \frac{c}{\lambda_e\left(1 - G\left(z^*\right)\right)}. \tag{A8}$$

A.1.2 Wage determination

For the division of the match value between the firm and the worker, I use the Nash bargaining framework. The worker's reservation wage is equal to her flow value of unemployment, U, while the firm's reservation profit flow is equal to the flow value of holding a vacancy, which in equilibrium is equal to zero due to the free entry condition. Therefore, for any match with value $z \geq z^*$, the flow surplus S is defined as

$$S\left(z\right) = \left[w - U\right] + \left[z - w\right] = z - U,$$

and the wage is given by

$$w = \arg\max_{w}\left[w - U\right]^{\beta}\left[z - w\right]^{1-\beta}, \text{ that is,}$$

$$w\left(z\right) = \beta z + \left(1 - \beta\right)U, \tag{A9}$$

where β is the worker's bargaining power. The worker receives his reservation wage, U, plus a fraction β of the match specific flow surplus, S, while the firm receives its reservation profit flow, $V = 0$, plus a share $(1 - \beta)$ of the flow surplus S.

From the above, it is straightforward that a match with productivity z is mutually acceptable if and only if the wage and the profit flow are greater than or equal to the respective outside options of the paired worker-firm, which implies that the reservation productivity is given by

$$z^* = U. \tag{A10}$$

In this setting, both the reservation productivity and the reservation wage are equal to the workers' outside option, U.

A.1.3 Equilibrium

At stationary equilibrium, a mass of workers are unemployed. The unemployment rate, u, is found by equalizing the flow of entries into and exits from the unemployment pool. At every point in time, an unemployed worker finds a vacant job with probability λ_w and is hired if the match productivity value exceeds the threshold $z^* = U$. The exit rate from unemployment is equal to $\lambda_w\left[1 - G\left(U\right)\right]$ and the number of unemployed workers finding a job amounts to $\lambda_w [1 - G(U)] u$. The number of job destructions at every point in time is given by $\delta (1 - u)$. Equating these flows gives equilibrium unemployment/ unemployment rate:

$$u = \frac{\delta}{\delta + \lambda_w [1 - G(U)]}. \tag{A11}$$

So far, the flow value of unemployment was considered as exogenously given. Having determined the wage offer and the reservation productivity, I can now calculate the flow value of search:

$$U = b + \beta \lambda_w \int_U^{\overline{z}} \left(\frac{z - U}{\delta} \right) dG(z). \tag{A12}$$

Similarly, the job creation condition (equation [A8]) can be expressed as follows

$$c = (1 - \beta) \lambda_e \int_U^{\overline{z}} \left(\frac{z - U}{\delta} \right) dG(z). \tag{A13}$$

Combining equations (A12) and (A13), I get a reduced-form equilibrium relation between the value of unemployed workers and market tightness

$$U = b + \frac{\beta}{(1 - \beta)} c\theta, \tag{A14}$$

which, when substituted into (A13), gives

$$\frac{\delta c}{\lambda_e [1 - G(U)]} = (1 - \beta)(z^e - b) - \beta c\theta,$$

where $z^e = E(z | z \geq z^*)$. Substituting λ_e from equation, yields the job creation condition

$$(1 - \beta)(z^e - b) = \beta c\theta^* + \frac{\delta c}{q(\theta^*) [1 - G(z^*)]}, \tag{A15}$$

which can be solved implicitly for tightness at decentralised labour market equilibrium.

Equilibrium market tightness can be substituted into (A14) to compute the flow value of unemployment, which is equal to the reservation productivity and the reservation wage. These values can then be used in combination with equation (A11), to calculate equilibrium unemployment and vacancies.

Definition 4 *A labour market equilibrium consists of the triplet* (u^*, U^*, θ^*)*, which satisfies equations (A11), (A14), and (A15).*

In equilibrium, for any match value on the acceptable range, the wage offer is given by (A9). Using the wage offer equation and the distribution of match values, I calculate the wage distribution, denoted $H(w)$. First, I solve the wage equation (A9) for match specific productivity, z:

$$z = x(w) = \frac{w - (1 - \beta) U}{\beta},$$

where $x(w)$ is the inverse function of $w(z)$. Meetings of firms with workers lead to acceptable matches if the realized match value z is greater than or equal to the reservation match value, z^*, and the wage is greater than or equal to the reservation wage, U; since $z^* = U$, a match is accepted with probability $[1 - G(U)]$. The equilibrium distribution of wages is a simple mapping from the match productivity distribution, $G(z)$, into $H(w)$, conditional on the match value (and the wage) being on the acceptable range

$$H(w|w \geq U) = \frac{G(x(w)) - G(U)}{1 - G(U)}. \tag{A16}$$

After simple differentiation, I get the following expression for the wage density

$$h(w) = \left\{ \begin{array}{cc} \frac{\beta^{-1} g(x(w))}{1-G(U)} & \text{for } w \geq U \\ 0 & \text{for } w < U \end{array} \right\}. \tag{A17}$$

Therefore, the accepted wage density is determined by: the workers' bargaining power, β; the lowest accepted productivity/wage, which is equal to the flow value of unemployment, $w^* = z^* = U$; and the primitive parameters of the match specific productivity distribution.

A.2 Equilibirum under a Binding Minimum Wage

Using expression (5) to substitute the average wage, w^e, into equation (1), it is possible to express the flow value of an unemployed worker as follows:

$$U_m = b + \varepsilon \lambda_w (1 - G(m)) \frac{z^e - U_m}{\delta} \tag{A18}$$

which means that in equilibrium the value of being unemployed depends on market tightness, the minimum wage, the "effective" bargaining power, and the primitive parameters of the model. Assuming that the minimum wage is exogenously determined, equation (A18) gives an equilibrium condition in terms of three endogenous variables: the value of unemployment, market tightness, and the "effective" bargaining power.

Examining the equilibrium behaviour of firms with vacancies leads to an additional equilibrium condition in terms of market tightness and the value of search. Free entry and exit guarantees that in equilibrium all profits from new jobs are exhausted, which implies that the expected profit from filling a vacant job is equal to the cost of posting a vacancy or equivalently the equilibrium value of holding a vacancy is equal to zero, $V_m = 0$. Taking into account that $V_m = 0$ and substituting the average wage, w^e, from (5) into (2) yields the job creation condition:

$$c = (1 - \varepsilon) \lambda_e (1 - G(m)) \frac{z^e - U_m}{\delta}. \tag{A19}$$

Combining equations (A18) and (A19), I get a relation between the value of unemployed workers, market tightness, and the "effective" bargaining power that holds in the constrained equilibrium

$$U_m = b + \frac{\varepsilon}{(1 - \varepsilon)} c\theta, \tag{A20}$$

which, when substituted into (A19), gives

$$\frac{\delta c}{q(\theta)\left[1-G(m)\right]} = (1-\varepsilon)\left(z^e - b\right) - \varepsilon c \theta. \tag{A21}$$

A.2.1 Equilibrium Wage Distribution

Let us now examine how wages are distributed in the constrained equilibrium. In the presence of a binding minimum wage, wages are given by (4). Contacts with match values on $[\hat{z}, \overline{z}]$ result in jobs paying wages weakly greater than the minimum. Solving the wage function in terms of the match-specific productivity

$$z = x(w;m) = \frac{w - (1-\beta)U_m}{\beta},$$

where $x(w;m)$ is the inverse function of $w(z)$ for any given minimum wage, m. Meetings of firms with workers lead to acceptable matches if the realized match value z is greater than or equal to the minimum wage, which happens with probability $\left[1-G(m)\right]$. Therefore, the equilibrium distribution of wages that are strictly greater than the minimum can be expressed as a simple mapping from the match specific productivity distribution, $G(z)$, into $F(w)$, conditional on the match value (and wage) being on the acceptable range

$$F(w|w>m) = \frac{G(x(w;m)) - G(m)}{1-G(m)}.$$

All contacts with $z \in [m, \hat{z})$ result in jobs offering the minimum wage. This implies that the observed wage distribution will exhibit a mass point/ spike at the legislated minimum wage. The magnitude of the spike is given by the mass of matches offering the minimum wage, $G(\hat{z}) - G(m)$, conditional on them being on the acceptable range, that is,

$$F(w|w=m) = \frac{G(\hat{z}) - G(m)}{1-G(m)}.$$

From the above analysis, it is straightforward that the density function of observed wages in the constrained steady state is

$$f(w;m) = \begin{cases} \frac{\beta^{-1} g(x(w;m))}{1-G(m)} & \text{for } w > m, \\ \frac{G(\hat{z})-G(m)}{1-G(m)} & \text{for } w = m, \\ 0 & \text{for } w < m. \end{cases} \tag{A22}$$

B The Social Welfare Function and the Planner's Problem

Define social welfare as aggregate output net of search costs −the cost of posting vacancies:

$$SW_m = ub + (1-u)\, z^e - vc.$$

Note that in this definition of welfare I include the return on leisure received by job seekers, b, the surplus of accepted matches −which is divided between the firm and its employee− expressed in terms of expected productivity, $z^e = E\left(z | z \geq z_m^*\right)$, and finally, I subtract the costs incurred by firms holding a vacancy. Recalling that $v = \theta u$, I obtain a more convenient expression for social welfare:

$$SW_m = ub + (1-u)\, z^e - \theta uc, \tag{B1}$$

B.1 The Problem of Social Planner Constrained by m

Consider the case of a Social Planner constrained by an exogenous minimum wage, as in Section 2.4. The planner solves the following problem:

$$\max_{\theta, z_m^*, u} \{SW_m = ub + (1-u)\, z^e - \theta uc\}, \tag{B2}$$

subject to

$$u = \frac{\delta}{\delta + \theta q(\theta) \left[1 - G\left(z_m^*\right)\right]}$$

$$z_m^* \geq m$$

The Lagrangian is:

$$\mathcal{L} = ub + (1-u)\, z^e - \theta uc + \kappa_1 \left[u - \frac{\delta}{\delta + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right]}\right] + \kappa_2 \left[z_m^* - m\right] \tag{B3}$$

The solution to the planner's problem satisfies the following conditions:

$$\mathcal{L}_u = b - z^e - \theta c + \kappa_1 = 0 \tag{B4}$$

$$\mathcal{L}_\theta = -uc + \kappa_1 \frac{\delta\left[q(\theta) + \theta q'(\theta)\right]\left[1 - G\left(z_m^*\right)\right]}{\left[\delta + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right]\right]^2} = 0 \tag{B5}$$

$$\mathcal{L}_{z_m^*} = -z_m^* g(z_m^*)(1-u) - \kappa_1 \frac{\theta q(\theta) g\left(z_m^*\right)}{\left[\delta + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right]\right]^2} + \kappa_2 = 0 \tag{B6}$$

$$\mathcal{L}_{\kappa_1} = u - \frac{\delta}{\delta + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right]} = 0 \tag{B7}$$

$$\mathcal{L}_{\kappa_2} = z_m^* - m \geq 0, \quad \kappa_2 \geq 0, \quad \kappa_2 \mathcal{L}_{\kappa_2} = 0 \tag{B8}$$

Combining conditions (B5) and (B7), solving for κ_1, and substituting κ_1 in condition (B4) (taking into account that $\eta = -\frac{q'(\theta)\theta}{q(\theta)}$) leads to:

$$\frac{\delta c}{q(\theta)\left[1 - G\left(z_m^*\right)\right]} = (1-\eta)\left(z^e - b\right) - \eta c \theta. \tag{B9}$$

Combining conditions (B5) and (B6), implies that $\kappa_2 > 0$, which in turn implies (from (B8)):

$$z_m^* = m \tag{B10}$$

B.2 The Value of Unemployment as a Welfare Criterion

Using the welfare function, given by B1, substitute u from equilibrium equation (6):

$$SW_m = \frac{\delta b + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right] z^e - \delta c \theta}{\delta + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right]}.$$

To proceed, we need to use the two conditions characterising the equilibrium behaviour of workers and firms searching for a match, given by (A18) and (A19), and rewritten here for ease of exposition:

$$U_m = b + \varepsilon \lambda_w \left(1 - G\left(z_m^*\right)\right) \frac{z^e - U_m}{\delta},$$
$$c = (1-\varepsilon)\lambda_e \left(1 - G\left(z_m^*\right)\right) \frac{z^e - U_m}{\delta}.$$

Using condition (A19), implied by the free entry of firms in equilibrium, yields:

$$SW_m = \frac{\delta b + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right] z^e - \theta q(\theta)\left[1 - G\left(z_m^*\right)\right](1-\varepsilon)\left(z^e - U_m\right)}{\delta + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right]}.$$

Recalling that U_m is given by (A18), implies

$$SW_m = \frac{\delta b + \delta(U_m - b) + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right] U_m}{\delta + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right]} = U_m. \tag{B11}$$

In equilibrium, social welfare is equal to the value of unemployed search. This implies that the social planner's problem reduces to the maximisation of the value of unemployed search.

This equilibrium relationship between social welfare and the value of unemployed search holds both in the constrained and in the unconstrained cases. The derivation in the unconstrained case follows the same steps but assumes that the minimum wage is not binding, that is, m is weakly smaller than the value of unemployed search.

C Equilibrium Effects of Changes in the Minimum Wage

A change in the minimum wage affects simultaneously the "effective" bargaining power of workers, the value of search, market tightness, and unemployment. The impact of the minimum wage on the equilibrium values of these variables is determined endogenously: one needs to examine the simultaneous effect of the minimum wage on the set of equilibrium equations (5), (6), (7), and (8). Taking the total differential of each equilibrium equation yields a system of four equations, which can be solved in terms of $\frac{d\varepsilon}{dm}$, $\frac{dU_m}{dm}$, $\frac{d\theta}{dm}$, and $\frac{du}{dm}$.

To consider the effects of minimum wage changes on equilibrium unemployment, take the total differential of equation (6):

$$\frac{du}{dm} = -\Delta_1 \frac{d\theta}{dm} + \Delta_2, \tag{C1}$$

where

$$\Delta_1 = \frac{u\left(1 - G\left(m\right)\right) q\left(\theta\right)\left(1 - \eta\right)}{\left[\delta + \theta q\left(\theta\right)\left(1 - G\left(m\right)\right)\right]} \text{ and} \tag{C2}$$

$$\Delta_2 = \frac{u\theta q\left(\theta\right) g\left(m\right)}{\left[\delta + \theta q\left(\theta\right)\left(1 - G\left(m\right)\right)\right]}. \tag{C3}$$

Clearly, the variation of equilibrium unemployment in response to minimum wage changes is dependent on the effect of m on θ. If $\frac{d\theta}{dm} < 0$, equilibrium unemployment increases after a minimum wage hike; if $\frac{d\theta}{dm} > 0$, the effect of m on u is ambiguous.

Let us now examine the implications of a varying minimum wage for the other endogenously determined variables. Take the total differential of the reduced-form relation between the value of search and market tightness, given by equation (7),

$$\frac{dU_m}{dm} = \frac{c\varepsilon}{\left(1 - \varepsilon\right)} \frac{d\theta}{dm} + \frac{c\theta}{\left(1 - \varepsilon\right)^2} \frac{d\varepsilon}{dm}. \tag{C4}$$

Similarly, the total differential of the job creation condition, given by equation (8), is:

$$\begin{aligned} - \left[1 - G\left(m\right)\right]\left\{\eta\left[\left(1 - \varepsilon\right)\left(z^e - U_m\right)\right] + \theta\varepsilon c\right\} \frac{d\theta}{dm} - \\ - \left[1 - G\left(m\right)\right]\theta\left[\left(z^e - b\right) + \theta c\right] \frac{d\varepsilon}{dm} - \\ - g\left(m\right)\theta\left[\left(1 - \varepsilon\right)\left(m - U_m\right)\right] = 0. \end{aligned} \tag{C5}$$

It is evident that both $\frac{dU_m}{dm}$ and $\frac{d\theta}{dm}$ depend on the impact of m on the "effective" bargaining power of workers; the "effective" bargaining power of workers is defined by equation (5). Substituting the average wage and average match specific productivity into (5) gives

$$\varepsilon = \frac{\left(m - U_m\right)\left(G\left(\hat{z}\right) - G\left(m\right)\right) + \beta \int_{\hat{z}}^{\overline{z}} \left(z - U_m\right) dG\left(z\right)}{\int_m^{\overline{z}} \left(z - U_m\right) dG\left(z\right)}. \tag{C6}$$

The total differential of the above equation is:

$$\int_m^{\overline{z}} (z - U_m)\, dG(z) \frac{d\varepsilon}{dm} = -\left[(1-\varepsilon)(1-G(m)) - (1-\beta)(1-G(\hat{z}))\right] \frac{dU_m}{dm} + \\ + \left[G(\hat{z}) - G(m) - (1-\varepsilon)(m - U_m) g(m)\right]. \quad \text{(C7)}$$

Exploiting equations (C4), (C5), and (C7) it is possible to determine how the equilibrium values of U_m and θ vary with m. First, use equation (C5) to substitute $\frac{d\varepsilon}{dm}$ into (C4) and solve for $\frac{d\theta}{dm}$

$$\frac{d\theta}{dm} = -\left(\frac{(1-\varepsilon)^2 (z^e - b + \theta c)}{c(\eta - \varepsilon)\left[(1-\varepsilon)(z^e - b) - \theta\varepsilon c\right]}\right)\frac{dU_m}{dm} - \\ - \left(\frac{\theta(1-\varepsilon)(m - U_m) g(m)}{(1 - G(m))(\eta - \varepsilon)\left[(1-\varepsilon)(z^e - b) - \theta\varepsilon c\right]}\right). \quad \text{(C8)}$$

From equation (C8), substitute $\frac{d\theta}{dm}$ into equation (C4)

$$\frac{dU_m}{dm} = \frac{c\theta}{(1-\varepsilon)} \frac{[1 - G(m)](z^e - U_m)[\eta - \varepsilon]}{[1 - G(m)]\{\eta[(1-\varepsilon)(z^e - b) - \theta\varepsilon c] + \theta\varepsilon c\}} \frac{d\varepsilon}{dm} - \\ - \frac{c\theta}{(1-\varepsilon)} \frac{\varepsilon(1-\varepsilon)(m - U_m) g(m)}{[1 - G(m)]\{\eta[(1-\varepsilon)(z^e - b) - \theta\varepsilon c] + \theta\varepsilon c\}}. \quad \text{(C9)}$$

Now use equation (C7) to substitute $\frac{d\varepsilon}{dm}$ into equation (C9)

$$B_1 \frac{dU_m}{dm} = B_2\left[(G(\hat{z}) - G(m))(\eta - \varepsilon) - \eta(1-\varepsilon)(m - U_m) g(m)\right], \quad \text{(C10)}$$

where

$$B_1 = 1 + \frac{c\theta}{(1-\varepsilon)} \frac{[\eta - \varepsilon]\left[(1-\varepsilon)(1 - G(m)) - (1-\beta)(1 - G(\hat{z}))\right]}{[1 - G(m)]\{\eta[(1-\varepsilon)(z^e - b) - \theta\varepsilon c] + \theta\varepsilon c\}} \text{ and} \quad \text{(C11)}$$

$$B_2 = \frac{c\theta}{(1-\varepsilon)[1 - G(m)]\{\eta[(1-\varepsilon)(z^e - b) - \theta\varepsilon c] + \theta\varepsilon c\}}. \quad \text{(C12)}$$

Equation (C10) gives a reduced-form relation between $\frac{dU_m}{dm}$ and its determinants. The following Lemma simplifies the relation that determines the dependence of U_m on m:

Lemma 1 *The direction of change in the value of search after an increase in the minimum wage is given by*

$$\frac{dU_m}{dm} \overset{sgn}{=} (G(\hat{z}) - G(m))(\eta - \varepsilon) - \eta(1-\varepsilon)(m - U_m) g(m). \quad \text{(C13)}$$

Proof Clearly, it has to be the case that $B_1 \overset{sgn}{=} B_2$, where B_1 and B_2 are given by equation (C11) and equation (C12), respectively. Inspection of equation (C12) suggests that B_2 is positive, so for (C13) to hold, B_1 must be positive.

B_1 can be expressed as follows

$$B_1 = \frac{(1-\varepsilon)\left[1-G(m)\right]\eta\left[(1-\varepsilon)(z^e-b)-\theta\varepsilon c\right]+(1-\beta)(1-G(\hat{z}))\,\theta\varepsilon c}{(1-\varepsilon)\left[1-G(m)\right]\left\{\eta\left[(1-\varepsilon)(z^e-b)-\theta\varepsilon c\right]+\theta\varepsilon c\right\}}+$$
$$+\eta\theta c\frac{(1-\varepsilon)\left[1-G(m)\right]-(1-\beta)(1-G(\hat{z}))}{(1-\varepsilon)\left[1-G(m)\right]\left\{\eta\left[(1-\varepsilon)(z^e-b)-\theta\varepsilon c\right]+\theta\varepsilon c\right\}},$$

which is positive if

$$(1-\varepsilon)\left[1-G(m)\right]-(1-\beta)\left[1-G(\hat{z})\right]\geq 0. \tag{C14}$$

Combining equations (5) and (A19), we can rewrite the job creation condition as follows:

$$c = q(\theta)\left(\frac{\int_m^{\hat{z}}(z-m)\,dG(z)+(1-\beta)\int_{\hat{z}}^{\overline{z}}(z-U_m)\,dG(z)}{\delta}\right). \tag{C15}$$

Using equations (A19) and (C15), we can substitute $(1-\varepsilon)\left[1-G(m)\right]$ and $(1-\beta)\left[1-G(\hat{z})\right]$ into (C14):

$$(1-\varepsilon)\left[1-G(m)\right]-(1-\beta)(1-G(\hat{z}))=$$
$$=\frac{\delta c}{q(\theta)\,\frac{\int_m^{\overline{z}}(z-U_m)dG(z)}{[1-G(m)]}}-\frac{\delta c-q(\theta)\int_m^{\hat{z}}(z-m)\,dG(z)}{q(\theta)\,\frac{\int_{\hat{z}}^{\overline{z}}(z-U_m)dG(z)}{[1-G(\hat{z})]}}=$$
$$=\frac{\left[1-G(\hat{z})\right]\int_m^{\hat{z}}(z-m)\,dG(z)}{\int_{\hat{z}}^{\overline{z}}(z-U_m)\,dG(z)}+$$
$$+\frac{\delta c}{q(\theta)}\left[\left(\frac{\int_m^{\overline{z}}(z-U_m)\,dG(z)}{\left[1-G(m)\right]}\right)^{-1}-\left(\frac{\int_{\hat{z}}^{\overline{z}}(z-U_m)\,dG(z)}{\left[1-G(\hat{z})\right]}\right)^{-1}\right].$$

Given that $\frac{\int_l^L zdG(z)}{[1-G(l)]}$ is decreasing in l, the above expression is positive for $\hat{z}\geq m$. Therefore, B_1 is positive. ■

Exploiting Lemma 1, I determine the sufficient condition for $\frac{d\theta}{dm}<0$, summarised in Lemma 2:

Lemma 2 *If the value of search is increasing in the minimum wage* $\left(\frac{dU_m}{dm}>0\right)$, *market tightness is decreasing in the minimum wage* $\left(\frac{d\theta}{dm}<0\right)$.

Proof Inspection of equation (C8) suggests that $\frac{dU_m}{dm}>0$ and $(\eta-\varepsilon)>0$ imply $\frac{d\theta}{dm}<0$. From (C13), $(\eta-\varepsilon)>0$ is a necessary condition for $\frac{dU_m}{dm}>0$. Therefore, if $\frac{dU_m}{dm}>0$, then $(\eta-\varepsilon)>0$. Hence, $\frac{dU_m}{dm}>0$ is a sufficient condition for $\frac{d\theta}{dm}<0$. ■

C.1 Properties of the "Effective" Bargaining Power

The "effective" bargaining power of workers is defined as the average share of the match surplus received by employed workers, equation (5). Substituting the average wage and average match specific productivity from Definition 1 into (5) gives the relation between the "effective" bargaining power and its determinants, equation (C6), which is reproduced here for convenience:

$$\varepsilon = \frac{(m - U_m)(G(\hat{z}) - G(m)) + \beta \int_{\hat{z}}^{\overline{z}} (z - U_m)\, dG(z)}{\int_m^{\overline{z}} (z - U_m)\, dG(z)}.$$

It is evident that ε is a function of β, m, and U_m: $\varepsilon = \phi(\beta, m, U_m)$. If the minimum wage is not binding ($m \leq U_m = U$), then $\varepsilon = \phi(\beta, m, U) = \beta$. The intuition is that in the absence of a binding wage floor all workers capture a share of the surplus equal to their bargaining power, β. If the minimum wage is binding ($m > U_m$), the workers capture a share of the surplus greater than their bargaining power, so $\varepsilon = \phi(\beta, m, U_m) > \beta$.

To examine how $\varepsilon = \phi(\beta, m, U_m)$ varies with β, m, and U_m, I take its total differential

$$d\varepsilon = \frac{\partial \phi(\beta, m, U_m)}{\partial \beta} d\beta + \frac{\partial \phi(\beta, m, U_m)}{\partial m} dm + \frac{\partial \phi(\beta, m, U_m)}{\partial U_m} dU_m$$

or alternatively

$$\begin{aligned}\int_m^{\overline{z}} (z - U_m)\, dG(z)\, d\varepsilon = &\left[\int_{\hat{z}}^{\overline{z}} (z - U_m)\, dG(z)\right] d\beta + \\ &+ \left[G(\hat{z}) - G(m) - (1 - \varepsilon)(m - U_m)\, g(m)\right] dm - \\ &- \left[(1 - \varepsilon)(1 - G(m)) - (1 - \beta)(1 - G(\hat{z}))\right] dU_m. \end{aligned} \tag{C16}$$

Clearly, the "effective" bargaining power of workers is decreasing in the value of search (U_m). Any factor that affects U_m in the constrained decentralised equilibrium should also have an indirect effect on ε. In the context of this paper, I only consider the equilibrium effects of changes in the minimum wage, so $d\beta = 0$ and all other factors (e.g. unemployment benefits) that could influence ε indirectly through U_m are assumed to be given.

To account for this indirect effect of the minimum wage on ε, I substitute dU_m from equation (C9) into equation (C16). This gives

$$Q_1 d\varepsilon = Q_2 dm, \tag{C17}$$

where

$$\begin{aligned} Q_1 = \frac{c\theta}{(1 - \varepsilon)} &\frac{\left[(1 - \varepsilon)(1 - G(m)) - (1 - \beta)(1 - G(\hat{z}))\right]\left[1 - G(m)\right](z^e - U_m)\left[\eta - \varepsilon\right]}{\left[1 - G(m)\right]\left\{\eta\left[(1 - \varepsilon)(z^e - b) - \theta\varepsilon c\right] + \theta\varepsilon c\right\}} + \\ &+ \int_m^{\overline{z}} (z - U_m)\, dG(z), \end{aligned}$$

and

$$Q_2 = \frac{c\theta\left[(1-\varepsilon)(1-G(m)) - (1-\beta)(1-G(\hat{z}))\right]\varepsilon(m-U_m)g(m)}{[1-G(m)]\left\{\eta\left[(1-\varepsilon)(z^e-b)-\theta\varepsilon c\right]+\theta\varepsilon c\right\}} + \\ + \left[G(\hat{z}) - G(m) - (1-\varepsilon)(m-U_m)g(m)\right]. \tag{C18}$$

Using the proof of Lemma 1, it is straightforward to show that Q_1 is positive. Therefore, the dependence of ε on the minimum wage is only influenced by the sign of Q_2. Rewrite equation (C18) as follows

$$Q_2 = Q_{2A} + Q_{2B},$$

where

$$Q_{2A} = \frac{c\theta\left[(1-\varepsilon)(1-G(m)) - (1-\beta)(1-G(\hat{z}))\right]\varepsilon(m-U_m)g(m)}{[1-G(m)]\left\{\eta\left[(1-\varepsilon)(z^e-b)-\theta\varepsilon c\right]+\theta\varepsilon c\right\}} \text{ and} \tag{C19}$$

$$Q_{2B} = \left[G(\hat{z}) - G(m) - (1-\varepsilon)(m-U_m)g(m)\right]. \tag{C20}$$

Clearly, $Q_{2A} > 0$ (see the proof of Lemma 1), so the sign of Q_2 depends on the sign of Q_{2B}. If $Q_{2B} \geq 0$, then ε is increasing in m. If $Q_{2B} < 0$, then the dependence of ε on the minimum wage is ambiguous in sign. Therefore,

$$\left[G(\hat{z}) - G(m) - (1-\varepsilon)(m-U_m)g(m)\right] \geq 0 \tag{C21}$$

is a sufficient condition for $\frac{d\varepsilon}{dm} > 0$.

Combining this sufficient condition for the "effective" bargaining power to be increasing in the minimum wage with Lemma 1 leads to the following result:

Lemma 3 *If the value of search is increasing in the minimum wage, the "effective" bargaining power of workers is also increasing in the minimum wage.*

Proof Lemma 1 suggests that $\frac{dU_m}{dm} > 0$ if and only if

$$\left[G(\hat{z}) - G(m)\right](\eta-\varepsilon) - \eta(1-\varepsilon)(m-U_m)g(m) > 0,$$

which implies

$$\left[G(\hat{z}) - G(m)\right] - (1-\varepsilon)(m-U_m)g(m) > \frac{\varepsilon\left[G(\hat{z}) - G(m)\right]}{\eta} > 0.$$

Therefore, if $\frac{dU_m}{dm} > 0$, (C21) holds, so $\frac{d\varepsilon}{dm} > 0$. ■

D Proofs

Proof of Proposition 1. Under a binding minimum wage, $m > U$ and $\hat{z} > m$. If $\varepsilon > \eta$, then

$$\left(G(\hat{z}) - G(m)\right)(\eta-\varepsilon) - \eta(1-\varepsilon)(m-U_m)g(m) < 0,$$

which implies (from equation [14])

$$\frac{dU_m}{dm} < 0.$$

Therefore, if $\varepsilon > \eta$, then U_m and social welfare decrease with the minimum wage. ■

Proof of Proposition 2. Suppose $\frac{dU_m}{dm} \geq 0$. As demonstrated in Proposition 1 and Corollary 1, $\eta > \varepsilon$ is a necessary condition for $\frac{dU_m}{dm} \geq 0$. For ease of exposition, rewrite inequality (16):

$$\frac{dU_m}{dm} \overset{sgn}{=} \eta \times \underbrace{\left[\frac{(1-\beta)}{\beta} g(x_m) - g(m)\right]}_{\text{expression F1}} - \varepsilon \times \underbrace{\left[\frac{(1-\beta)}{\beta} g(x_m) - \eta g(m)\right]}_{\text{expression F2}} \geq 0 \tag{D1}$$

Simple inspection of expressions (F1) and (F2) in inequality (D1) suggests that $(F1) \leq (F2)$. This observation allows us to prove (by contradiction) that $(F1) > 0$ is a necessary condition for $\frac{dU_m}{dm} \geq 0$. Suppose that $F1 < 0$; from the left hand-side of inequality (D1), we have:

$$\eta \times (F1) - \varepsilon \times (F2) \leq \eta \times (F1) - \varepsilon \times (F1) = (\eta - \varepsilon) \times (F1) < 0, \tag{D2}$$

which means that inequality (D1) does not hold and $\frac{dU_m}{dm} < 0$. Therefore, a minimum wage increase can only improve welfare, $\frac{dU_m}{dm} \geq 0$, if $0 < (F1) \leq (F2)$, which implies:

$$\frac{(1-\beta)}{\beta} > \frac{g(m)}{g(x_m)} \tag{D3}$$

■

Proof of Proposition 3. Condition (18) includes two inequalities. Inequality

$$\frac{\eta\left[g\left(x_m\right) - \beta g\left(x_m\right) - \beta g\left(m\right)\right]}{g\left(x_m\right) - \beta g\left(x_m\right) - \beta \eta g\left(m\right)} = \frac{\eta \times (F1)}{(F2)} \leq \eta$$

encompasses the necessary condition presented in Proposition 2, i.e. $(F1) > 0$. Starting from inequality (16), or equivalently inequality (D1), we have

$$0 < \varepsilon(m) \leq \frac{\eta \times (F1)}{(F2)} \leq \eta \Rightarrow 0 < (F1) \leq (F2)$$

A binding minimum wage m is welfare improving if m and the corresponding “effective” bargaining power $\varepsilon\left(m\right)$ satisfy inequality (16), or equivalently inequality (D1), which can be solved in terms of ε (given $(F1) > 0$):

$$\varepsilon(m) \leq \frac{\eta\left[g\left(x_m\right) - \beta g\left(x_m\right) - \beta g\left(m\right)\right]}{g\left(x_m\right) - \beta g\left(x_m\right) - \beta \eta g\left(m\right)}. \tag{D4}$$

Consider a binding minimum wage m and the corresponding $\varepsilon(m) > \beta$. If $\varepsilon(m)$ satisfies condition (D4), then (given (F1)¿0) it is straightforward to show that $\frac{dU_m}{dm} \geq 0$. ■

Proof of Proposition 4 Consider the effect of a minimum wage change from m_1 to

m_2, assuming that $m_2 > m_1$. Wages under the two different levels of the wage floor m_1 and m_2 are denoted w_1 and w_2, respectively, and are given by equation (4), reproduced below

$$w_i(z) = \left\{ \begin{array}{c} \beta z + (1-\beta) U_{m_i}, \text{ for } z \in [\hat{z}_i, \overline{z}] \\ m_i > \beta z + (1-\beta) U_{m_i}, \text{ for } z \in [m_i, \hat{z}_i) \end{array} \right\}, \tag{D5}$$

where $i \in \{1, 2\}$.

Suppose that the minimum wage increase has no spillover effects. Conditional on a worker being matched with the same employer under both minimum wage levels, his earnings under the new minimum, $w_2(z)$, should equal his initial earnings, $w_1(z)$, plus the compliance effect, denoted $CP(z; m_1, m_2)$. Algebraically,

$$w_2(z) = w_1(z) + CP(z; m_1, m_2) =$$
$$= w_1(z) + \max(m_2 - w_1, 0) = \left\{ \begin{array}{c} w_1(z), \text{ for } z \in [\hat{z}_2, \overline{z}] \\ m_2, \text{ for } z \in [m_2, \hat{z}_2) \end{array} \right\}.$$

All matches with productivity $z \in [\hat{z}_2, \overline{z}]$ pay the same wage under both minimum wages. Simple comparison of the wage equations on the this range of z values suggests that

$$U_{m_1} = U_{m_2}.$$

In other words, if there are no spillover effects, the minimum wage uprating does not affect the value of search.

Let us now assume that the minimum wage uprating considered above has spillover effects. Conditional on a worker being matched with the same employer under both minimum wage levels, the minimum wage uprating has a spillover effect on this worker's wage if and only if

$$w_2(z) \neq w_1(z) \text{ for } w_1(z), w_2(z) > m_2 > m_1.$$

Equation (D5) implies that this can only happen if

$$U_{m_1} \neq U_{m_2}.$$

The magnitude of the spillover effect, denoted $SP(z; m_1, m_2)$, is given by

$$SP(z; m_1, m_2) = w_2(z) - w_1(z) = (1-\beta)(U_{m_2} - U_{m_1}), \text{ for } z \in [\hat{z}_2, \overline{z}].$$

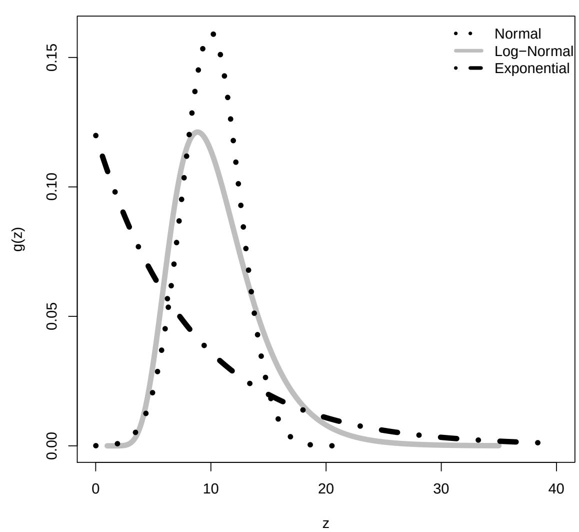
Therefore, the minimum wage uprating has positive spillover effects when the value of search increases $(U_{m_1} < U_{m_2})$ and negative spillover effects when the value of search decreases $(U_{m_1} > U_{m_2})$. ■

Proof of Proposition 5. Consider (C1) in Appendix C. It is evident that when $\frac{d\theta}{dm} < 0$, equilibrium unemployment increases in response to a minimum wage increase $\left(\frac{du}{dm} > 0\right)$. Lemma 2 in Appendix C shows that if the value of search increases $\left(\frac{dU_m}{dm} > 0\right)$ with the minimum, market tightness will decrease $\left(\frac{d\theta}{dm} < 0\right)$. This is only a sufficient condition for increasing equilibrium unemployment: market tightness may decrease even in cases where $\frac{dU_m}{dm} < 0$. ■

E Numerical Illustration

To illustrate the properties of the labour market equilibrium with and without a binding minimum wage, I consider a number of parametric examples. This exercise is similar to Example 2 in Chapter 3 of Flinn (2011), where similar parametric examples are presented and discussed; a notable difference is that in the framework considered by Flinn (2011) the decision to participate in the workforce is endogenous. I compute equilibrium outcomes under three assumptions regarding the match-specific productivity distribution: (i) productivity is normally distributed with mean 10 and standard deviation 2.5; (ii) log-productivity is normally distributed with mean $\ln 10$ and standard deviation 0.35; (iii) productivity follows an exponential distribution with parameter 0.12. Figure E2 presents the probability density function (PDF) of each one of these three match-specific productivity distributions.

Figure E2: Match-Specific Productivity PDF.



Tables E1 and E2 present equilibrium outcomes for various combinations of the bargaining power parameter (β), the elasticity of the matching function with respect to unemployment (η), and the minimum wage. The equilibrium outcomes considered are: the value of unemployment (U_m), which is a measure of social welfare; the unemployment rate (u), the level of labour market tightness θ, and the "effective" bargaining power (ε). The last two columns of Tables E1 and E2 report whether the two efficiency conditions are satisfied: (18) in Proposition 3 and condition (20) in Corollary 2. If the efficiency condition for incremental changes in the minimum wage ($\beta < \frac{\eta}{1+\eta}$) is satisfied, then there is room for the minimum wage to improve social welfare. If the more general efficiency condition ($\varepsilon < \overline{\varepsilon}$)[18] is satisfied, then a discrete change in the minimum wage will be welfare improving. Finally, if $\varepsilon = \overline{\varepsilon}$, then the minimum wage is optimal /welfare maximising.

To highlight the differential impact of the minimum wage when it "bites" at different parts of the same distribution, I consider two different parameterisations:

- in Table E1, the flow cost of holding a vacancy (c) is 60, the utility flow in unemployment (b) is -5, and the job separation rate (δ) is 0.038
- in Table E2, the flow cost of holding a vacancy (c) is 180, the utility flow in unemployment (b) is -45, and the job separation rate (δ) is 0.038.

First, consider the environments presented in Table E1. If workers have a low bargaining power (i.e. low β) and make small contributions to the creation of match opportunities (i.e. low η), the introduction of a minimum wage has a negative impact on unemployment (u) and social welfare (U_m); this is evident if one compares environments 1 and 2, or 13 and 14, or 25 and 26. If the bargaining power is lower than the elasticity of matching with respect to unemployment, then a binding minimum wage can lead to improvements in social welfare: compare environments 3 and 4, or 15 and 16, or 27 and 28.

The intuition for these results is as follows: when the contribution of workers to the creation of match opportunities (η) exceeds the share of the surplus they capture (β), firms make high profits and create an inefficiently high number of vacancies; at the same time, the value of unemployment is inefficiently low leading to high job acceptance. Introducing a minimum wage in this context has the dual effect of adjusting job creation and job acceptance to levels that are more in line with an efficient labour market equilibrium. For example, in environment 3, unemployment is inefficiently low, $u = 4.76\%$; the efficient allocation would be achieved if β was raised to the level of η, as in environment 7, where unemployment is more than double in magnitude, $u = 9.59\%$. If the planner has no control over the division of the match surplus, the minimum wage can be used as a policy tool to push the equilibrium allocation closer to the efficient level; for example, in environment 4 the allocation achieved through an optimally chosen minimum wage dominates the allocation in environment 3; however, the optimal allocation with a minimum wage in environment 4 is dominated by the optimal allocation without a minimum wage in environment 7. Specifically, environment 4 is characterised by: inefficiently low job acceptance, the lowest acceptable productivity in environment 4 is the minimum wage $m = 9.36$, whereas in environment 7 it is given by the value of unemployed search $U^* = 8.21$; inefficiently high job creation, for a similar level of unemployment, market tightness in environment 4 is much higher ($\theta = 0.35$) than in environment 7 ($\theta = 0.22$). These results are not sensitive to the shape of the underlying productivity distribution: minimum wage changes have qualitatively similar effects in all three panels of Table E1. Figure E3 illustrates that an optimally chosen minimum wage can improve welfare in an inefficient economy and bring the allocation closer to the socially optimal level: starting from an inefficient equilibrium allocation without a minimum wage

[18]Note that $\overline{\varepsilon}$ is the right hand-side of (18) in Proposition 3.

(environment 15 where welfare is $U = 8.4357$), we can see how introducing and then gradually increasing the minimum wage leads to higher welfare; when $m = 10.49$ welfare under a minimum wage reaches a maximum ($U_m^* = 8.7865$), this is environment 16; the socially efficient equilibrium allocation is given by environment 19, where the Hosios condition for efficiency is satisfied ($\beta = \eta = 0.5$) and welfare is $U^* = 9.1844$. The minimum wage can correct in part existing inefficiencies in the equilibrium allocation and bring the economy closer to the social optimum. However, as Figure E3 depicts the socially optimal allocation under a minimum wage is dominated by the socially optimal allocation without a minimum wage.

Figures E4 and E5 present the paths of minimum wages that maximise welfare for the parameterisations corresponding to environments 13-24 (lognormal match-specific productivity distribution) and environments 25-36 (exponential match-specific productivity distribution). These figures illustrate that the welfare impact of minimum wage changes is not particularly sensitive to changes in the match-specific productivity distribution: for given parameter values, the shape of social welfare responses to changes in the minimum wage is very similar in two Figures.

Suppose now that the bargaining power of workers exceeds the bargaining power of firms ($\beta > \frac{1}{2}$): for example, environments 10-12, 22-24, and 34-36 in Table E1. In these environments, the minimum wage does not have a welfare improving effect; the conditions for a welfare improving minimum wage are violated as indicated by the last two columns of Table E1. The equilibrium allocation in these environments is such that the density of acceptable productivities is either everywhere decreasing (i.e. $g'(z) < 0$) or increasing for a small range of values of z and then decreasing. Therefore, in such a setting, the costs of a minimum wage hike would outweigh the benefits: a higher m would imply a loss of worker surplus, $g(m) \times \beta$, much greater than the extra surplus captured by the average worker earning the new minimum, $g(x_m) \times (1 - \beta)$. This is due to the already high bargaining power of workers ($\beta > \frac{1}{2}$), and the not sufficiently increasing density of match-specific productivities.

In Table E2, I consider a different parameterisation and examine the impact of the minimum wage when it "bites" at different parts of the match-specific productivity distribution, where the density is more likely to be increasing. Under this parameterisation, a minimum wage hike can improve welfare even in environments where the bargaining power of workers exceeds the bargaining power of firms (for example, see environment 40). The intuition for this result is that the increasing match-specific productivity density implies that a much bigger mass of workers capture a bigger share of the surplus due to the higher minimum wage than the mass of workers forced into unemployment due to their matches no longer meeting the productivity threshold.

Table E1: Equilibrium Outcomes with & without a minimum wage, Experiment A

Environment	Parameters β	η	Equilibrium $m^{\dagger}$	U_m	u	θ	ε	Condition satisfied $\varepsilon \leq \overline{\varepsilon}$ §	$\beta < \frac{\eta}{1+\eta}$ §§
Normal $G(z)$: $z \sim \mathcal{N}(10, 2.5^2)$									
(1)	0.200	0.2	0.0^*	7.45541	0.04956	0.83036	0.200	NO	NO
(2)	0.200	0.2	7.60	7.45420	0.05069	0.82326	0.201	NO	NO
(3)	0.200	0.5	0.0	7.61355	0.04755	0.84090	0.200	YES	YES
(4)	0.200	0.5	9.36^*	7.97727	0.09699	0.34709	0.384	YES	YES
(5)	0.330	0.5	0.0	8.04996	0.06812	0.44159	0.330	YES	YES
(6)	0.330	0.5	8.79^*	8.05347	0.08112	0.39448	0.355	YES	YES
(7)	0.500	0.5	0.0^*	8.20769	0.09593	0.22013	0.500	NO	NO
(8)	0.330	0.8	0.0	8.62544	0.05890	0.46106	0.330	YES	YES
(9)	0.330	0.8	10.61^*	8.89864	0.11196	0.23075	0.499	YES	YES
(10)	0.501	0.8	0.0^*	9.21425	0.07525	0.23596	0.501	NO	NO
(11)	0.501	0.8	9.40	9.21272	0.07862	0.23505	0.502	NO	NO
(12)	0.800	0.8	0.0^*	9.59387	0.10542	0.06081	0.800	NO	NO
Log-Normal $G(z)$: $z \sim$Lognormal$(\ln(10), 0.35^2)$									
(13)	0.200	0.2	0.0^*	8.31252	0.05626	0.88750	0.200	NO	NO
(14)	0.200	0.2	8.50	8.30956	0.05869	0.87370	0.201	NO	NO
(15)	0.200	0.5	0.0	8.43570	0.05525	0.89571	0.200	YES	YES
(16)	0.200	0.5	10.49^*	8.78650	0.11430	0.43588	0.345	YES	YES
(17)	0.330	0.5	0.0	8.98135	0.08175	0.47311	0.330	YES	YES
(18)	0.330	0.5	9.27^*	8.98151	0.08671	0.46698	0.333	YES	YES
(19)	0.500	0.5	0.0^*	9.18441	0.11592	0.23641	0.500	NO	NO
(20)	0.330	0.8	0.0	9.66777	0.07509	0.49633	0.330	YES	YES
(21)	0.330	0.8	12.02^*	9.89896	0.13810	0.31031	0.445	YES	YES
(22)	0.501	0.8	0.0^*	10.48511	0.10052	0.25705	0.501	NO	NO
(23)	0.501	0.8	10.70	10.48317	0.10538	0.25623	0.502	NO	NO
(24)	0.800	0.8	0.0^*	11.04196	0.14385	0.06684	0.800	NO	NO
Exponential $G(z)$: $z \sim$ Exp(0.12)									
(25)	0.200	0.2	0.0^*	9.05022	0.10604	0.93668	0.200	NO	NO
(26)	0.200	0.2	9.20	9.04903	0.10799	0.93370	0.201	NO	NO
(27)	0.200	0.5	0.0	9.17648	0.10520	0.94510	0.200	YES	YES
(28)	0.200	0.5	13.28^*	9.73331	0.20720	0.51206	0.324	YES	YES
(29)	0.330	0.5	0.0	10.22645	0.15298	0.51524	0.330	YES	YES
(30)	0.330	0.5	10.45^*	10.22648	0.15671	0.51410	0.331	YES	YES
(31)	0.500	0.5	0.0^*	10.62945	0.21048	0.26049	0.500	NO	NO
(32)	0.330	0.8	0.0	11.39238	0.14366	0.55469	0.330	YES	YES
(33)	0.330	0.8	16.31^*	11.74512	0.24614	0.37919	0.424	YES	YES
(34)	0.501	0.8	0.0^*	13.04451	0.18790	0.29954	0.501	NO	NO
(35)	0.501	0.8	13.20	13.04408	0.19077	0.29943	0.502	NO	NO
(36)	0.800	0.8	0.0^*	14.22000	0.25753	0.08008	0.800	NO	NO

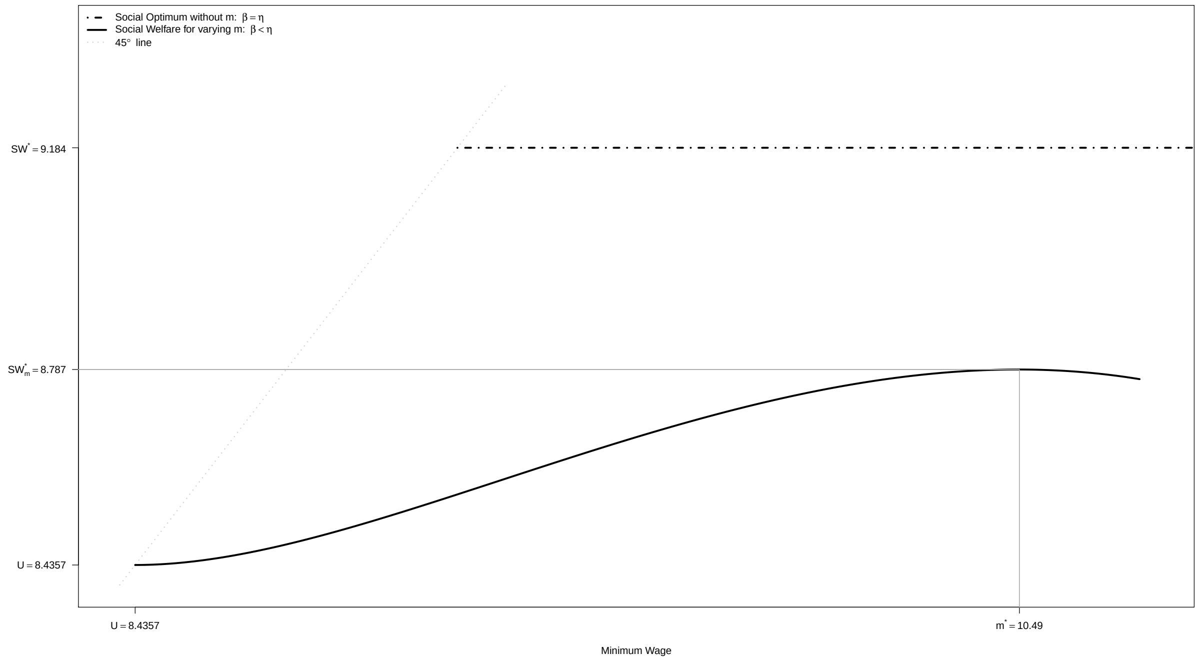
Parameter values: $c = 60$, $\delta = 0.038$, $b = -5$.

† The * symbol indicates that this level of m maximises social welfare for the given parameter values.

§ $\overline{\varepsilon}$ is the value of the right hand-side of condition (18) in Proposition 3; for the parameterisation presented in this Table, $\beta \geq 0.5$ implies $\overline{\varepsilon} \leq 0$.

§§ This is condition (20) in Corollary 2.

Figure E3: Social Optimum with and without a minimum wage.



Notes: A comparison of welfare in Environments 15-16 vs Environment 19.

Table E2: Equilibrium Outcomes with & without a minimum wage, Experiment B

Environment	Parameters β	η	Equilibrium $m^{\dagger}$	U_m	u	θ	ε	Condition satisfied $\varepsilon < \overline{\varepsilon}$ §	$\beta < \frac{\eta}{1+\eta}$ §§
Normal $G(z)$: $z \sim \mathcal{N}(10, 2.5^2)$									
(37)	0.330	0.8	0.0	3.68469	0.04131	0.54914	0.330	YES	YES
(38)	0.330	0.8	8.49^*	4.91315	0.06693	0.20520	0.574	YES	YES
(39)	0.501	0.8	0.0	5.11161	0.04797	0.27729	0.501	NO	NO
(40)	0.501	0.8	7.44^*	5.11898	0.05621	0.24170	0.535	YES	NO
(41)	0.800	0.8	0.0^*	5.94529	0.06376	0.07076	0.800	NO	NO
Log-Normal $G(z)$: $z \sim$Lognormal$(\ln(10), 0.35^2)$									
(42)	0.330	0.8	0.0	4.25554	0.04128	0.55558	0.330	YES	YES
(43)	0.330	0.8	8.62^*	5.07078	0.06864	0.28123	0.497	YES	YES
(44)	0.501	0.8	0.0^*	5.71311	0.04928	0.28061	0.501	NO	NO
(45)	0.501	0.8	5.90	5.71274	0.04984	0.28049	0.502	NO	NO
(46)	0.800	0.8	0.0^*	6.59468	0.06796	0.07166	0.800	NO	NO
Exponential $G(z)$: $z \sim$ Exp(0.12)									
(47)	0.330	0.8	0.0	2.472468	0.05476	0.53546	0.330	YES	YES
(48)	0.330	0.8	7.47^*	2.906472	0.10244	0.36161	0.424	YES	YES
(49)	0.501	0.8	0.0^*	4.487621	0.07780	0.27383	0.501	NO	NO
(50)	0.501	0.8	4.70	4.486648	0.07966	0.27366	0.502	NO	NO
(51)	0.800	0.8	0.0^*	5.896670	0.11581	0.07069	0.800	NO	NO

Parameter values: $c = 180$, $\delta = 0.038$, $b = -45$.

† The * symbol indicates that this level of m maximises social welfare for the given parameter values.

§ $\overline{\varepsilon}$ is the value of the right hand-side of condition (18) in Proposition 3.

§§ This is condition (20) in Corollary 2.

Figure E4: Minimum Wage Paths, Lognormal Productivity.

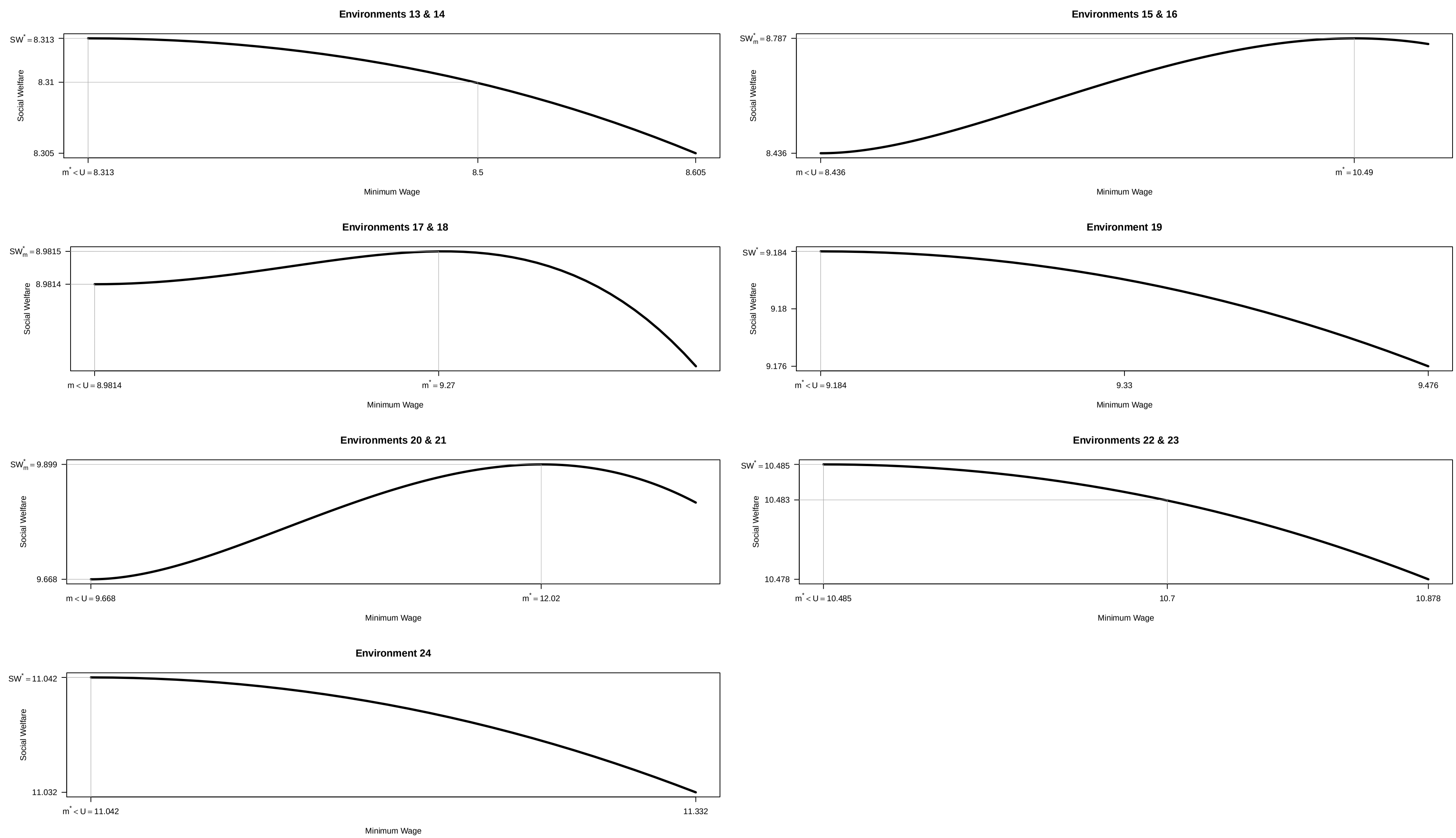
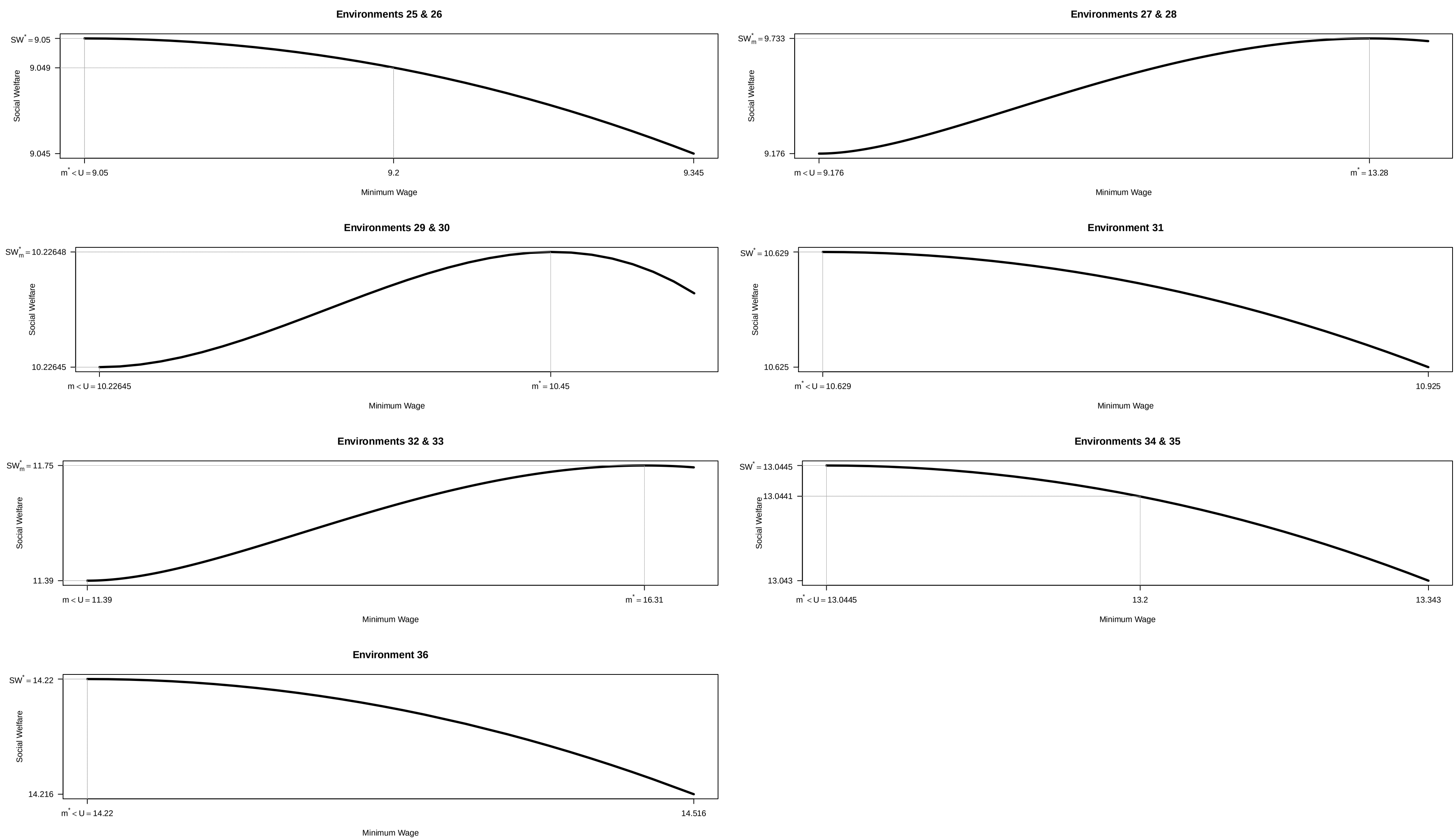


Figure E5: Minimum Wage Paths, Exponential Productivity.



F The Model with discounting: $r > 0$

For any match with value z, the surplus is

$$S(z, m) = [W - U] + [J - V],$$

and corresponding wage offer is given by the solution to

$$w(z, U) = \arg\max_{w \geq m} [W(w) - U]^{\beta} \left[\frac{z - w}{r + \delta}\right]^{1-\beta}.$$

If we ignore the constraint, we have

$$w(z, U) = \beta z + (1 - \beta) rU = rU + \beta S(z, m). \tag{FF1}$$

We can invert the wage function solving for z

$$z = x(w) = \frac{w - (1 - \beta) rU}{\beta}.$$

There is a match value $\hat{z}$ for which

$$w(\hat{z}, U) = m. \tag{FF2}$$

Combining equations (FF1) and (FF2), we have

$$\hat{z} = \frac{m - (1 - \beta) rU}{\beta}. \tag{FF3}$$

The firm would never accept a match with value $\hat{z} < m$, since that would imply a negative instantaneous profit $[\hat{z} - m < 0]$. Therefore, we only examine the case where $\hat{z} \geq m$. For match values $z \in [m, \hat{z})$, the firm offers a wage less than m. However, when confronted with the choice of giving some of its surplus to the worker versus a return of zero, the firm pays a wage of m for all the match values in the above range. The wage offer is

$$w(z) = \left\{ \begin{array}{c} rU + \beta S(z, m), \text{ for } z \in [\hat{z}, \overline{z}] \\ m > rU + \beta S(z, m), \text{ for } z \in [m, \hat{z}) \end{array} \right\}.$$

In this way, for values of the match specific productivity that lie in the range $[m, \hat{z})$, the minimum wage constraint allows the worker to appropriate a share of the surplus strictly greater than his bargaining power.

The value of a filled job is

$$J = \left\{ \begin{array}{c} (1 - \beta) S(z, m), \text{ for } z \in [\hat{z}, \overline{z}] \\ \left(\frac{z-m}{r+\delta}\right) < (1 - \beta) S(z, m), \text{ for } z \in [m, \hat{z}) \end{array} \right\}.$$

Moreover, the decision of a firm to enter the labour market and post a vacancy depends on the cost of a vacancy and on the expected value of filling this vacancy, that is

$$\begin{aligned} rV &= -c+\lambda_e\left[1-G\left(m\right)\right]\max\left\{E\left(J\right)-V,0\right\} \\ rV &= -c+\lambda_e\left[1-G\left(m\right)\right]\left[E\left(J-V|z\geq m\right)\right]. \end{aligned}$$

The free entry of firms guarantees an equilibrium $V=0$, which implies

$$c=\lambda_e*\left[1-G\left(m\right)\right]*\left\{\frac{\int_m^{\hat{z}}\left(\frac{z-m}{r+\delta}\right)dG\left(z\right)+\left(1-\beta\right)\int_{\hat{z}}^{\bar{\bar{z}}}S\left(z,m\right)dG\left(z\right)}{\left[1-G\left(m\right)\right]}\right\}. \tag{FF4}$$

In the steady-state, the unemployment rate is given by equating the flows into and out of the unemployment pool::

$$u=\frac{\delta}{\delta+\lambda_w*\left[1-G\left(m\right)\right]}. \tag{FF5}$$

The value of unemployment is

$$rU=b+\lambda_w\left\{\int_m^{\hat{z}}\left(S\left(z,m\right)-\frac{z-m}{r+\delta}\right)dG\left(z\right)+\beta\int_{\hat{z}}^{\bar{\bar{z}}}S\left(z,m\right)dG\left(z\right)\right\}. \tag{FF6}$$

From the above, it is straightforward that the density function of observed wages is

$$\chi\left(w|m\right)=\left\{\begin{array}{cc} \frac{\beta^{-1}g\left(\frac{w-(1-\beta)rU}{\beta}\right)}{1-G(m)} & \text{for } w\geq m \\ \\ \frac{G\left(\frac{m-(1-\beta)rU}{\beta}\right)-G(m)}{1-G(m)} & \text{for } w=m \\ \\ 0 & \text{for } w<m \end{array}\right\}.$$

F.1 The Planner's Problem

Social welfare is defined as aggregate output net of search costs. Output includes the return on leisure received by job seekers, b and the surplus of accepted matches $-$which is divided between the firm and its employee$-$ expressed in terms of expected productivity, $z^e=E\left(z|z\geq z_m^*\right)$. Search costs are the costs incurred by firms for holding a vacancy, c. Recalling that $v=\theta u$, social welfare for an infinitely lived economy is given by:

$$WL=\int_0^{\infty}e^{-rt}\left[ub+\left(1-u\right)z^e-\theta uc\right]dt.$$

The evolution of unemployment in this market is given by considering the difference between the flows into and out of the unemployment pool:

$$\dot{u}=\delta(1-u)-\theta q(\theta)\left[1-G\left(z_m^*\right)\right]u$$

The planner is faced with the same constraints as workers and firms in this economy.

Therefore, the planner's problem is:

$$\max_{u,\theta,z_m} WL \tag{FF7}$$

subject to

$$\dot{u} = \delta(1-u) - \theta q(\theta)\left[1 - G\left(z_m^*\right)\right] u \tag{FF8}$$

$$z_m^* \geq m \tag{FF9}$$

Assuming that κ_1 and κ_2 are co-state variables, the Lagrangian is:

$$\begin{aligned}\mathcal{L} = \int_0^\infty e^{-rt}\left[ub + (1-u)\, z^e - \theta uc\right] dt \\ &+ \int_0^\infty \kappa_1 \left[\dot{u} - \delta(1-u) + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right] u\right] dt \\ &+ \int_0^\infty \kappa_2 \left[z_m^* - m\right] dt\end{aligned} \tag{FF10}$$

The Lagrangian can be rewritten as follows:

$$\begin{aligned}\mathcal{L} = \int_0^\infty e^{-rt}\left[ub + (1-u)\, z^e - \theta uc\right] dt \\ &+ \int_0^\infty \kappa_1 \dot{u}\, dt \\ &- \int_0^\infty \kappa_1 \left[\delta(1-u) - \theta q(\theta)\left[1 - G\left(z_m^*\right)\right] u\right] dt \\ &+ \int_0^\infty \kappa_2 \left[z_m^* - m\right] dt\end{aligned} \tag{FF11}$$

Integrating $\int_0^\infty \kappa_1 \dot{u}\, dt$ by parts:

$$\begin{aligned}\mathcal{L} = \int_0^\infty e^{-rt}\left[ub + (1-u)\, z^e - \theta uc\right] dt \\ &+ \left[\kappa_1 u\right]_0^\infty - \int_0^\infty \dot{\kappa}_1 u\, dt \\ &- \int_0^\infty \kappa_1 \left[\delta(1-u) - \theta q(\theta)\left[1 - G\left(z_m^*\right)\right] u\right] dt \\ &+ \int_0^\infty \kappa_2 \left[z_m^* - m\right] dt\end{aligned} \tag{FF12}$$

The solution to the planner's problem satisfies:

$$\mathcal{L}_u = e^{-rt}\left[b - z^e - \theta c\right] + \kappa_1 \left[\delta + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right]\right] - \dot{\kappa}_1 = 0 \tag{FF13}$$

$$\mathcal{L}_\theta = -e^{-rt} uc + \kappa_1 \left[q(\theta) + \theta q'(\theta)\right]\left[1 - G\left(z_m^*\right)\right] u = 0 \tag{FF14}$$

$$\mathcal{L}_{z_m^*} = -e^{-rt} z_m^* g(z_m^*)(1-u) - \kappa_1 \theta q(\theta) g\left(z_m^*\right) u - \kappa_2 = 0 \tag{FF15}$$

$$\mathcal{L}_{\kappa_1} = \dot{u} - \delta(1-u) + \theta q(\theta)\left[1 - G\left(z_m^*\right)\right] u = 0 \tag{FF16}$$

$$\mathcal{L}_{\kappa_2} = z_m^* - m \geq 0, \quad \kappa_2 \geq 0, \quad \kappa_2 \mathcal{L}_{\kappa_2} = 0 \tag{FF17}$$

Solving condition (FF14) for κ_1, substituting it in condition (FF13) and evaluating it in the steady-state (taking into account that $\eta = -\frac{q'(\theta)\theta}{q(\theta)}$) leads to:

$$\frac{\delta c}{q(\theta)\left[1 - G(m)\right]} = (1-\eta)\left(z^e - b\right) - \eta c\theta. \tag{FF18}$$

Similarly, evaluating condition (FF16) in the steady state, leads to

$$u = \frac{\delta}{\delta + \lambda_w \left[1 - G(m)\right]}. \tag{FF19}$$

Combining conditions (FF14) and (FF15), implies that $\kappa_2 > 0$, which in turn implies (from (FF17)):

$$z_m^* = m \tag{FF20}$$

F.2 Comparative statics

Suppose now that the planner can only set the minimum wage -they have no control over any of the other endogenous variables in this economy. This means that given the minimum wage set by the planner sets the minimum wage the agents behaviour in the economy leads to a decentralised equilibrium allocation characterised by this level of the minimum wage. To determine the conditions for a minimum wage to be welfare improving, conduct a comparative statics exercise.

From equation (FF6), define

$$A = rU - b - \frac{\lambda_w}{r+\delta}\left\{\int_m^{\hat{z}} \left[m - rU\right] dG(z) + \beta \int_{\hat{z}}^{\bar{z}} \left[z - rU\right] dG(z)\right\} = 0.$$

Take the total differential of A

$$\frac{\partial A}{\partial rU} d(rU) + \frac{\partial A}{\partial \theta} d\theta + \frac{\partial A}{\partial m} dm = 0 \Rightarrow$$

$$\begin{gathered}
\{1 - \frac{\lambda_w}{r+\delta}[\int_m^{\hat{z}} (-1)\, dG(z) + (m - rU) * g(\hat{z}) * \frac{\partial \hat{z}}{\partial (rU)} + \\
+\beta \int_{\hat{z}}^{\bar{z}} (-1)\, dG(z) - \beta(\hat{z} - rU) * g(\hat{z}) \frac{\partial \hat{z}}{\partial (rU)}]\} * d(rU) - \\
-\left\{\int_m^{\hat{z}} \frac{[m - rU]}{r+\delta} dG(z) + \beta \int_{\hat{z}}^{\bar{z}} \left(\frac{z - rU}{r+\delta}\right) dG(z)\right\}\left(\frac{\partial \lambda_w}{\partial \theta}\right) d\theta - \\
-\frac{\lambda_w}{r+\delta}\left[\int_m^{\hat{z}} (1)\, dG(z) - [m - rU]\, g(m)\right] dm = 0 \Rightarrow
\end{gathered}$$

$$\left\{1-\frac{\lambda_w}{r+\delta}\left[\int_m^{\hat{z}}(-1)\,dG(z)+\beta\int_{\hat{z}}^{\overline{z}}(-1)\,dG(z)\right]\right\}d(rU)-$$
$$-\left\{\int_m^{\hat{z}}\frac{[m-rU]}{r+\delta}dG(z)+\beta\int_{\hat{z}}^{\overline{z}}\left(\frac{z-rU}{r+\delta}\right)dG(z)\right\}\left(\frac{\partial\lambda_w}{\partial\theta}\right)d\theta-$$
$$=\frac{\lambda_w}{r+\delta}\left[\int_m^{\hat{z}}(1)\,dG(z)-[m-rU]\,g(m)\right]dm\Rightarrow$$

$$\left\{1+\frac{\lambda_w}{r+\delta}\left(G(\hat{z})-G(m)+\beta\left[1-G(\hat{z})\right]\right)\right\}\frac{d(rU)}{dm}-$$
$$-\left\{\int_m^{\hat{z}}\left(\frac{m-rU}{r+\delta}\right)dG(z)+\beta\int_{\hat{z}}^{\overline{z}}\left(\frac{z-rU}{r+\delta}\right)dG(z)\right\}\left(\frac{\partial\lambda_w}{\partial\theta}\right)\frac{d\theta}{dm} \tag{FF21}$$
$$=\frac{\lambda_w}{r+\delta}\left[G(\hat{z})-G(m)-[m-rU]\,g(m)\right]$$

where $\left(\frac{\partial\lambda_w}{\partial\theta}\right)>0$ because of the constant returns to scale matching technology.

From equation (FF4), define

$$B=-c+\frac{\lambda_e}{r+\delta}\left\{\int_m^{\hat{z}}(z-m)\,dG(z)+(1-\beta)\int_{\hat{z}}^{\overline{z}}[z-rU]\,dG(z)\right\}=0.$$

Take the total differential of B

$$\frac{\partial B}{\partial rU}d(rU)+\frac{\partial B}{\partial\theta}d\theta+\frac{\partial B}{\partial m}dm=0\Rightarrow$$

$$\frac{\lambda_e}{r+\delta}[(\hat{z}-m)\,g(\hat{z})\frac{\partial\hat{z}}{\partial(rU)}+(1-\beta)\int_{\hat{z}}^{\overline{z}}(-1)\,dG(z)-$$
$$-(1-\beta)(\hat{z}-rU)*g(\hat{z})\frac{\partial\hat{z}}{\partial(rU)}]*d(rU)+$$
$$+\left\{\int_m^{\hat{z}}\left(\frac{z-m}{r+\delta}\right)dG(z)+(1-\beta)\int_{\hat{z}}^{\overline{z}}\left(\frac{z-rU}{r+\delta}\right)dG(z)\right\}\left(\frac{\partial\lambda_e}{\partial\theta}\right)d\theta+$$
$$+\frac{\lambda_e}{r+\delta}\left[\int_m^{\hat{z}}(-1)\,dG(z)\right]dm=0\Rightarrow$$

$$\frac{\lambda_e}{r+\delta}\left[(1-\beta)\int_{\hat{z}}^{\overline{z}}(-1)\,dG(z)\right]d(rU)+$$
$$+\left\{\int_m^{\hat{z}}\left(\frac{z-m}{r+\delta}\right)dG(z)+(1-\beta)\int_{\hat{z}}^{\overline{z}}\left(\frac{z-rU}{r+\delta}\right)dG(z)\right\}\left(\frac{\partial\lambda_e}{\partial\theta}\right)d\theta+$$
$$+\frac{\lambda_e}{r+\delta}\left[\int_m^{\hat{z}}(-1)\,dG(z)\right]dm=0\Rightarrow$$

$$
\begin{aligned}
&\frac{\lambda_e}{r+\delta}(1-\beta)\left[1-G(\hat{z})\right]\frac{d(rU)}{dm}- \\
&-\left\{\int_m^{\hat{z}}\left(\frac{z-m}{r+\delta}\right)dG(z)+(1-\beta)\int_{\hat{z}}^{\overline{z}}\left(\frac{z-rU}{r+\delta}\right)dG(z)\right\}\left(\frac{\partial\lambda_e}{\partial\theta}\right)\frac{d\theta}{dm}+ \\
&=-\frac{\lambda_e}{r+\delta}\left[G(\hat{z})-G(m)\right]
\end{aligned}
\tag{FF22}
$$

where $\left(\frac{\partial\lambda_e}{\partial\theta}\right)<0$ because of the constant returns to scale matching technology.

We have a system of two equations ($[FF21]$ and $[FF22]$) with two unknowns, $\frac{d\theta}{dm}$ and $\frac{d(rU)}{dm}$. For the sake of exposition, we define

$$
\begin{aligned}
\chi &= \left\{1+\frac{\lambda_w}{r+\delta}\left(G(\hat{z})-G(m)+\beta\left[1-G(\hat{z})\right]\right)\right\}>0 \\
\psi &= \left\{\underbrace{\int_m^{\hat{z}}\frac{[m-rU]}{r+\delta}dG(z)+\beta\int_{\hat{z}}^{\overline{z}}\left(\frac{z-rU}{r+\delta}\right)dG(z)}_{\text{expected surplus of worker}}\right\}\left(\frac{\partial\lambda_w}{\partial\theta}\right)>0 \\
\omega &= \frac{\lambda_w}{r+\delta}\left[G(\hat{z})-G(m)-[m-rU]\,g(m)\right]\gtreqless 0 \\
\mu &= \frac{\lambda_e}{r+\delta}(1-\beta)\left[1-G(\hat{z})\right]>0 \\
\nu &= -\left\{\underbrace{\int_m^{\hat{z}}\left(\frac{z-m}{r+\delta}\right)dG(z)+(1-\beta)\int_{\hat{z}}^{\overline{z}}\left(\frac{z-rU}{r+\delta}\right)dG(z)}_{\text{expected surplus of firm}}\right\}\left(\frac{\partial\lambda_e}{\partial\theta}\right)>0 \\
\xi &= \frac{\lambda_e}{r+\delta}\left[G(\hat{z})-G(m)\right]>0
\end{aligned}
$$

All expressions are positive except for $\omega\gtreqless 0$.

Hence, the system can be written as follows

$$
\begin{aligned}
\chi\times\frac{d(rU)}{dm}-\psi\times\frac{d\theta}{dm}&=\omega \\
\mu\times\frac{d(rU)}{dm}+\nu\times\frac{d\theta}{dm}&=-\xi.
\end{aligned}
$$

Solve using Cramer's rule

$$
\begin{aligned}
\frac{d(rU)}{dm}&=\frac{D_U}{D}\text{ and} \\
\frac{d\theta}{dm}&=\frac{D_\theta}{D}
\end{aligned}
$$

where

$$D = \det \begin{bmatrix} \chi & -\psi \\ \mu & \nu \end{bmatrix} = \chi\nu + \psi\mu > 0,$$

$$D_U = \det \begin{bmatrix} \omega & -\psi \\ -\xi & \nu \end{bmatrix} = \omega\nu - \psi\xi \gtrless 0, \text{ and}$$

$$D_\theta = \det \begin{bmatrix} \chi & \omega \\ \mu & -\xi \end{bmatrix} = -\chi\xi - \omega\mu \gtrless 0.$$

Clearly,

$$\frac{d\,(rU)}{dm} = \frac{D_U}{D} > 0$$

when $D_U > 0$, that is,

$$\omega\nu - \psi\xi > 0 \Rightarrow$$

$$-\left\{\frac{\lambda_w}{r+\delta}\left[G\left(\hat{z}\right) - G\left(m\right) - \left[m - rU\right] g\left(m\right)\right]\right\} \times E\left(\text{firm's surplus}\right) \times \left(\frac{\partial \lambda_e}{\partial \theta}\right) >$$

$$\frac{\lambda_e}{r+\delta}\left[G\left(\hat{z}\right) - G\left(m\right)\right] \times E\left(\text{worker's surplus}\right)\left(\frac{\partial \lambda_w}{\partial \theta}\right),$$

which implies that the necessary and sufficient condition for the increase in the value of unemployment after a change in the minimum wage is

$$\frac{-\lambda_w\left(\frac{\partial \lambda_e}{\partial \theta}\right)}{\lambda_e\left(\frac{\partial \lambda_w}{\partial \theta}\right)} \frac{\left[G\left(\hat{z}\right) - G\left(m\right) - \left[m - rU\right] g\left(m\right)\right]}{\left[G\left(\hat{z}\right) - G\left(m\right)\right]} > \frac{E\left(\text{worker's surplus}\right)}{E\left(\text{firm's surplus}\right)}. \tag{FF23}$$

For this inequality to hold, it has to be the case that $\omega > 0$, which implies that $D_\theta < 0$. Therefore, when the value of unemployment increases in response to a minimum wage increase, market tightness must decrease. How does this affect unemployment? Equation (6) suggests that the unemployment rate is decreasing in the arrival rate of offers for unemployed workers and the constant returns to scale matching technology implies that the arrival rate of offers for unemployed workers is increasing in market tightness. Therefore, the decrease in market tightness entails an increase in the unemployment rate. This is the result presented in Proposition 5.

Taking into account that $\eta = -\frac{q'(\theta)\theta}{q(\theta)}$ we can express inequality (FF23) as follows

$$\frac{\eta}{(1-\eta)} \frac{\left[G\left(\hat{z}\right) - G\left(m\right) - \left[m - rU\right] g\left(m\right)\right]}{\left[G\left(\hat{z}\right) - G\left(m\right)\right]} - \frac{E\left(\text{worker's surplus}\right)}{E\left(\text{firm's surplus}\right)} > 0. \tag{FF24}$$

Therefore, a minimum wage hike increases the value of unemployment if and only if condition (FF24) is satisfied. This is the necessary and sufficient condition for a minimum wage hike to be welfare improving. This condition is equivalent to condition (16) presented in Section 3.1.1.

Evaluating the limit of condition (FF24) as $m \rightarrow rU$, we get a version of this necessary and sufficient condition for a welfare-improving minimum wage that does not depend on the match-specific productivity distribution:

$$\lim_{m \to rU} \left\{ \frac{\eta}{(1-\eta)} \frac{[G(\hat{z}) - G(m) - [m - rU] g(m)]}{[G(\hat{z}) - G(m)]} - \frac{E\,(\text{worker's surplus})}{E\,(\text{firm's surplus})} \right\} =$$

$$= \lim_{m \to rU} \left\{ \frac{\eta}{(1-\eta)} \frac{\left[g(\hat{z}) \frac{1}{\beta} - g(m) - [m - rU] g'(m) - g(m) \right]}{g(\hat{z}) \frac{1}{\beta} - g(m)} - \frac{\beta \int_{rU}^{\bar{z}} \left(\frac{z - rU}{r + \delta} \right) dG(z)}{(1-\beta) \int_{rU}^{\bar{z}} \left(\frac{z - rU}{r + \delta} \right) dG(z)} \right\} =$$

$$= \frac{\eta}{(1-\eta)} \frac{\left[g(rU) \frac{1}{\beta} - 2g(rU) \right]}{g(rU) \frac{1}{\beta} - g(rU)} - \frac{\beta \int_{rU}^{\bar{z}} \left(\frac{z - rU}{r + \delta} \right) dG(z)}{(1-\beta) \int_{rU}^{\bar{z}} \left(\frac{z - rU}{r + \delta} \right) dG(z)} =$$

$$= \frac{\eta}{(1-\eta)} \frac{(1-2\beta)}{(1-\beta)} - \frac{\beta}{(1-\beta)} = \frac{\eta - 2\eta\beta - \beta + \eta\beta}{(1-\eta)(1-\beta)} = \frac{\eta - \beta - \eta\beta}{(1-\eta)(1-\beta)}.$$

Therefore, a marginal increase in the minimum wage can be welfare improving if and only if:

$$\frac{\eta - \beta - \eta\beta}{(1-\eta)(1-\beta)} > 0 \Rightarrow$$
$$\beta < \frac{\eta}{(1+\eta)}.$$

This is the result presented in Corollary 2 and discussed in Sections 3.1.1 and 3.1.3