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Brief paper

Frequency domain solution to delay-type Nehari problem

Qing-Chang Zhong

Department of Electrical and Electronic Engineering, Imperial College, Exhibition Road, London, SW7 2BT, UK

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Abstract

This paper generalizes the frequency-domain results on the delay-type Nehari problem in the stable case to the unstable case. The solvability condition of the delay-type Nehari problem is formulated in terms of the nonsingularity of three matrices. The optimal value $\gamma_{\text{opt}}$ is the maximal $\gamma \in (0, \infty)$ such that one of the three matrices becomes singular. All sub-optimal compensators are parameterized in a transparent structure incorporating a modified Smith predictor.

Keywords: $H_\infty$ control; Smith predictor; Nehari problem; Time delay systems; Riccati equation; $L_2[0,h]$-induced norm

1. Introduction

The $H_\infty$ control of processes with delay(s) has been an active research area since the mid-1980s. There are mainly three kinds of methods: operator-theoretic methods (Foias, Ózbay, & Tannenbaum, 1996; Dym, Georgiou, & Smith, 1995; Zhou & Khargonekar, 1987), state-space methods (Nagpal & Ravi, 1997; Tadmor, 1997a, 2000; Başar & Bernhard, 1995) and frequency-domain methods (Mirkin, 2000; Meinsma & Zwart, 2000; Meinsma, Mirkin, & Zhong, 2002; Zhong, 2003). It is well known that a large class of $H_\infty$ control problems can be reduced to Nehari problems (Francis, 1987). This is still true in the case for systems with delay(s), where the simplified problem is a delay-type Nehari problem. Some papers, e.g. (Zhou & Khargonekar, 1987; Flamm & Mitter, 1987), were devoted to calculate the infimum of the delay-type Nehari problem in the stable case. It was shown in (Zhou & Khargonekar, 1987) that this problem in the stable case is equivalent to calculating an $L_2[0,h]$-induced norm. However, for the unstable case, it becomes much more involved. Tadmor (1997b) presented a state space solution to this problem in the unstable case, in which a differential/algebraic matrix Riccati equation-based method was used. The optimal value relies on the solution of a differential Riccati equation. Although the expositions are very elegant, the suboptimal solution is too complicated and the structure is not transparent.

Motivated by the idea of Meinsma and Zwart (2000), this paper presents a frequency domain solution to the delay-type Nehari problem in the unstable case. The main tool used here is the $J$-spectral factorization of generic para-Hermitian matrices (Kwakernaak, 2000; Zhong, 2001). The optimal value $\gamma_{\text{opt}}$ of the delay-type Nehari problem is formulated in a simple, clear way: it is the maximal $\gamma$ such that one of three matrices becomes singular when $\gamma$ decreases from $+\infty$ to 0. Hence, one need no longer solve a differential Riccati equation. A prominent advantage is that the suboptimal solutions have a transparent structure incorporating a modified Smith predictor.

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* Tel.: +44-20-759-46295; fax: +44-20-759-46282.
  E-mail address: zhongqc@ic.ac.uk (Q.-C. Zhong).
  URL: http://members.fortunecity.com/zhongqc

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Given a matrix $A$, $A^T$ and $A^*$ denote its transpose and complex conjugate transpose, respectively, and $A^{-*}$ stands for $(A^{-1})^*$ when the inverse $A^{-1}$ exists. A rational transfer matrix $G(s) = D + C(sI - A)^{-1}B$ is frequently denoted as
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
and its conjugate is defined as
\[
G^*(-s^*) = \begin{bmatrix}
-A^* & -C^* \\
B^* & D^*
\end{bmatrix}.
\]

A completion operator $\pi_h$ is defined as
\[
\pi_h\{G\} = \begin{bmatrix}
A & B \\
Ce^{-4h} & 0
\end{bmatrix} - e^{-sh} \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \doteq \hat{G}(s) - e^{-sh}G(s),
\]
where $h \geq 0$. This follows (Mirkin, 2000), except for a small adjustment in the notation. This operator maps any rational transfer matrix $G$ into an finite impulse response (FIR) block.

Let $N = \begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix}$ be a $2 \times 2$ block transfer matrix. The upper linear fractional transformation is defined as $\mathcal{F}_u(N,Q) = N_{22} + N_{21}Q(I - N_{11}Q)^{-1}N_{12}$ provided that $(I - N_{11}Q)^{-1}$ exists. Another two less-used linear fractional transformations, called homographic transformations (Kimura, 1996; Delsarte, Genin, & Komp, 1979), are defined as follows provided that the corresponding inverse exists:
\[
\mathcal{H}_l(N,Q) = (N_{11}Q + N_{12})(N_{21}Q + N_{22})^{-1},
\]
\[
\mathcal{H}_r(N,Q) = -(N_{11} - QN_{21})^{-1}(N_{12} - QN_{22}),
\]
where the subscript $l$ stands for left and $r$ for right. These transformations correspond to the matrix multiplication, or chain scattering (Kimura, 1996), from the left side and from the right side, respectively:
\[
\mathcal{H}_l(N_1,\mathcal{H}_l(N_2,Q)) = \mathcal{H}_l(N_1N_2,Q),
\]
\[
\mathcal{H}_l(N_1,\mathcal{H}_r(N_2,Q)) = \mathcal{H}_r(N_2N_1,Q),
\]
where $N_1$ and $N_2$ are $2 \times 2$ block transfer matrices with appropriate dimensions.

A signature matrix is defined as
\[
J_{p,q} = \begin{bmatrix}
I_p & 0 \\
0 & -I_q
\end{bmatrix}
\]
and is reduced to $J$ when the indices $p$ and $q$ are obvious or irrelevant. A $\gamma$-related signature matrix is defined as
\[
J_\gamma = \begin{bmatrix}
I & 0 \\
0 & -\gamma^2I
\end{bmatrix}.
\]

The following Hamiltonian matrices are frequently used in this paper:
\[
H_c = \begin{bmatrix}
A & \gamma^{-2}BB^* \\
0 & -A^*
\end{bmatrix}, \quad H_o = \begin{bmatrix}
A & 0 \\
-C^*C & -A^*
\end{bmatrix},
\]
\[
H = \begin{bmatrix}
A & \gamma^{-2}BB^* \\
-C^*C & -A^*
\end{bmatrix}.
\]

$H_c$ relates to the control matrix $B$ and $H_o$ relates to the observation matrix $C$. The Hamiltonian-matrix exponential with respect to $H$ is defined as
\[
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix} = e^{Jh},
\]
where $h \geq 0$ in this paper. This matrix plays quite an important role in $H_\infty$-control of dead-time systems. This matrix holds some nice properties, see the technical-report version (Zhong, 2001) of this paper.

2. Problem formulation

The Delay-type Nehari Problem (NPh) is described as follows: Given a minimal state-space realization

\[ G_\beta(s) = \begin{bmatrix} A & B \\ -C & 0 \end{bmatrix}, \]

which is not necessarily stable and $h \geq 0$, characterize the optimal value

\[ \gamma_{\text{opt}} = \inf \{ \| G_\beta(s) + e^{-sh}K(s) \|_{L_\infty} : K(s) \in H_\infty \} \]

and for a given $\gamma > \gamma_{\text{opt}}$, parameterize the suboptimal set of proper and causal $K(s) \in H_\infty$ such that

\[ \| G_\beta(s) + e^{-sh}K(s) \|_{L_\infty} \leq \gamma. \]

A similar Nehari problem with distributed delays, which can be reformulated as $\| e^{-sh}G_\beta(s) + K(s) \|_{\infty} < \gamma$ in the single delay case, was studied in (Tadmor, 1995). Although this question looks similar to (2) in their forms, they are entirely different in essence and the problem studied here and in (Tadmor, 1997b) is much more complicated.

It is well known (Gohberg, Goldberg, & Kaashoek, 1993) that this problem is solvable if

\[ \gamma > \gamma_{\text{opt}} \equiv \| G_\beta(s) \|_{L_2[0,h]}, \]

where $\Gamma$ denotes the Hankel operator. Inspecting the transfer matrix $e^{sh}G_\beta(s)$, one can see that $\gamma_{\text{opt}}$ is not less than the $L_2[0,h]$-induced norm of $G_\beta(s)$ (Foias et al., 1996; Zhou & Khargonekar, 1987; Gu, Chen, & Toker, 1996), i.e.,

\[ \gamma_{\text{opt}} \geq \gamma_h = \| G_\beta(s) \|_{L_2[0,h]}. \]

Hence, it is assumed that $\gamma > \gamma_h$ in the sequel. Under this condition, matrix $\Sigma_{22}$, i.e. the $(2,2)$-block of $\Sigma$, is always nonsingular. A simple formula to compute the $L_2[0,h]$-induced norm $\gamma_h$ was given in (Zhou & Khargonekar, 1987) as

\[ \gamma_h = \max \{ \gamma : \det \Sigma_{22} = 0 \}, \]

i.e., the maximal $\gamma$ that makes $\Sigma_{22}$ singular or the maximal root of $\det \Sigma_{22} = 0$.

3. Main result and the proof

**Theorem 1.** For a given minimally-realized transfer matrix

\[ G_\beta(s) = \begin{bmatrix} A & B \\ -C & 0 \end{bmatrix} \]

having neither jω-axis zero nor jω-axis pole, the following algebraic Riccati equations:

\[ \begin{bmatrix} -L_c & I \end{bmatrix} H_c \begin{bmatrix} I \\ L_c \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} I & -L_o \end{bmatrix} H_o \begin{bmatrix} L_o \\ I \end{bmatrix} = 0 \]

always have unique solutions $L_c \leq 0$ and $L_o \leq 0$ such that $A + \gamma^{-2}BB^*L_c$ and $A + L_oC^*C$ are stable, respectively. The optimal value $\gamma_{\text{opt}}$ of the delay-type Nehari problem (2) is

\[ \gamma_{\text{opt}} = \max \{ \gamma_h, \gamma_1, \gamma_2 \}, \]

where

\[ \gamma_h = \max \{ \gamma : \det \begin{bmatrix} 0 & I \\ I & \Sigma \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = 0 \}, \]

\[ \gamma_1 = \max \{ \gamma : \det \begin{bmatrix} 0 & I \\ I & \Sigma \end{bmatrix} \begin{bmatrix} L_o \\ I \end{bmatrix} = 0 \}, \]

\[ \gamma_2 = \max \{ \gamma : \det \begin{bmatrix} -L_c & I \\ I & \Sigma \end{bmatrix} \begin{bmatrix} L_o \\ I \end{bmatrix} = 0 \}. \]
Furthermore, for a given $\gamma > \gamma_{opt}$, all $K(s) \in H_\infty$ satisfying (2) can be parameterized as

$$K(s) = \mathcal{H}_I \left( \begin{bmatrix} I & 0 \\ \Delta(s) & I \end{bmatrix} W^{-1}(s), Q(s) \right),$$

(4)

where

$$\Delta(s) = -\pi_k \left\{ \mathcal{F}_u \left( \begin{bmatrix} G_\beta(s) I \\ I \end{bmatrix}, \gamma^{-2}G_\beta^*(s) \right) \right\},$$

(5)

$$W^{-1}(s) = \begin{bmatrix} A + \gamma^{-2}BB^*L_c & (I - L_{oh}L_c)^{-1}L_{oh}C^* (I - L_{oh}L_{c1})^{-1}(L_{oh}\Sigma_{21} - \Sigma_{11})B \\ -C & I \\ \gamma^{-2}B^*(\Sigma_{21} - \Sigma_{11}L_c) & 0 & I \end{bmatrix}$$

(6)

with $L_{oh} = \mathcal{H}_1(\Sigma, L_{oh})$, and $\|Q(s)\|_{H_\infty} < \gamma$ is a free parameter.

**Remark 2.** Here, $\gamma > \gamma_h$ ensures the nonsingularity of $\Sigma_{22}$; $\gamma > \gamma_1$ ensures the existence of $L_{oh}$ and $\gamma > \gamma_2$ ensures the existence of the J-spectral factorization and the stability of $K(s)$.

The structure of $K(s)$ is shown in Fig. 1 (the dashed box). It consists of an infinite-dimensional block $\Delta(s)$, which is a FIR block (i.e. a modified Smith predictor), and a finite-dimensional block $W^{-1}(s)$. The right-upper tag means $W^{-1}(s)$ maps the right variables to the left variables while a left-upper tag, if any, means the matrix maps the left variables to the right variables.

**Proof.** Using the chain-scattering representation (Kimura, 1996),

$$G_\beta(s) + e^{-sh} K(s) = \mathcal{H}_I(G(s), K(s)),$$

where

$$G(s) = \begin{bmatrix} e^{-sh}I & G_\beta(s) \\ 0 & I \end{bmatrix}.$$

It has already been well known (Kimura, 1996; Meinsma & Zwart, 2000; Green, Glover, Limebeer, & Doyle, 1990) that the $H_\infty$ control problem

$$\|\mathcal{H}_I(G(s), K(s))\|_{H_\infty} < \gamma$$

is equivalent to that $G^{-}(s)J_I G(s)$ has a J-spectral factor $V(s)$ such that the (2,2)-block of $G(s)V^{-1}(s)$ is bistable. Hence, in this proof, we characterize the conditions to meet these requirements. Here,

$$G^{-}(s)J_I G(s) = \begin{bmatrix} I & e^{sh}G_\beta(s) \\ e^{-sh}G_\beta^*(s) & G_\beta^*(s)G_\beta(s) - \gamma^2I \end{bmatrix}.$$

The main idea underlying here is to find a unimodular matrix to equivalently rationalize the system and then to find the J-spectral factorization of the rationalized system. This idea was firstly proposed in (Meinsma & Zwart, 2000) for a two-block problem. The result there was obtained for a stable $G(s)$ and cannot be directly used here because $G(s)$ here is...
which is rationalized to be not necessarily stable. We borrowed some ideas from there but we use an elementary tool (the similarity transformation) to find the realization of the rationalized system.

\[ A(s) \text{ given in (5) can be decomposed into two parts } A(s) = F_0(s) + F_1(s)e^{-sk} \text{ with} \]

\[
F_1(s) = \mathcal{F}_a \left( \begin{bmatrix} G_\beta(s) & I \\ I & 0 \end{bmatrix}, \gamma^{-2}G_\beta(s) \right).
\]  

(7)

Since \( A(s) \) is an FIR block, finding the \( J \)-spectral factorization of \( G^{-}(s)J\gamma G(s) \) is equivalent to finding the \( J \)-spectral factorization of the following matrix (Kimura, 1996; Meinsma & Zwart, 2000):

\[
\Theta(s) \triangleq \begin{bmatrix} I & \Delta(s)^{-}\gamma^{-2}(s)J\gamma \end{bmatrix} G^{-}(s)J\gamma G(s) \begin{bmatrix} I \\ \Delta(s) \end{bmatrix},
\]

which is rationalized to be

\[
\Theta(s) = \begin{bmatrix} \gamma^2((\gamma^2 I - G_\beta G_\beta^{-1}) + F_0^{-}(G_\beta^{-} G_\beta - \gamma^2 I)F_0^{-} G_\beta^{-}(G_\beta^{-} G_\beta - \gamma^2 I) & F_0^{-} G_\beta^{-} G_\beta - \gamma^2 I \\ (G_\beta^{-} G_\beta - \gamma^2 I)F_0^{-} G_\beta^{-} G_\beta - \gamma^2 I & \end{bmatrix}.
\]

(8)

Substitute (1) into (7) and use the star product formula (Zhou & Doyle, 1997), then the realization of \( F_1(s) \) can be obtained as

\[
F_1(s) = \begin{bmatrix} A & \gamma^{-2}BB^* \\ -C^*C & -A^* \\ 0 & \gamma^{-2}B^* \end{bmatrix}.
\]

(9)

Accordingly, \( F_0(s) \) has the following realization according to the definition of the operator \( \pi_h \):

\[
F_0(s) = \begin{bmatrix} A & \gamma^{-2}BB^* \\ -C^*C & -A^* \\ \gamma^{-2}B^* \Sigma_{21} & -\gamma^{-2}B^* \Sigma_{11} \end{bmatrix}.
\]

(10)

Substitute (1), (9) and (10) into (8), then the realization of \( \Theta^{-1}(s) \) can be obtained after tedious matrix manipulations, see Appendix, as

\[
\Theta^{-1}(s) = \begin{bmatrix} A & \gamma^{-2}BB^* \\ -C & -A^* \\ \gamma^{-2}B^* \Sigma_{21} & -\gamma^{-2}B^* \Sigma_{11} \end{bmatrix}.
\]

Using the \( J \)-spectral factorization theory of generic para-Hermitian matrices (Zhong, 2001), \( \Theta^{-1}(s) \) has a \( J \)-spectral co-factorization if and only if the two Riccati equations

\[
\begin{bmatrix} -L_c & I \end{bmatrix} H_c \begin{bmatrix} I \\ L_c \end{bmatrix} = 0 \quad \text{and}
\]

\[
\begin{bmatrix} I & -L_{ob} \end{bmatrix} \Sigma H_o \Sigma^{-1} \begin{bmatrix} L_{ob} \\ I \end{bmatrix} = 0
\]

have unique symmetric stabilizing solutions \( L_c \leq 0 \) and \( L_{ob} \), respectively, and \( \det(I - L_{ob}L_c) \neq 0 \). Since the Hamiltonian matrix in the last Riccati equation is similar to \( H_o \), the unique stabilizing solution \( L_{ob} \) can also be obtained as

\[
L_{ob} = (\Sigma_{11}L_o + \Sigma_{12})(\Sigma_{21}L_o + \Sigma_{22})^{-1} = \mathcal{F}_t(\Sigma, L_o)
\]

provided that \( \Sigma_{21}L_o + \Sigma_{22} \) is nonsingular, where \( L_o \leq 0 \) is the unique stabilizing solution of

\[
\begin{bmatrix} I & -L_o \end{bmatrix} H_o \begin{bmatrix} L_o \\ I \end{bmatrix} = 0.
\]
The $J$-spectral co-factor of $\Theta^{-1}(s)$ is
\[
W_0^{-1}(s) = \begin{bmatrix} A + \gamma^{-2}BB^*L_c & (I - L_{oh}L_c)^{-1}L_{oh}C^* (I - L_{oh}L_c)^{-1}(L_{oh}\Sigma_{21} - \Sigma_{11})\gamma^{-1}B \\
\gamma^{-2}B^*(\Sigma_{21} - \Sigma_{11}L_c) & I \\
-\gamma^{-2}C & 0 \\
0 & \gamma^{-1}I \end{bmatrix}.
\]
Hence, the following equality is obtained:
\[
G^{-}(s)JFG(s) = \begin{bmatrix} I & -A(s)^{-1} \\
0 & I \end{bmatrix} W_0^{-}(s)JW_0(s) \begin{bmatrix} I & 0 \\
-A(s) & I \end{bmatrix}.
\]
Since $W_0(s)$ and
\[
\begin{bmatrix} I & 0 \\
-A(s) & I \end{bmatrix}
\]
are all bistable,
\[
W_0(s) \begin{bmatrix} I & 0 \\
-A(s) & I \end{bmatrix}
\]
is a $J$-spectral factor of $G^{-}(s)JFG(s)$. This means that any $K(s)$ in the form of
\[
K(s) = \mathcal{H}_1 \left( \begin{bmatrix} I & 0 \\
A(s) & I \end{bmatrix} W^{-1}(s), Q(s) \right),
\]
where $W^{-1}(s)$ is as given in (6) and $\|Q(s)\|_{\infty} < \gamma$ is a free parameter, satisfies
\[
\|G_p(s) + e^{-s}K(s)\|_{\infty} < \gamma.
\]
In order to guarantee $K(s) \in H_{\infty}$, the bistability of the (2,2)-block of the matrix
\[
\Pi(s) = \begin{bmatrix} \Pi_{11}(s) & \Pi_{12}(s) \\
\Pi_{21}(s) & \Pi_{22}(s) \end{bmatrix} \equiv G(s) \begin{bmatrix} I & 0 \\
A(s) & I \end{bmatrix} W^{-1}(s)
\]
is required. Using similar arguments of Meinsma and Zwart (2000), the bistability of $\Pi_{22}(s)$ is equivalent to the existence of $L_{oh}$ (equivalently, the nonsingularity of $\Sigma_{21}L_o + \Sigma_{22}$) and the nonsingularity of $I - L_{oh}L_c$ (equivalently, $I - L_cL_{oh}$) not only for $\gamma$ but also for any number larger than $\gamma$.

In summary, the solvability conditions are:
(i) $\gamma > \gamma_h$ so that $\Sigma_{22} = \begin{bmatrix} 0 & I \end{bmatrix} \Sigma \begin{bmatrix} 0 \\
I \end{bmatrix}$ is always nonsingular;

(ii) There exists a $\gamma_1 > 0$ such that $\Sigma_{21}L_o + \Sigma_{22} = \begin{bmatrix} 0 & I \end{bmatrix} \Sigma \begin{bmatrix} L_o \\
I \end{bmatrix}$ is always nonsingular for $\gamma > \gamma_1$;

(iii) There exists a $\gamma_2 > 0$ such that $I - L_cL_{oh}$ is always nonsingular for $\gamma > \gamma_2$. When condition (ii) is satisfied, the nonsingularity of $I - L_cL_{oh}$ is equivalent to that of $(I - L_cL_{oh})(\Sigma_{21}L_o + \Sigma_{22}) = \Sigma_{21}L_o + \Sigma_{22} - L_c(\Sigma_{11}L_o + \Sigma_{12}) = \begin{bmatrix} -L_c & I \end{bmatrix} \Sigma \begin{bmatrix} L_o \\
I \end{bmatrix}$.

The minimal $\gamma$ satisfying these conditions is the optimal value $\gamma_{opt} = \max\{\gamma_h, \gamma_1, \gamma_2\}$, as given in Theorem 1. This completes the proof.

4. Special cases

Case 1: If $A$ is stable, then $L_o = 0$, $L_c = 0$ and $L_{ob} = \Sigma_{12}\Sigma_{22}^{-1}$. Condition (iii) is always satisfied and condition (ii) becomes the same as condition (i), i.e., only the nonsingularity of $\Sigma_{22}$ is required. In this case, $\gamma_{opt} = \gamma_h$.

Corollary 3. For a given minimally realized stable transfer matrix
\[
G_p(s) = \begin{bmatrix} A \\
-C \end{bmatrix} \begin{bmatrix} B \\
0 \end{bmatrix}
\]
having neither jо-axis zero nor jо-axis pole, the delay-type Nehari problem (2) is solvable iff $\gamma > \gamma_h$ (or equivalently, $\Sigma_{22}$ is nonsingular not only for $\gamma$ but also for any number larger than $\gamma$). Furthermore, if this condition holds, then $K(s)$ is parameterized as (4) but

\[
W^{-1}(s) = \begin{bmatrix}
A & \Sigma_{12}\Sigma_{22}^{-1}C^* - \Sigma_{22}^{-1}B \\
C & I \\
\gamma^{-2}B^*\Sigma_{21} & 0 & I
\end{bmatrix}.
\]

**Remark 4.** The cost to be paid for the instability of $G_\beta(s)$ is the nonsingularity of the matrices $\begin{bmatrix} 0 & I \end{bmatrix} \Sigma \begin{bmatrix} L_o \\ I \end{bmatrix}$ and $\begin{bmatrix} -L_c & I \end{bmatrix} \Sigma \begin{bmatrix} L_o \\ I \end{bmatrix}$, in addition to the matrix $\begin{bmatrix} 0 & I \end{bmatrix} \Sigma \begin{bmatrix} 0 \\ I \end{bmatrix}$ for the stable case, for $\gamma$ and any number larger than $\gamma$.

**Case 2:** If delay $h = 0$, then $\Sigma = I$ and $L_{oh} = L_o$. Conditions (i) and (ii) are always satisfied and $L_{oh}$ always exists. Hence, the conditions are reduced to the nonsingularity of $I - L_oL_c$ for any $\gamma > \gamma_2$. In this case, $\gamma_{opt} = \gamma_2 = \max\{\gamma : \det(I - L_oL_c) = 0\}$.

**Corollary 5.** For a given minimally-realized transfer matrix

\[
G_\beta(s) = \begin{bmatrix} A \\ C \\ 0 \end{bmatrix}
\]

having neither jо-axis zero nor jо-axis pole, the delay-free Nehari problem (to find $K(s) \in H_\infty$ such that $\|G_\beta(s) + K(s)\|_{L_\infty} < \gamma$) is solvable iff $\gamma > \max\{\gamma : \det(I - L_oL_c) = 0\}$. Furthermore, if this condition holds, then $K(s)$ is parameterized as

\[
K(s) = \mathcal{H}_\gamma(W^{-1}(s), Q(s)),
\]

where

\[
W^{-1}(s) = \begin{bmatrix}
A + \gamma^{-2}BB^*L_c & (I - L_oL_c)^{-1}L_oC^* & -(I - L_oL_c)^{-1}B \\
-C & I \\
-\gamma^{-2}B^*L_c & 0 & I
\end{bmatrix}
\]

and $\|Q(s)\|_{H_\infty} < \gamma$ is a free parameter.

**Remark 6.** This is an alternative solution to the well-known Nehari problem which has been addressed extensively, e.g. in (Francis, 1987; Green et al., 1990). The $A$-matrix $A$ is not split here. In common situations, it was handled by modal decomposition, see, e.g. (Green et al., 1990), and the $A$-matrix $A$ was split into a stable part and an anti-stable part.

**Case 3:** If delay $h = 0$ and $A$ is stable, then $L_0 = 0$, $L_c = 0$, $L_{oh} = 0$, and $\Sigma = I$. The conditions are always satisfied for any $\gamma > 0$. $K(s)$ is then parameterized as $K(s) = \mathcal{H}_\gamma(W^{-1}(s), Q(s))$, where

\[
W^{-1}(s) = \begin{bmatrix} I & -G_\beta \\ 0 & I \end{bmatrix}.
\]

This is obvious. For the stable delay-free Nehari problem, the solution is definitely $K(s) = -G_\beta(s) + Q(s)$ for any $\gamma > 0$, where $\|Q(s)\|_{H_\infty} < \gamma$ is a free parameter. In this case, $\gamma_{opt} = 0$.

5. **Conclusion**

The solution to the delay-type Nehari problem in the stable case is extended to the unstable case. The additional cost paid for the instability is the nonsingularity of two additional matrices. All sub-optimal compensators are parameterized in a transparent structure incorporating a modified Smith predictor, which is the only infinite-dimensional part in the controller.
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Appendix. Realizations of $\Theta^{-1}(s)$ and $\Theta(s)$

Although $\begin{bmatrix} I & F_0^* \cr 0 & I \end{bmatrix}$ is unstable, $\Theta(s)$ can be calculated as follows without changing the result after a series of simplification (only for calculation, not for implementation):

$$\Theta(s) = \begin{bmatrix} I & F_0^* \cr 0 & I \end{bmatrix} \begin{bmatrix} \gamma^2 I (\gamma^2 I - G_p G_p^* )^{-1} & 0 \\ 0 & G_p G_p^* - \gamma^2 I \end{bmatrix} \begin{bmatrix} I & 0 \\ F_0 & I \end{bmatrix}.$$ (11)

Substituting the realizations of $G_p(s)$ and $F_0(s)$ into (11), $\Theta(s)$ is

$$\Theta(s) = \begin{bmatrix} A & \gamma^{-2} B B^* \\ -C^* C & -A^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A & \gamma^{-2} B B^* \\ -C^* C & -A^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In order to further simplify $\Theta(s)$, invert it:

$$\Theta^{-1}(s) = \begin{bmatrix} A & \gamma^{-2} B B^* \\ -C^* C & -A^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
\[
\begin{bmatrix}
A & \gamma^{-2}BB^* & 0 & 0 & 0 & 0 & 0 & \gamma^{-2}B \\
-C^*C & -A^* & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -A^* & 0 & 0 & 0 & C^* & -\gamma^{-2}\Sigma_{21}B \\
0 & 0 & -\gamma^{-2}BB^* & A & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -C^*C & -A^* & \gamma^{-2}BB^* & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\gamma^{-2}\Sigma_{21}B & 0 \\
0 & 0 & -\gamma^{-2}B^* & \gamma^{-2}B^*\Sigma_{11} & \gamma^{-2}B^*\Sigma_{21} & \gamma^{-2}B^*\Sigma_{11} & 0 & 0 \\
0 & 0 & 0 & -C & 0 & 0 & 0 & I \\
0 & -\gamma^{-2}B^* & \gamma^{-2}B^*\Sigma_{11} & \gamma^{-2}B^*\Sigma_{21} & -\gamma^{-2}B^*\Sigma_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\gamma^{-2}B^* & \gamma^{-2}B^*\Sigma_{11} & \gamma^{-2}B^*\Sigma_{21} & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-A^* & 0 & C^* & -\gamma^{-2}\Sigma_{11}B \\
-\gamma^{-2}BB^* & A & 0 & 0 \\
0 & -C & I & 0 \\
\gamma^{-2}B^*\Sigma_{11} & \gamma^{-2}B^*\Sigma_{21} & 0 & -\gamma^{-2}I \\
\end{bmatrix}
\]

\[
\Theta(s) = \begin{bmatrix}
A & 0 & -\Sigma_{11}C^*B \\
-C^*C & -A^* & \Sigma_{11}C^* \\
-C\Sigma_{11} & -C\Sigma_{12} & I & 0 \\
0 & B^* & 0 & 0 & -\gamma^{-2}I \\
\end{bmatrix}
\]

References


Qing-Chang Zhong was born in 1970 in Sichuan, China. He obtained the M.S. degree in Electrical Engineering in 1997 from Hunan University, China and the Ph.D. degree in 2000 in Control Theory and Engineering from Shanghai Jiao Tong University, China. He held a postdoctoral position at Technion-Israel Institute of Technology, Israel. He is currently a Research Associate at the Department of Electrical and Electronic Engineering of Imperial College of Science, Technology and Medicine, London, UK. His current research focuses on $H$-infinity control of time-delay systems, control in power electronics, process control, control using delay elements and control in communication networks.