Lecture 7:
Basics of magnetic resonance imaging (MRI): one dimensional Fourier imaging

Lecture aims to explain:

1. Basic aims of magnetic resonance imaging
2. Signal demodulation in magnetic resonance
3. Frequency encoding and Fourier transform
4. Simple two-spin example for MR imaging
Basic aims of Magnetic Resonance Imaging

The **goal of MR imaging** is not only to establish the presence of different nuclei, but also to determine the spatial distribution of a given species within the sample.

In experiment this can be done by determining the “frequency content” of the resulting MR signal, provided a well-defined spatial field variation is superimposed on the homogeneous static field.

In this lecture we will consider a large external field along z-axis $H_0$ and additional (time-dependent) 1D magnetic field gradient $G \cdot z$.

**Larmor precession** of the spins (once flipped in the xy-plane) will depend on their spatial coordinate. The angle of rotation will be given by

$$\theta = \gamma (H_0 + Gz) t$$
Signal demodulation in magnetic resonance
FID signal in laboratory frame

Reminder: in free induction decay spins are first flipped in xy plane by a $\pi/2$-pulse, and their Larmor precession is detected by a pick-up coil.

Signal obtained by the coil: $EMF = e^{-t/T_2} \cos(\omega_0 t)$

Here inhomogeneity of the spin ensemble (or magnetic field) is ignored.
Demodulation corresponds to the multiplication of the signal by sinusoid or cosinusoid with a frequency at or near $\omega_0$.

Demodulated signal $\propto$

$$\sin(\omega_0 + \delta\omega)t \cdot \sin \omega_0 t = \frac{1}{2} \left[ \cos \delta\omega t - \cos(2\omega_0 + \delta\omega)t \right]$$

Low pass filtering applied to the demodulated signal eliminates the high frequency component.

Note, the demodulated signal corresponds to the rotating frame signal in the frame with $\Omega = \omega_0 + \delta\omega$.
Demodulation at a detuned frequency leads to a faster decay of the signal with additional oscillations ("beating").

Example shows demodulated FID signal for one "detuned" spin.

If a large number of spins is considered all with differing frequencies the "beating" will be smeared out and a faster decay of the total magnetization will be observed revealing $T_2^*$. 

$e^{-t/T_2}$
In practice demodulation in two “channels” is used, producing a complex demodulated signal: demodulation using \( \cos \) and \( \sin \) is applied.

In the case when a distribution of spins at different frequencies is considered the demodulated signal at time \( t \) after \( \pi/2 \)-pulse will be given by:

\[
s(t) = \int dz \rho(z) e^{i\varphi(z,t)}
\]

\( \rho \) - is the effective spin density taken to be a simplified 1D function in our discussion.

The angle \( \varphi \) is the accumulated phase during time \( t \):

\[
\varphi(z,t) = -\omega(z)t
\]
Frequency encoding and Fourier transform
Frequency encoding in a simple imaging experiment

Magnetization is tipped into the xy-plane with a $\pi/2$-pulse prior to introduction of the field gradient. Field gradient is switched on when magnetization is freely precessing with $\omega_0$.

Once the gradient is switched on, the Larmor frequency is given by:

$$\omega_G(z,t) = \gamma z G(t)$$

for

$$H_z(z,t) = H_0 + zG(t)$$

The use of a gradient to establish a relation such as given above between the position of spins along some direction and their precessional rates is referred to as frequency encoding along that direction.
The 1D imaging equation

Demodulated signal: \( s(t) = \int dz \, \rho(z) e^{i\varphi_G(z,t)} \)

With accumulated phase due to the field gradient \( G \):
\[
\varphi_G(z,t) = -\gamma z \int_0^t G d\tau
\]

The explicit \( z \)-dependence in the phase for the linear field gives:
\[
s(k) = \int dz \, \rho(z) e^{-i2\pi k z}
\]

With the spatial frequency \( k=k(t) \) given by:
\[
k(t) = \frac{\gamma}{2\pi} \int_0^t G d\tau
\]

These equations show that, when linear gradients are implemented, the signal \( s(k) \) is the Fourier transform of the effective spin density of the sample. The spin density of the sample is found by taking the inverse Fourier transform of the signal:
\[
\rho(z) = \int dk \, s(k) e^{+i2\pi k z}
\]
Simple two-spin example for MR imaging
Two spin example: initialisation (no gradient applied)

A pair of spins lying along at $z=\pm z_0$, external magnetic field along $z$-axis

Signals shown are before demodulation (i.e. In laboratory frame). If demodulation were applied, the signal on the right would be constant.
Gradient \( G_z \) is applied at \( 0 < t < t_2 \) (let \( t_1 = 0 \)) leading to the phase at \( z_0 \) and \( -z_0 \):

\[
\varphi(\pm z_0, t) = \mp \gamma G z_0 (t - t_1)
\]

\[
s(t) = s_0 e^{-iz_0\gamma G t} + s_0 e^{iz_0\gamma G t}
\]

\[
= 2s_0 \cos(z_0\gamma G t)
\quad \text{for } 0 < t < t_2
\]

Can be rewritten for \( 0 < k < \gamma G t_2 / 2\pi \) as

\[
s(k) = 2s_0 \cos(2\pi k z_0)
\]

The beat frequency \( \gamma G z_0 \) implies separation between spins of \( 2z_0 \).

The exact positions are defined by the inverse Fourier transform of \( s(k) \) giving two \textbf{delta functions} centred at \( z_0 \) and \( -z_0 \).
The goal of MR imaging is not only to establish the presence of different nuclei, but also to determine the spatial distribution of a given species within the sample. In experiment this can be done by determining the “frequency content” of the resulting MR signal, provided a well-defined spatial field variation is superimposed on the homogeneous static field.

**Signal demodulation** is a techniques for obtaining MR signal. Demodulation corresponds to the multiplication of the signal by sinusoid or cosinusoid with a frequency at or near $\omega_0 = \gamma H_0$. Using both $\sin$ and $\cos$ a complex demodulated signal is obtained, expressed via the spin density $\rho$ and the accumulated phase due to inhomogeneous field (in a simple 1D case) as:

\[
s(t) = \int dz \rho(z) e^{i \varphi(z,t)}
\]

\[
s(k) = \int dz \rho(z) e^{-i 2 \pi k z}
\]

\[
k(t) = \frac{\gamma}{2\pi} \int_0^t G d\tau
\]

It’s possible to show that, when linear gradients $G$ are implemented, the signal $s(k)$ (given on the left) is the Fourier transform of the spin density of the sample. The spin density of the sample is found by taking the inverse Fourier transform of the signal:

\[
\rho(z) = \int dk \ s(k) e^{+i 2 \pi k z}
\]