Lecture 11: Fraunhofer diffraction by a circular aperture

Lecture aims to explain:

1. Diffraction problem for a circular aperture

2. Diffraction pattern produced by a circular aperture, Airy rings

3. Importance of diffraction for imaging: Rayleigh criterion
Diffraction problem for a circular aperture
Standard approach for diffraction by a circular aperture

We take an element of area $dA$ which radiates in all directions. We need to sum all contributions, over all such elements, at some angle $\theta$.

Consider a strip at a distance $y$ from a centre line, having a path difference $y \sin \theta$.

As previously the phase difference between this point and the centre line is

$$\delta = \frac{2\pi}{\lambda} y \sin \theta$$
Diffraction pattern produced by a circular aperture
Analytical expression for diffraction pattern

Angular distribution of intensity of light diffracted by a circular aperture:

\[
\frac{I(\theta)}{I(0)} = \left[ \frac{2J_1(\rho)}{\rho} \right]^2
\]

where

\[
\rho = \frac{\left( \frac{2\pi \sin \theta}{\lambda} \right)^2 a}{2}
\]

\[a \text{ – diameter of the aperture}\]

\(J_1(\rho)\) is known as a Bessel function of the 1\(^{\text{st}}\) kind and 1\(^{\text{st}}\) order. The general definition with \(n\) – the order

\[
J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} \left( \frac{x}{2} \right)^{2m+n}
\]
Diffraction pattern: Airy rings

The amplitude distribution for diffraction due to a circular aperture forms an intensity pattern with a bright central band surrounded by concentric circular bands of rapidly decreasing intensity (Airy pattern).

The 1\textsuperscript{st} maximum is roughly 1.75\% of the central intensity. 84\% of the light arrives within the central peak called the \textbf{Airy disk}. 
Angular width of the central diffraction maximum

For \( \frac{I(\theta)}{I(0)} = \left[ \frac{2J_1(\rho)}{\rho} \right]^2 \)

the first zero occurs at

\[
\rho = 3.832 = \left( \frac{a}{2} \right) \left( \frac{2\pi}{\lambda} \right) \sin \theta
\]

In small angle approximation the total angular width of the first maximum is:

\[
\Delta \theta_{\text{total}} = 2\theta_{\text{first-zero}} \approx 2.44 \frac{\lambda}{a}
\]
Importance of diffraction for imaging: Rayleigh criterion
Example 11.1

(a) The Sun has the angular size of 0.5 degree. Calculate the size of the image produced by the Sun in the focal plane of the telescope objective. The focal length of the objective is 30 cm, and its diameter is 5 cm. Is the diffraction effect important?

(b) Betelgeuse is the eighth brightest star in the night sky and second brightest star in the constellation of Orion. Its angular size is 0.05 arcseconds.

(i) Can its size be measured with the same telescope as in (a)?

(ii) Find the image size of this star produced by the telescope.

(ii) How the diameter of the objective should be changed so that the angular size of Betelgeuse is measured accurately?
Example 11.2

Light of wavelength 1µm is incident on an adjustable aperture in an otherwise opaque screen

(a) If the aperture size is 100 µm, calculate the smallest light spot that can be obtained in the focal plane of a lens with 10 cm focal length positioned behind the screen

(b) How should the size of the aperture be adjusted so that light is focussed to a 100 µm spot in the focal plane of the lens?

(c) A lens with what focal length should be used in the case of 100 µm aperture so that light is focussed into 100 µm spot in its focal plane.
**Effect of diffraction on image resolution**

Individual images (peaks) on the detector plane with and without diffraction (red and black/grey, respectively).

**Intensity measured by the detector (or eye) is the sum of the individual intensities.** The intensity profile in the case without diffraction retains features of individual images. In the case with diffraction a single peak is observed, hence information on individual images is lost.
Rayleigh criterion

The image of two stars is two independent Airy patterns. Resolving these images depends upon the size of the Airy disks and their overlap.

The Rayleigh criterion: two images are just resolved if the centre of the first Airy pattern is superimposed on the 1st dark ring of the 2nd pattern.

Thus, the minimum resolvable angular separation or angular limit of resolution is given by

\[ \Delta \theta = \frac{1.22 \lambda}{D} \]

\( \lambda \) wavelength and \( D \) is the diameter of the lens.
Example 11.3

A road crossing is filmed by a camera from a distance of 5 km. What should be the diameter of the camera lens so that individual traffic lights are resolved in the image.

Example 11.4

A mobile phone camera has a diameter of 2 mm. What is the maximum distance at which this camera can be used to resolve facial features.

Example 11.5

Find maximum angular resolution of the long range air traffic control radar having a size of 10 m and operating at a frequency of 1 GHz. At what largest distance can this radar resolve two aircrafts flying in adjacent altitude corridors 300 meters from each other?
SUMMARY

The amplitude distribution for diffraction by a circular aperture forms an **Airy intensity pattern**. The 1st maximum is roughly 1.75% of the central intensity. **84%** of the light arrives within the central peak, the **Airy disk**

In small angle approximation the total angular width of the first maximum is:

\[ \Delta \theta_{total} = 2 \theta_{first-zero} \approx 2.44 \frac{\lambda}{\alpha} \]

The **Rayleigh criterion**: two images are *just resolved* if the centre of the first Airy pattern is superimposed on the 1st dark ring of the 2nd pattern.

According to Rayleigh criterion, **the minimum resolvable angular separation or angular limit of resolution** is given by:

\[ \Delta \theta = \frac{1.22}{D} \lambda \]

\( \lambda \) wavelength and \( D \) is the diameter of the lens