On Feedback Stabilizability of Linear Systems With State and Input Delays in Banach Spaces

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Abstract—The feedback stabilizability of a general class of well-posed linear systems with state and input delays in Banach spaces is studied in this paper. Using the properties of infinite dimensional linear systems, a necessary condition for the feedback stabilizability of delay systems is presented, which extends the well-known results for finite dimensional systems to infinite dimensional ones. This condition becomes sufficient as well if the semigroup of the delay-free system is immediately compact and the control space is finite dimensional. Moreover, under the condition that the Banach space is reflexive, a rank condition in terms of eigenvectors and control operators is proposed. When the delay-free state space and control space are all finite dimensional, a very compact rank condition is obtained. Finally, the abstract results are illustrated with examples.

Index Terms—Banach spaces, feedback stabilizability, Hautus criterion, rank condition, regular systems, time-delay systems.

I. INTRODUCTION

A. Literature Review and Motivation

Delay-differential systems arise in the study of many problems with theoretical and practical importance. The behavior of these systems was studied in the seventies, especially in the book by Hale [17], where the semigroup theory was applied to reformulate delay systems as distributed parameter systems (see also Bellman and Cooke [4] and Halanay [16] for an introduction to this approach). The recent well-developed theory for such systems is gathered in several books e.g., Bátkaí and Piazzera [2], Bensoussan et al. [3], and Hale and Verduyn Lunel [18]. More recently it is shown in [10], [11], [14] that delay systems form a subclass of well-posed and regular infinite dimensional linear systems in the Salamon-Weiss sense [30–33], [37].

It has been recognized that the delay presence in the state and/or input could induce bad performance (even instability) and complicate controller design and system analysis [39]. This fact makes the investigation of the stability of delay systems very interesting and motivates many researchers to look for conditions guaranteeing stabilizability of such systems. By analyzing the existing theory in this area one can note that the results are obtained with some special techniques, which do not fit into a general theory as that developed for delay-free systems. Also, most of the works have been devoted to feedback stabilizability of state delay systems with a finite dimensional delay-free state space e.g., [4], [6], [20] or with an infinite dimensional space [25]. The feedback stabilizability of state-input delay systems is investigated in [19], [22], [26]–[29], where the delay-free state space is finite dimensional. It is shown by Olbrot [26] that the feedback stabilizability of the state-input delay system

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - 1) + P u(t) + P_1 u(t - 1)
\]

is equivalent to the condition

\[
\text{Rank}[\Delta(\lambda) = \begin{bmatrix} P + e^{-\lambda P_1} & \end{bmatrix} = n \quad \text{for } \lambda \in \mathbb{C} \text{ with } \Re \lambda \geq 0
\]

where \(n\) is the dimension of the delay-free system and \(\Delta(\lambda) := \lambda I - A_0 - A_1 e^{-\lambda}\).

We are interested in developing a more general approach to the feedback stabilizability of state-input delay systems in Banach spaces. We shall introduce a general class of feedback laws which stabilize the closed-loop system. This is based on the recent theory of regular linear systems [21], [31], [32], [37], and the recent work on state-input delay systems in Banach spaces [11], [24], where it has been proved that systems with general distributed state, input and output delays can be reformulated as regular linear systems.

Notation and Problem Statement

Throughout this paper we use the following notation. For a Banach space \(Z\), a real number \(r > 0\), and a function \(f : [-r, \infty) \to Z\), the history function \(f_t : [-r, 0) \to Z\) of \(f\) is defined as \(f_t(\theta) = f(t + \theta)\) for \(t \geq 0\) and \(\theta \in [-r, 0)\). We denote by \(C([-r, 0], Z)\) the space of all continuous functions from \([-r, 0]\) to \(Z\). For \(p \in [1, \infty)\) we denote by \(L^p([-r, 0], Z)\) the Lebesgue space of all \(p\)-integrable functions \(\varphi : [-r, 0] \to Z\) [7], by \(W^{1,p}([-r, 0], Z)\) the space of indefinite integrals of \(p\)-integrable functions, and by \(L(E, F)\) the Banach space of all linear and bounded operators from a Banach space \(E\) to another Banach space \(F\) with \(L(E) := L(E, E)\).

In this paper we consider the state-input delay system

\[
\begin{align*}
\dot{z}(t) &= A_0 z(t) + L x(t) + B u(t), \\
x(t) &= z, \quad x(t) = \varphi(t), \quad u(t) = \psi(t), \quad -r \leq t \leq 0.
\end{align*}
\]

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Here $A$ is the generator of a $C_0$-semigroup $T := (T(t))_{t \geq 0}$ on a Banach space $X$ and the delay operators $L : W^{1,p}([-\tau,0], X) \to X$ and $B : W^{1,p}([-\tau,0], U) \to X$ are linear and bounded, where $U$ is a (control) Banach space. The function $x_J$ is the history function of the function $x : [-\tau, \infty) \to X$ and $u_J$ is the history function of the function $u : [-\tau, \infty) \to U$. The initial conditions are $z \in X, \varphi \in IP([-\tau,0], X)$ and $\psi \in IP([-\tau,0], U)$, which form the initial state of the system. If $X$ and $U$ are finite dimensional spaces then the operators $L$ and $B$ can take the explicit form of Riemann-Stieltjes integrals

$$L\hat{x} = \int_{-\tau}^{0} d\mu(\theta)\hat{x}(\theta), \quad B\hat{y} = \int_{-\tau}^{0} d\nu(\theta)\hat{y}(\theta) \quad (3)$$

for $\hat{x} \in C([-\tau,0], X)$ and $\hat{y} \in C([-\tau,0], U)$, where $\mu : [-\tau,0] \to L(X)$ and $\nu : [-\tau,0] \to L(U, X)$ are functions of bounded variations [3, p.249]. In this case the system (2) is well-posed in the sense that it can be rewritten as a well-posed open-loop system [11], [13].

In this paper $X$ is an arbitrary Banach space and the operators $L$ and $B$ are not assumed to be of the form (3). In this case, it is shown in [11] that the system (2) can be reformulated as a well-posed control system on the state space $Z := X \times IP([-\tau,0], X) \times IP([-\tau,0], U)$ in the sense of [35] if $L$ and $B$ are observation operators of regular linear systems governed by the left shift semigroup (see [13] for more details on such systems). These conditions on $L$ and $B$ are always satisfied if $X$ and $U$ are finite dimensional, see [13].

**B. Overview of the Major Contributions**

After discussing left shift semigroups, transforming the system (2) into a well-posed open-loop system on $Z$ and $U$ based on [11] and characterizing the structure of feedback stabilizable operators, we introduce a necessary condition for the stabilizability of the system (2), using the generalization of the Hautus criterion for the stabilizability of distributed-parameter linear systems [38, Proposition 3.5]. This generalizes the condition (1) to the case of general Banach spaces with general delay operators (see Theorem 7). This necessary condition does not require any regularity on the semigroup $T$ or on the geometry of the state and control spaces. However, to show the sufficiency of this condition, we need to assume that the control space $U$ is finite dimensional and the semigroup $T(t)$ is compact for $t > 0$, i.e., immediately compact. Under these conditions, the delay system (2) is feedback stabilizable if and only if

$$\text{Im}(\lambda - A - LE_{\lambda}) + \text{Im}(BE_{\lambda}) = X$$

holds for all $\lambda$ with $\text{Re}\lambda \geq 0$, which belongs to the finite unstable spectrum of the operator $A + LE_{\lambda}$, where $e_{\lambda} : Z \to LP([-\tau,0], Z)$ is defined as $(e_{\lambda} z)(\theta) = e^{\lambda \theta} z, z \in Z, \theta \in [-\tau,0]$ with $Z = X$ or $U$. Furthermore, a rank condition, derived from the above condition, is given in terms of eigenvectors and control operators in the case with a reflexive Banach space $X$. This extends the result in [25] where a rank condition was given for systems with state delays only. When the delay-free state space and control space are finite dimensional, a compact rank condition equivalent to that of Olbrot [26] is obtained. We illustrate our abstract results with examples involving a single delay, multiple delays, distributed delays and elliptic operators in bounded domains of the Euclidean space.

**C. Organization of the Paper**

The organization of the paper is as follows. In Section II we recall some preliminaries about well-posed and regular infinite dimensional systems. In Section III we discuss left shift semigroups, state the main assumptions and reformulate the delay system (2) to an infinite dimensional open loop system, we present the definition of feedback stabilizability and characterize the structure of feedback stabilizable operators. Section IV is devoted to state and prove the main results on feedback stabilizability of the delay system (2) and in Section V we present some examples. Conclusions are made in Section VI, followed by two appendices, in which the frequently-cited known results are gathered.

**II. PRELIMINARIES: WELL-POSED AND REGULAR INFINITE DIMENSIONAL LINEAR SYSTEMS**

Here, we briefly recall the framework of infinite dimensional well-posed and regular linear systems in the Salamon-Weiss sense from [31], [32], [34]–[36], [33], [37].

**A. Basic Concepts**

Let $G : D(G) \subseteq Z \to Z$ be the generator of a $C_0$-semigroup $V := (V(t))_{t \geq 0}$ on a Banach space $(Z, \| \cdot \|)$. We denote by $\rho(G)$ the resolvent set of $G$, i.e., the set of all $\lambda \in \mathbb{C}$ such that $\lambda - G$ is invertible. The spectrum of $G$ is by definition $\sigma(G) = \mathbb{C} \setminus \rho(G)$. We define the resolvent operator of $G$ as $R(\lambda, G) := (\lambda - G)^{-1} \lambda \in \rho(G)$. The domain $D(G)$ endowed with the graph norm $\| z \|_D := \| z \|_Z + \| Gz \|$, $z \in D(G)$, is a Banach space. We also define the norm $\| z \|_{-1} := \| R(\lambda, G) z \|$ for some $\lambda \in \rho(G)$. The completion of $Z$ with respect to the norm $\| \cdot \|_{-1}$ is a Banach space denoted by $Z_{-1}$, which is called the extrapolation space associated with $Z$ and $G$. Moreover, the continuous injection $Z \hookrightarrow Z_{-1}$ holds. The semigroup $V$ can be naturally extended to a strongly continuous semigroup $V_{\lambda 1} = (V_{\lambda 1}(t))_{t \geq 0}$ on $Z_{-1}$, of which the generator $G_{\lambda 1} : Z \to Z_{-1}$ is the extension of $G$ from $D(G)$ to $Z$; see [8] for more details.

The pair $(V, \Phi) := (V(t), (\Phi(t))_{t \geq 0})$ is called a control system on $Z, U$ if $(V(t))_{t \geq 0}$ is a $C_0$-semigroup on $Z$ and $\Phi(t) : LP([0,t], U) \to Z, t \geq 0$, is a linear bounded operator satisfying, for $u \in LP([0,t], U)$ and $t, s \geq 0$.

$$\Phi(t+s)u = \Phi(t)(u(\cdot + s)|_{[0,t]}) + V(t)\Phi(s)(u|_{[0,t]}) \quad (4)$$

See [35], [36] for more details on the maps $\Phi(t)$. By the representation theorem due to Weiss [35, Theorem 3.9], there exists a unique operator $B \in LU(Z_{-1}, Z_{-1})$, called an admissible control operator for $G$ (or $V$), such that for any $t \geq 0$ and $u \in LP([0,t], U)$,

$$\Phi(t)u = \int_{0}^{t} V_{\lambda 1}(t-\sigma)Bu(\sigma) d\sigma \quad (5)$$

where the integral exists in $Z_{-1}$. We say that $(V, \Phi)$ is represented by the control operator $B$. Each control system $(V, \Phi)$
with the representing control operator $B$ is completely determined by an abstract differential equation of the form
\[ \dot{x}(t) = Gx(t) + Bu(t), \quad t \geq 0. \] (6)
It is well-posed in the sense that it has a unique strong solution, called the state trajectory with initial state $x(0)$, given by
\[ x(t) = V(t)x(0) + \Phi(t)u, \quad t \geq 0. \] (7)

An operator $C \in \mathcal{L}(\mathcal{D}(G), Y)$ is called an admissible observation operator for $G$ (or $V$) if
\[ \int_0^T \|CV(t)x\|^p dt \leq \gamma(\tau)^p \|x\|^p \] (8)
holds for any $x \in \mathcal{D}(G)$ and for some constants $\gamma > 0$ and $\gamma(\tau) > 0$. For simplicity, denote
\[ \mathcal{O}^p_Y(G) := \{ C \in \mathcal{L}(\mathcal{D}(G), Y) : C \text{ is an admissible observation operator for } G \}. \] (9)

For $C \in \mathcal{O}^p_Y(G)$, the linear system
\[ \dot{x}(t) = Gx(t), \quad x(0) = \eta \]
\[ y(t) = Cx(t), \quad t \geq 0 \] (10)
is well-posed in the sense that the observation function $y(t)$ can be extended to a function in $L^p_{\text{loc}}(\mathbb{R}^+, Z)$. In fact, from (8), the map
\[ \Psi : \mathcal{D}(G) \to L^p_{\text{loc}}(\mathbb{R}^+, Y), \quad x \mapsto CV(x) \] (11)
can be extended to a linear bounded operator from $Z$ to $L^p_{\text{loc}}(\mathbb{R}^+, Y)$, called the extended output map. Then we can set $y(t) = (\Psi\infty x)(t)$ for any $x \in Z$ and almost every $t \geq 0$. If we define
\[ (\Psi(t)x)(\tau) = (\Psi\infty x)(\tau) \] (12)
for $x \in \mathcal{D}(G)$ and $0 \leq \tau \leq t$, and for $\tau > t$, then $(\Psi(t))_{t \geq 0}$ is a family of bounded linear operators from $Z$ to $L^p_{\text{loc}}(\mathbb{R}^+, Y)$. We say that $(V, \Psi) := (V, (\Psi(t))_{t \geq 0})$ is an observation system on $Z, Y$, and the observation equation $y(t)$ in (10) satisfies $y(\tau) = (\Psi(t)x)(\tau)$ for almost every $(a.e.)$ $0 \leq \tau \leq t$ and all $x \in Z$. Note that $y(t)$ is extended in the abstract way. In order to obtain a pointwise representation in terms of the observation operator $C$, consider the Yosida extension[34] of $C$ with respect to $G$ defined as
\[ \tilde{C}z := \lim_{\lambda \to +\infty} \lambda C\mathcal{L}(\lambda, G)z \]
\[ \mathcal{D}(\tilde{C}) := \{ z \in Z : \text{the above limit exists in } Y \} \] (13)

Let $\mathcal{D}(G)$ be endowed with the graph norm. For $z \in \mathcal{D}(G)$,
\[ \|C\mathcal{L}(\lambda, G)z - Cz\| \leq \|C\| \lambda \|\mathcal{L}(\lambda, G)z - z\|_G \]
\[ = \|C\| \left( \|\lambda \mathcal{L}(\lambda, G)z - z\|_G + \|\mathcal{L}(\lambda, G)Gz - Gz\| \right) \text{.} \]

The right hand side goes to 0 when $\lambda \to +\infty$, according to Lemma 3.4 in [8, p.73]. Hence, $\mathcal{D}(G) \subset \mathcal{D}(\tilde{C})$ and $\tilde{C}z = Cz$ for any $z \in \mathcal{D}(G)$. Note that the definition of $\tilde{C}$ does not require that $C \in \mathcal{O}^p_Y(G)$. However, if this is the case, $V(t)z \in \mathcal{D}(\tilde{C})$ and $\Psi\mathcal{L}(t)z := CV(t)z$ for all $z \in Z$ and $a.e. \ t \geq 0$, (see [34, Theorem 4.5]).

For $f \in L^p_{\text{loc}}(\mathbb{R}^+, Z)$, denote its convolution with the semi-group $G(t)$ as
\[ (G \ast f)(t) := \int_0^t G(t-r)f(r)dr, \quad t \geq 0. \]
Then, we have the following result [9, Proposition 3.3].

Proposition 1: Assume $C \in \mathcal{O}^p_Y(G)$ and $f \in L^p_{\text{loc}}(\mathbb{R}^+, Z)$. Define
\[ \mathcal{E}^p(\tilde{C}) := \{ h \in L^p_{\text{loc}}(\mathbb{R}^+, Z) : h(t) \in \mathcal{D}(\tilde{C}) \text{ for } a.e. \ t \geq 0, \text{ and } \mathcal{C}h(t) \in L^p_{\text{loc}}(\mathbb{R}^+, Z) \}. \]
Then $G \ast f \in \mathcal{E}^p(\tilde{C})$ and
\[ \|\mathcal{C}(G \ast f)\|_{L^p([0, \alpha]; Z)} \leq c(\alpha)\|f\|_{L^p([0, \alpha]; Z)} \]
for $\alpha > 0$ and a constant $c(\alpha) > 0$, which is independent of $f$ and approaches 0 as $\alpha \to 0$.

B. Well-Posed and Regular Systems

Here, we focus on a control system $(V, \Phi)$ represented by $B$ and an observation system $(V, \Psi)$ represented by $C$. We say that the linear system
\[ \dot{x}(t) = Gx(t) + Bu(t), \quad t \geq 0 \]
\[ y(t) = Cx(t), \quad t \geq 0 \] (14)
with the initial state $x(0)$, is well-posed on $Z, U, Y$ if there exists a family $F := (F(t))_{t \geq 0}$ of bounded linear operators from $L^p([0, t], U)$ to $L^p([0, t], Y)$, $t \geq 0$, satisfying
\[ [F(t + s)u](\tau) = [F(t)u(\cdot + s)](\tau - s) \]
\[ + [\Psi(t)\Phi(s)(u(\cdot))(\tau - s)] \] (15)
for $\tau \in [s, s + t], \ t, s \geq 0$, and $u \in L^p([0, s + t], U)$. For more details on operators $F(t)$ we refer to [36], [37]. In this case we also say that the quadruple $\Sigma := (T, \Phi, \Psi, F)$ is a well-posed system on $Z, U, Y$.

Let $P_T$ be the operator of truncation to $[0, \tau]$, that is $(P_T f)(t) = f(t)$ for $t \in [0, \tau]$ and zero otherwise. The operators $F(t)$ are compatible in the sense that $P_T F(t) = F(t)$ for $t \in [0, \tau]$ and zero otherwise. This property provides a unique operator $F_{\infty} : L^p_{\text{loc}}(\mathbb{R}^+, U) \to L^p_{\text{loc}}(\mathbb{R}^+, Y)$, called the extended input-output map, which satisfies $F(t) = P_T F_{\infty} = P_T F_{\infty} P_T$ for $\tau \geq 0$. We recall from [36, Theorem 3.6] that there exist $c \in \mathbb{R}$ and a unique bounded and analytic function $H(\cdot) : \{ \lambda \in \mathbb{C} : \Re \lambda > c \} \to \mathcal{L}(U, Y)$ such that
\[ \tilde{y}(\lambda) = H(\lambda)\tilde{u}(\lambda), \quad \Re \lambda > c \]
if $y = F_{\infty} u$. Here $\tilde{y}$ and $\tilde{u}$ are the Laplace transform of $y$ and $u$, respectively. Here, $H$ is called the transfer function of $\Sigma$. 

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We say that the well-posed system $\Sigma$ is regular (with zero feedthrough) if the limit
\[ \lim_{t \to 0} \frac{1}{t} \int_0^t \left( F(\infty, u_0)(\tau) \right) d\tau = 0 \]
ezists in $Y$ for the constant input $u_0(t) = z$, $z \in U$, $t \geq 0$.

The following definitions will be used throughout this paper.

**Definition 1:** Let $B$ and $C$ be the admissible control and observation operators issued from $(V, \Phi)$ and $(V, \Psi)$, respectively. We say that the triple $(G, B, C)$ generates a regular system $\Sigma$ if there exists a bounded operator $F_{\infty}^\mu : L^p_{\text{loc}}(\mathbb{R}_+, U) \to L^p_{\text{loc}}(\mathbb{R}_+, Y)$ such that $\Sigma := (V, \Phi, \Psi, F)$ is regular on $Z$, $U$, $Y$.

**Definition 2:** Let $\Sigma$ be a regular system with input-output operators $F(\cdot)$. An operator $\Gamma \in \mathcal{L}(Y, U)$ is called an admissible feedback for $\Sigma$ if $I - F(\cdot)\Gamma$ has uniformly bounded inverse.

Note that in the case of Hilbert spaces and $p = 2$ one can use transfer functions instead of input-output operators for the definition of admissible feedback operators [36].

We now state a very general perturbation theorem due to Weiss in Hilbert spaces [37] and due to Staffans in general Banach spaces [32, Chap.7].

**Theorem 3:** Assume that $(G, B, C)$ generates a regular system $\Sigma$ with admissible feedback operator $\Gamma$. Then the operator
\[ G_{\Gamma} = G_{\Gamma-1} + B\mathcal{L}(C) \]
\[ D(G_{\Gamma}) := \{ z \in D(C) : (G_{\Gamma-1} + B\mathcal{L}(C)) z \in Z \} \]
with the sum defined in $Z_{\Gamma-1}$ generates a $G_{\Gamma}$-semigroup $V_{\Gamma}$ on $Z$. Moreover, for a.e. $\sigma \geq 0$, $V_{\Gamma}(\sigma) z \in D(C)$ and, for $z \in Z$, $t \geq 0$,
\[ V_{\Gamma}(t) z = V(t) z + \int_0^t V_{\Gamma-1}(t - \sigma) B\mathcal{L}(C) V_{\Gamma}(\sigma) z d\sigma. \]

III. FEEDBACK STABILIZABILITY OF SYSTEM (2)

A. Left Shift Semigroups and Main Assumptions

Here, we follow Section II-A to present two left shift semigroups and then state the main assumptions on delay operators $L$ and $B$ of the system (2). Define
\[ Q_X := \frac{\partial}{\partial \theta} \]
\[ D(Q_X) := \{ \varphi \in W^{1,p}([\sigma, 0], X) : \varphi(0) = 0 \} \]
and, similarly, $Q_X$. It is well known that $(Q_X, D(Q_X))$ generates the left shift semigroup
\[ (S_X(t) \varphi)(\theta) = \begin{cases} 0, & t + \theta \geq 0 \\ \varphi(t + \theta), & t + \theta \leq 0 \end{cases} \]
for $t \geq 0$, $\theta \in [-\sigma, 0]$ and $\varphi \in L^p([\sigma, 0], X)$. Similarly, we have $S_X$. The pair $(S_X, \Phi_X)$ with
\[ (\Phi_X(t)x)(\theta) = \begin{cases} x(t + \theta), & t + \theta \geq 0 \\ 0, & t + \theta \leq 0 \end{cases} \]
for the control function $x \in L^p_{\text{loc}}(\mathbb{R}_+, X)$ is a control system on $L^p([\sigma, 0], X)$ and $X$, which is represented by the (strictly) unbounded admissible control operator [13]
\[ B_X := (\lambda - (Q_X)^{-1}) \mathcal{J}_X, \quad \lambda \in \mathbb{C} \]
where $(Q_X)^{-1}$ is the generator of the extrapolation semigroup associated with $S_X(t)$ and $\mathcal{J}_X$ is the linear bounded operator $\mathcal{J}_X : X \to L^p([\sigma, 0], X)$ with $\mathcal{J}(\varphi)(\theta) = e^{\lambda \sigma} \varphi$.

In fact, $B_X$ is the adjoint of the delta operator at zero. Similarly, we can have operators $\Phi_U$ and $B_U$.

For the control function $x \in L^p_{\text{loc}}([0, \infty), (S_X, \Phi_X)$ with $x(\theta) = \varphi(\theta)$ for a.e. $\theta \in [-\sigma, 0]$, the state trajectory of $(S_X(t) \Phi_X)$ is the history function of $x$ given by
\[ x_t = S_X(t) \varphi(t) + \Phi_X(t)x, \quad t \geq 0. \]
Similarly, for the control system $(S_U, \Phi_U)$ represented by $B_U$, we have
\[ u_t = \Phi_U(t) \psi(t) + \Phi_U(t) u, \quad t \geq 0 \]
with $u(\theta) = \psi(\theta)$ for a.e. $\theta \in [-\sigma, 0]$.

According to (11), if $L$ is an admissible observation operator of $S_X$, we define
\[ \Psi^L \varphi = L S_X(\cdot) \varphi \]
and then $\Psi^L$ according to (12). Similarly, we define $\Psi^B$ and $\Psi^B$ for operator $B$. The input-output operator $F^L$ associated with $\Phi_X$ and $\Psi^L$ will be defined according to (15), replacing $\Psi$ with $\Phi^B \Phi_X$. The input-output operator $F^B$ is similarly defined for operator $B$.

From now on we assume that the operators $L$ and $B$ in system (2) satisfy:
1. $(A_1)$ $L \in \mathcal{O}_X^\infty(Q_X)$ and the triple $(Q_X, B_X, L)$ generates a regular system $\Sigma^L := (S_X, \Phi_X, \Psi^L, F^L)$ on $L_p([-\sigma, 0], X)$, $X$, $X$.
2. $(A_2)$ $B \in \mathcal{O}_X^\infty(Q_U)$ and the triple $(Q_U, B_U, B)$ generates a regular system $\Sigma^B := (S_U, \Phi_U, \Psi^B, F^B)$ on $L_p([-\sigma, 0], U)$, $U$, $X$.

**Remark 4:** Here, we give an example of operators $L$ and $B$ that satisfy the above assumptions. Let $\mu : [-\sigma, 0] \to \mathcal{L}(X)$ and $\nu : [-\sigma, 0] \to \mathcal{L}(U, X)$ be functions of bounded variations such that the total variations of $\mu$ and $\nu$ approach 0 when the interval goes to 0. If the operators $L$ and $B$ take the form of
\[ L \varphi = \int_{-\sigma}^{0} d\mu(\theta) \varphi(\theta) \quad \text{and} \quad B \psi = \int_{-\sigma}^{0} d\nu(\theta) \psi(\theta) \]
for $\varphi \in C([-\sigma, 0], X)$ and $\psi \in C([-\sigma, 0], U)$, where $X$ and $U$ are not necessarily finite dimensional, then $L$ satisfies $(A_1)$ and $B$ satisfies $(A_2)$; see [13].

B. Reformulation of System (2)

We recall from [11], [15] how the delay system (2) can be reformulated as a linear distributed parameter system on some appropriate Banach spaces.
Consider the Banach space $\mathcal{X} := X \times L^p([−r,0], X)$ endowed with the norm $\left\| (\hat{z}, \varphi) \right\| = \| z \| + \| \varphi \|_p$. It is well-known (see e.g., [2]) that the delay system (2) with $B \equiv 0$ is closely related to the following linear operator

$$A_L = \begin{pmatrix} A & L \\ 0 & 0 \end{pmatrix},$$

$$D(A_L) := \left\{ \begin{pmatrix} z \\ \varphi \end{pmatrix} \in D(A) \times W^{1,p}([−r,0], X) : \varphi(0) = z \right\}.$$  \hfill (20)

It is not difficult to show that $L \in \mathcal{O}^p_X(Q_X)$ if and only if

$$\int_0^t \left\| \left( T(t)z + S_X(t)\varphi \right) \right\|_p \, dt \leq \kappa \left\| \begin{pmatrix} z \\ \varphi \end{pmatrix} \right\|_p$$ \hfill (21)

for $\left( \begin{pmatrix} z \\ \varphi \end{pmatrix} \right) \in D(A_L)$ and some constants $\tau > 0$ and $\kappa > 0$ [9], where $T_t : X \to L^p([−r,0], X)$ is defined as

$$T_t z(\theta) = \begin{cases} T(t+\theta)z, & t + \theta \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$ \hfill (22)

According to the Hölder inequality, (21) implies that

$$\int_0^t \left\| \left( T(t)z + S_X(t)\varphi \right) \right\|_p \, dt \leq \gamma(\tau) \left\| \begin{pmatrix} z \\ \varphi \end{pmatrix} \right\|_p,$$ \hfill (23)

for $\left( \begin{pmatrix} z \\ \varphi \end{pmatrix} \right) \in D(A_L)$ and $\gamma(\tau) = \tau^{1/q} \kappa$ with $1/p + 1/q = 1$. Hence, the operator $A_L$ generates a strongly continuous semigroup $T_L := (T(t)L)_{t \geq 0}$ on $\mathcal{X}$, according to [1].

The following remark is important for the proof of the main results in this paper.

**Remark 5**: From the discussion above, we have seen that for $L \in \mathcal{O}^p_X(Q_X)$ the estimate (23) holds with $\gamma(\tau) \to 0$ as $\tau \to 0$. If in addition the operator $A$ generates an immediately compact semigroup on $X$ (i.e., compact for $t > 0$), then $A_L$ generates an eventually compact semigroup on $\mathcal{X}$ (i.e., compact for $t > t_0 > 0$, with $t_0 \equiv r$ in this paper) [23].

In order to consider the well-posedness of the state-input delay systems (2), define the operators

$$A_{L,B} = \begin{pmatrix} A_L & 0 \\ 0 & B \end{pmatrix},$$

$$D(A_{L,B}) := D(A_L) \times D(Q_U).$$ \hfill (24)

and

$$B = \begin{pmatrix} 0 & 0 \\ 0 & B_{\hat{U}} \end{pmatrix}^\top. \hfill (25)$$

It has been shown in [11, Theorem 3.1] that when $L$ satisfies the condition (23) and $B \in \mathcal{O}^p_X(Q_U)$, the operator $A_{L,B}$ generates the following $C_0$-semigroup on $\mathcal{Z} = \mathcal{X} \times L^p([−r,0], U)$.

$$T_{L,B}(t) = \begin{pmatrix} T_L(t) & \mathcal{R}(t) \\ \hat{U}(t) & S_U(t) \end{pmatrix}.$$ \hfill (26)

with $\mathcal{R}(t) : L^p([−r,0], U) \to \mathcal{X}$ given by

$$\mathcal{R}(t)z = \int_0^t T_L(t−\tau)\mathcal{R}([\psi_B z](\tau)) \, d\tau$$ \hfill (27)

for $t \geq 0$ and $z \in L^p([−r,0], U)$, where $\mathcal{R} : \mathcal{X} \to \mathcal{X}$ is the extended output map associated with $S_U$ and $B$. Furthermore, we have the following result from [11, Proposition 3.5] and [15, Proposition 3.2].

**Proposition 2**: Assume that $L$ satisfies (23) and $B$ satisfies the condition (A2). Then the operator $B$ defined in (25) is an admissible control operator for $A_{L,B}$ defined in (24).

According to (5), the control maps associated with the control operator $B$ are given by

$$\Phi_{L,B}(t)u = \int_0^t (T_{L,B}(t−\tau)B_{\hat{U}}(\tau)u) \, d\tau$$ \hfill (28)

for $t \geq 0$ and $u \in L^p_{loc}(\mathbb{R}_+, U)$. According to [11, Proposition 3.2], this is equivalent to

$$\Phi_{L,B}(t)u = \begin{pmatrix} \int_0^t T_L(t−\tau) & \mathcal{R}(\mathcal{F}_B^\infty u)(\tau) \end{pmatrix} \begin{pmatrix} \mathcal{F}_B^\infty u(\tau) \\ \mathcal{F}_B^\infty u(0) \end{pmatrix} \mathcal{T}_{L,B}(t)$$ \hfill (29)

for $t \geq 0$ and $u \in L^p([0,t], U)$, where $\mathcal{F}_B^\infty$ is the extended input-output map of the regular system $\Sigma^B$. Now, the open loop system

$$\begin{cases} \dot{x}(t) = A_{L,B}x(t) + Bu(t), & t \geq 0 \\ x(0) = (z, \varphi, \psi)^\top, \end{cases}$$ \hfill (30)

is well-posed according to Proposition 2. Combining (26) and

$$\mathcal{R}(t) = \begin{pmatrix} \mathcal{R}(t) & \mathcal{R}(t) \end{pmatrix},$$ \hfill (31)

we can see that the state trajectory of the system (30), for the initial state $x(0) = (z, \varphi, \psi)^\top$, is given by

$$\mathcal{R}(t) = \begin{pmatrix} \mathcal{R}(t) & \mathcal{R}(t) \end{pmatrix} = \mathcal{T}_{L,B}(t)x(0) + \mathcal{F}_{L,B}(t)u, \quad t \geq 0$$ \hfill (32)

according to (7). This is the same as the state trajectory of the system (2), see [11, Proposition 3.3]. This gives a natural connection between the systems (2) and (30). Let $u \in L^p_{loc}(\mathbb{R}_+, U)$ and $\lambda$ be sufficiently large. According to [13], the Laplace transform of $\mathcal{F}_B^\infty u$ in $\lambda$ is equal to $Be_{\hat{U}}(\lambda)$ and that of $\mathcal{F}_B(\cdot)u$ in $\lambda$ is exactly $e_{\hat{U}}(\lambda)$. Hence, by taking the Laplace transform on both sides of (28), we have

$$R(\lambda, A_{L,B}, \lambda)B_{\hat{U}}(\lambda) = \begin{pmatrix} R(\lambda, A_L)(Be_{\hat{U}}(\lambda)) \\ e_{\hat{U}}(\lambda) \end{pmatrix}.$$ \hfill (33)

C. The Structure of Feedback Stabilizable Operators

According to the reformulation of the delay system above, Proposition 2 and [38, Definition 2] (see also Appendix B), we introduce the following definition.

**Definition 6**: Let the conditions (A1) and (A2) be satisfied. We say that the state-input delay system (2) is feedback stabilizable if there exists an operator $\mathcal{C} \in \mathcal{L}(D(A_{L,B}), U)$ such that:

i) The triple $(A_{L,B}, B, C)$ generates a regular system $\Sigma$ on $\mathcal{Z}, U, \hat{U}$.

ii) The identity operator $I_U : U \to U$ is an admissible feedback operator for $\Sigma$. 

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iii) The semigroup generated by the operator \((A_{L,B})^{-1} + B\bar{C}\) endowed with the domain
\[ \mathcal{D}((A_{L,B})^{-1} + B\bar{C}) = \{ z \in \mathbb{C} : ((A_{L,B})^{-1} + B\bar{C})z \in \mathbb{C} \} \]
is exponentially stable.

In this case we say that the operator \(\bar{C}\) stabilizes the delay system (2). The following result shows an explicit structure of such operators \(\bar{C}\).

**Proposition 3:** Let the conditions (A1) and (A2) be satisfied. If \(C \in \mathcal{L}(\mathcal{D}(A_{L,B}), U)\) stabilizes the delay system (2) then it is of the form
\[ C = [C \ N \ D] \]
with \(C \in \mathcal{O}^D_U(A), N \in \mathcal{O}^D_U(Q_X), \) and \(D \in \mathcal{O}^D_U(Q_U)\) generated by the triple \((Q_U, B_U, D)\) generating a regular linear system with \(I_U\) as an admissible feedback operator.

**Proof:** Assume that \(C \in \mathcal{L}(\mathcal{D}(A_{L,B}), U)\) stabilizes the system (2), then at least we have \(C \in \mathcal{O}^D_U(A_{L,B})\). Decompose \(C\) into \(C = [Y \ D]\) with \(Y \in \mathcal{L}(\mathcal{D}(A_L), U)\) and \(D \in \mathcal{L}(\mathcal{D}(Q_U), U)\). Then, for any \((z, \varphi, 0) \in \mathcal{D}(A_{L,B})\) and \(\alpha > 0\), we have
\[ \int_0^\infty \left\| \mathcal{S}_{T_L}(\tau) \left( \begin{array}{c} z \\ \varphi \end{array} \right) \right\|^p d\tau = \int_0^\infty \left\| \mathcal{S}_{T_{L,B}}(\tau) \left( \begin{array}{c} z \\ 0 \end{array} \right) \right\|^p d\tau \leq \gamma^p(\alpha) \left\| \left( \begin{array}{c} z \\ \varphi \end{array} \right) \right\|^p \]
for \(\gamma > 0\). This means \(Y \in \mathcal{O}^D_U(A_L)\).

In order to characterize the operators \(Y \in \mathcal{O}^D_U(A_L)\), decompose \(A_L\) into \(A + P\) with
\[ A = \left( \begin{array}{cc} A & 0 \\ 0 & Q_X \end{array} \right), \quad D(\mathcal{A}) = \mathcal{D}(A_L) \]
and \(P = \left( \begin{array}{cc} 0 & L \\ 0 & 0 \end{array} \right), \quad D(\mathcal{P}) = \mathcal{D}(A_L)\).

From [1], the operator \(A\) is the generator of the \(C_0\)-semigroup
\[ T(t) = \left( \begin{array}{cc} T(t) & 0 \\ T(t) & S_X(t) \end{array} \right) \]
on \(\mathcal{X}\) for \(t \geq 0\), where the operators \(T(t)\) are defined in (22). Thus, the fact that \(L \in \mathcal{O}^D_U(Q_X)\) implies that \(P \in \mathcal{O}^D_U(A)\). According to the invariance of admissibility of observation operators [12, Theorem 3.1], which is reproduced in Appendix A for the reader’s convenience, there is \(\mathcal{O}^D_U(A_L) \subseteq \mathcal{O}^D_U(A)\). This allows us to concentrate on operators \(Y \in \mathcal{O}^D_U(A)\). Using (31), one can easily show that an operator \(Y \in \mathcal{O}^D_U(A)\) if and only if it is of the form
\[ Y = [C \ N] \]
with \(C \in \mathcal{O}^D_U(A)\) and \(N \in \mathcal{O}^D_U(Q_X)\). This shows that the operators \(C\) stabilizing the open loop system \((A_{L,B}, B)\) have at least the following form
\[ C = [C \ N \ D] \]
with \(C \in \mathcal{O}^D_U(A), \ N \in \mathcal{O}^D_U(Q_X)\) and \(D \in \mathcal{L}(\mathcal{D}(Q_U), U)\). Now prove the second part. Let \(\Sigma^D_{L,B}\) be the regular system generated by the triple \((A_{L,B}, B, C)\). According to [11, Theorem 5.1], \(D\) at least should be an observation operator of the regular system \(\Sigma^D := (S_U, \Phi_U, \Psi_U, F^D)\) generated by the triple \((Q_U, B_U, D)\) on \(L^p([-r, 0], U)\) and \(U\) such that \(I_U\) is an admissible feedback operator. Hence, \(D \in \mathcal{O}^D_U(Q_U)\). This ends the proof.

**IV. MAIN RESULTS**

**A. A Necessary Condition**

For each \(\lambda \in \mathbb{C}\), we redefine
\[ \Delta(\lambda) := (\lambda - A) - LE, \quad D(\Delta(\lambda)) = D(A) \]
for the system (2). From [2, p.56], we have
\[ \lambda \in \sigma(A + LE) \iff \lambda \in \sigma(A_L) \]
with \(A_L\) defined in (20). In view of this equivalence, \(\Delta(\lambda)\) is called the characteristic operator. The result below gives a necessary condition for the feedback stabilizability of the system (2).

**Theorem 7:** Assume the conditions (A1) and (A2) are satisfied. The feedback stabilizability of the delay system (2) implies the existence of \(\delta > 0\) such that
\[ \text{Im} \Delta(\lambda) + \text{Im}(B\mathcal{E}_X) = X \]
for any \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > -\delta\).

**Proof:** Denote
\[ \text{Ran}[\lambda - (A_{L,B})^{-1} B] = \{ (\lambda, z) : z \in \mathbb{C}, u \in U \} \subset \mathbb{C} \]
According to the generalized Hautus criterion [38, Prop. 3.5], which is reproduced in Appendix B for the reader’s convenience, the feedback stabilizability of system (29), and hence of (2), implies that there exists \(\delta > 0\) such that
\[ \text{Ran}[\lambda - (A_{L,B})^{-1} B] \supseteq \mathbb{C} \]
for any \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > -\delta\). This indicates, for any \((z, \varphi, \psi) \in \mathbb{C}\) and hence for any \(z \in X\), there exist \((z_0, \varphi_0, \psi_0) \in \mathbb{C}\) and \(u \in U\) such that, for \(\text{Re} \lambda > -\delta\),
\[ (z, \varphi, \psi)^T = (\lambda - (A_{L,B})^{-1})(z_0, \varphi_0, \psi_0)^T + Bu. \]
Let \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > -\delta\) and let \(\mu \in \rho(\mathcal{A}_L)\) be fixed. According to (30), we have
\[ Bu = (\mu - (A_{L,B})^{-1}) \left( \begin{array}{c} R(\mu, A_L) \\ \mu \end{array} \right) \]
\[ = (\lambda - (A_{L,B})^{-1}) \left( \begin{array}{c} R(\mu, A_L) \\ \mu \end{array} \right) \]
\[ \Rightarrow \text{Ran}[\lambda - (A_{L,B})^{-1} B] \supseteq \mathbb{C} \]
This is because \(\lambda\) does not always belong to \(\rho(\mathcal{A}_L)\).
Now, by (36), we obtain
\[
\begin{pmatrix}
  z \\
  \varphi \\
  \psi
\end{pmatrix} = \left(\lambda - (\mathcal{A}_L, B) \right)^{-1}
\begin{pmatrix}
  z_0 \\
  \varphi_0 \\
  \psi_0 + e_\mu u
\end{pmatrix} + \left(\mu - \lambda \right) R(\mu, \mathcal{A}_L) e_\mu u.
\]

It is clear that
\[
\begin{pmatrix}
  z_0 \\
  \varphi_0 \\
  \psi_0 + e_\mu u
\end{pmatrix} \in \mathcal{D}(\mathcal{A}_L, B) = \mathcal{D}(\mathcal{A}_L) \times \mathcal{D}(Q_U)
\]
which implies that
\[
\begin{pmatrix}
  z_0 \\
  \varphi_0
\end{pmatrix} \in \mathcal{D}(\mathcal{A}_L), \quad \psi_0 + e_\mu u \in \mathcal{D}(Q_U)
\]  
(38)
and, as a result
\[
\begin{pmatrix}
  z_0 \\
  \varphi_0 \\
  \psi_0 + e_\mu u
\end{pmatrix} \left(\lambda - (\mathcal{A}_L, B) \right)^{-1} + R(\mu, \mathcal{A}_L) e_\mu u = \mathcal{A}_L \begin{pmatrix}
  z_0 \\
  \varphi_0 \\
  \psi_0 + e_\mu u
\end{pmatrix}.
\]

Here, the identity \((\partial/\partial \theta) e_\mu u = \mu e_\mu u\) was used. Note that
\[
(\lambda - \mathcal{A}_L) R(\mu, \mathcal{A}_L)
\begin{pmatrix}
  B e_\mu u \\
  0
\end{pmatrix} = (\lambda - \mu) R(\mathcal{A}_L, B)
\begin{pmatrix}
  B e_\mu u \\
  0
\end{pmatrix} + \begin{pmatrix}
  B e_\mu u \\
  0
\end{pmatrix}.
\]

By combining (37), (39) and (40), we have
\[
(\lambda - \mathcal{A}_L) \begin{pmatrix}
  z_0 \\
  \varphi_0 \\
  \psi_0
\end{pmatrix} - B \psi_0 = \begin{pmatrix}
  z \\
  \varphi \\
  \psi
\end{pmatrix}
\]  
(41)
and
\[
(\lambda - \mathcal{A}_L) \varphi_0 = \varphi.
\]

The last identity implies \((\lambda - \partial/\partial \theta) \varphi_0 = e_\lambda z_0\), or
\[
\varphi_0 = e_\lambda z_0 + R(\lambda, Q_X) \varphi.
\]
Similarly, we have
\[
\psi_0 = - e_\lambda u + R(\lambda, Q_U) \psi.
\]
As we are only interested in any \(z \in X\), substitute \(\varphi_0 = \varphi\) with \(\varphi = 0, \psi = 0\) into (41), then
\[
(\lambda - A) z_0 - L e_\lambda z_0 + B e_\lambda u = z, \quad \text{or} \quad \Delta(\lambda) z_0 + B e_\lambda u = z.
\]

Hence, the condition (35) holds. This completes the proof. \(\blacksquare\)

B. A Necessary and Sufficient Condition

**Theorem 8:** Assume the conditions (A1) and (A2) are satisfied and the control space is finite dimensional. Moreover, assume that \(T(t)\) is compact for \(t > 0\). The delay system (2) is feedback stabilizable if and only if
\[
\text{Im}\Delta(\lambda) + \text{Im}(B e_\lambda) = X
\]
holds for all \(\lambda \in \sigma(\mathcal{A}_L)\) with \(\text{Re}\lambda \geq 0\).

**Proof:** The necessity has been proved in Theorem 7. Only the sufficiency will be shown here. At first, we will transform the system (2) into an equivalent system different from (29).

For the system (2), take \(x(\cdot) : [0, \infty) \to X\) and \(u(\cdot) : [0, \infty) \to U = \mathbb{C}^m\), respectively, and introduce the Banach space
\[
X = X \times U \quad \text{with norm} \quad \|x\| = \|x\|_X + \|u\|_U
\]
and the operator
\[
\mathcal{A} = \begin{pmatrix}
  A & 0 \\
  0 & 0
\end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times U
\]
which generates the \(C_0\)-semigroup
\[
\mathcal{T}(t) = \begin{pmatrix}
  T(t) & 0 \\
  0 & I_U
\end{pmatrix}, \quad t \geq 0
\]
on \(X\). Since \(T(t)\) is compact for \(t > 0\) and \(I_{\mathbb{C}^m}\) is compact, \(\mathcal{T}(t)\) is compact for \(t > 0\). Define
\[
\mathbb{L} = \begin{pmatrix}
  L & B \\
  0 & 0
\end{pmatrix} : W^{1,p}([-r, 0], \mathbb{X}) \to \mathbb{X}
\]
and \(w = (x, u) : [-r, \infty) \to \mathbb{X}\), and \(v = \hat{u}\)
where \(w(\cdot)\) is assumed to be smooth. Then the system (2) is converted into
\[
\begin{cases}
  \dot{x}(t) = \mathcal{A} w(t) + L u(t) + \mathbb{B} v(t), & t \geq 0 \\
  w(t) = (\varphi(t), \psi(t))^T, & a.e. \ t \in [-r, 0]
\end{cases}
\]
(43)
Here the new control operator \(\mathbb{B} : \mathbb{C}^m \to \mathbb{X}\) is defined by \(\mathbb{B} v = (0, v)^T\). Denote
\[
\mathbb{X} := X \times L^p([-r, 0], \mathbb{X})
\]
and define
\[
\mathcal{Q}_L = \begin{pmatrix}
  \mathcal{A} & \mathbb{L} \\
  0 & Q_m
\end{pmatrix}
\]
\[
\mathcal{D}(\mathcal{Q}_L) := \{(z, u, \varphi, \psi)^T \in \mathcal{D}(\mathcal{A}) \times W^{1,p}([-r, 0], \mathbb{X}) : \varphi(0) = z, \psi(0) = u\}
with
\[ Q_m \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \]
\[ \mathcal{D}(Q_m) = W^{1,p}([-r,0], X) \times W^{1,p}([-r,0], U). \]

Then, the operator
\[ Q \left( \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right) = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad \mathcal{D}(Q) = \mathcal{D}(Q_X) \times \mathcal{D}(Q_U) \]
is the generator of the following semigroup on \( L^p([-r,0], X) \):
\[ S_X(t) = \begin{pmatrix} S_X(t) & 0 \\ 0 & S_U(t) \end{pmatrix}, \quad t \geq 0. \]

Note that \( Q = Q_m \) on \( \mathcal{D}(Q) \). In particular, the assumptions (A1) and (A2) show that \( L \in C^0_c(Q). \) Hence \( \mathfrak{A}_L \) generates a strongly continuous semigroup on \( X \), according to [2, p.67], [9].

Define
\[ \mathfrak{B} := \begin{pmatrix} B \\ 0 \end{pmatrix} : U \to X \]
which is bounded, then the delay system (43), and hence (2), is equivalent to the open loop system
\[ \begin{cases} \dot{z}(t) = \mathfrak{A}_L z(t) + \mathfrak{B} u(t), & t \geq 0 \\ z(0) = (x, u(0), \varphi, \psi) \end{cases} \]
where the state \( z \) is defined as \( z = (w, w_\psi)^T = (x, u, x_t, u_t)^T \).

We have now converted the feedback stabilizability of the system (43) or system (2) to that of the above system, where \( \mathfrak{B} \) is linear bounded.

Now let us characterize the spectrum of \( \mathfrak{A}_L \). According to [2, p.56], we have
\[ \lambda \in \sigma(\mathfrak{A}_L) \iff \lambda \in \sigma(A + LE \lambda) \]

with
\[ E \lambda = \begin{pmatrix} e_\lambda & 0 \\ 0 & e_\lambda \end{pmatrix} \quad \text{and} \quad A + LE \lambda = \begin{pmatrix} A + Le_\lambda & Be_\lambda \\ 0 & 0 \end{pmatrix}. \]

Hence, by (34) we obtain
\[ \sigma(\mathfrak{A}_L) = \sigma(A_L) \cup \{0\}. \]

By assumption that \( T(t) \) is compact for \( t > 0 \), \( A_L \) generates an eventually compact semigroup on \( X \), according to Remark 5.

As a result, the following unstable set is finite:
\[ \sigma^+ := \{ \lambda \in \sigma(\mathfrak{A}_L) : \Re \lambda \geq 0 \}. \]

Take any \( (x, u)^T \in X \) and let \( \lambda \in \sigma^+ \). The condition (42) implies the existence of \( z \in X \) and \( v \in \mathbb{C}^m \) such that \( (\lambda - A)z - LEz + Be \lambda v = x \), which yields
\[ (\lambda - A) \begin{pmatrix} z \\ v \end{pmatrix} - L \begin{pmatrix} e_\lambda z \\ e_\lambda v \end{pmatrix} = \begin{pmatrix} x \\ u - \lambda v \end{pmatrix}. \]

Furthermore, take any \( \varphi \) and \( \psi \) such that \( (x, u, \varphi, \psi)^T \in \mathfrak{X} \) and define
\[ \varphi_0 = e_\lambda z + R(\lambda, Q_X) \varphi \quad \text{and} \quad \psi_0 = e_\lambda v + R(\lambda, Q_U) \psi \]
then
\[ (\lambda - Q_m) \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix}. \]

Substitute (46), after rearrangement, into (45), then
\[ (\lambda - A) \begin{pmatrix} z \\ v \end{pmatrix} - L \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} + \begin{pmatrix} 0 \\ u - \lambda v \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix} - LR(\lambda, Q) \begin{pmatrix} \varphi \\ \psi \end{pmatrix}. \]

Putting the last two identities together, we have
\[ (\lambda - A) (z, v, \varphi_0, \psi_0)^T + \mathfrak{B} u_0 \]
\[ = \begin{pmatrix} I & -LR(\lambda, Q) \\ 0 & I \end{pmatrix} (x, u, \varphi, \psi)^T \]
where \( u_0 = u - \lambda v \). This is equivalent to
\[ \text{Im}(\lambda - A) + \text{Im}(\mathfrak{B}) = \mathfrak{X}, \quad \lambda \in \sigma^+ \]
which means that the system (44), or system (2), is feedback stabilizable, according to [5, Theorem 1]. This ends the proof. \( \blacksquare \)

Remark 9: Denote by \( X^* \) the adjoint space of \( X \). Then the adjoint of the characteristic operator defined in (33) is given by
\[ \Delta(\lambda)^* = \overline{X} - A^* - (Le_\lambda)^* \]
where \( \overline{X} \) is the complex conjugate of \( X \). Using Theorem 7, [25, Prop.5.1] and the proof of Theorem 8 one can see that
\[ \text{Im} \Delta(\lambda) + \text{Im} (Be_\lambda) = X \iff \text{Ker} \Pi(\lambda) \cap \text{Ker} \mathfrak{B}^* = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \]

where
\[ \Pi(\lambda) := (\overline{X} - A^*) - (Le_\lambda)^* \]
A simple computation shows that
\[ \Pi(\lambda) = \begin{pmatrix} (\overline{X} - A^*) - (Le_\lambda)^* & 0 \\ -(Be_\lambda)^* & 0 \end{pmatrix} = \begin{pmatrix} \Delta(\lambda)^* & 0 \\ -(Be_\lambda)^* & 0 \end{pmatrix} \]
and
\[ \text{Ker} \Pi(\lambda) = [\text{Ker} \Delta(\lambda)^* \cap \text{Ker} (Be_\lambda)^*] \times \mathbb{C}^m. \]

Since
\[ \mathfrak{B}^* = [0 \ I], \quad \mathcal{D}(\mathfrak{B}^*) = X \times \mathbb{C}^m \]
we have
\[ \text{Ker} \mathfrak{B}^* = X \times \{0\}. \]
Combining (49), (51) and (48), the following holds under the conditions of Theorem 8:

\[ \text{Im}\Delta(\lambda) + \text{Im}(B\lambda) = X \iff \text{Ker}\Delta(\lambda)^* \cap \text{Ker}(B\lambda)^* = \{0\} \]

(52)

\[ \text{Rank}B\lambda_i = d_i, \quad i = 1, 2, \ldots, l \]  

(56)

with

\[
B\lambda_i = \begin{pmatrix}
\langle B_1 e_{\lambda 1}, \varphi_1 \rangle & \langle B_1 e_{\lambda 1}, \varphi_2 \rangle & \cdots & \langle B_1 e_{\lambda 1}, \varphi_{d_i} \rangle \\
\langle B_2 e_{\lambda 1}, \varphi_1 \rangle & \langle B_2 e_{\lambda 1}, \varphi_2 \rangle & \cdots & \langle B_2 e_{\lambda 1}, \varphi_{d_i} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle B_m e_{\lambda 1}, \varphi_1 \rangle & \langle B_m e_{\lambda 1}, \varphi_2 \rangle & \cdots & \langle B_m e_{\lambda 1}, \varphi_{d_i} \rangle
\end{pmatrix}
\]

(57)

for \( \lambda_i, \ i = 1, 2, \ldots, l \), given in (54).

\[ \langle B\lambda u, x \rangle = \sum_{k=1}^{m} \langle B_k e_{\lambda k} u_k, x \rangle = \sum_{k=1}^{m} \pi_k \langle B_k e_{\lambda 1}, x \rangle \]

we have

\[ \langle B\lambda^* x \rangle = \begin{pmatrix}
\langle B_1 e_{\lambda 1}, x \rangle \\
\langle B_2 e_{\lambda 1}, x \rangle \\
\vdots \\
\langle B_m e_{\lambda 1}, x \rangle
\end{pmatrix}, \quad x \in X^* \]

and, as a result,

\[ \text{Ker}(B\lambda)^* = \left\{ B_k e_{\lambda 1} : k = 1, 2, \ldots, m \right\}^\perp \]

(58)

Now, assume that the condition (56) does not hold. This means that there exists \( v = (v_1, \ldots, v_{d_i}) \in \mathbb{C}^{d_i} \setminus \{0\} \) such that

\[ B\lambda_i v = 0, \text{ i.e.,} \]

\[ \begin{pmatrix}
\sum_{j=1}^{d_i} v_j \langle B_1 e_{\lambda 1}, \varphi_j \rangle \\
\sum_{j=1}^{d_i} v_j \langle B_2 e_{\lambda 1}, \varphi_j \rangle \\
\vdots \\
\sum_{j=1}^{d_i} v_j \langle B_m e_{\lambda 1}, \varphi_j \rangle
\end{pmatrix} = \begin{pmatrix} 0 \\
0 \\
\vdots \\
0
\end{pmatrix}. \]  

(59)

Now since \((\varphi_1, \varphi_2, \ldots, \varphi_{d_i})\) is a basis of \( \text{Ker}\Delta(\lambda_i)^* \), we have

\[ \sum_{j=1}^{d_i} v_j \varphi_j^* \in \text{Ker}(\Delta(\lambda_i)^*). \]

On the other hand, by using (58) and (59) one can obtain

\[ \sum_{j=1}^{d_i} v_j \varphi_j^* \in \text{Ker}(\Delta(\lambda_i)^*). \]  

This is a contradiction, which shows that the equivalence (52) implies the rank conditions (56). The converse can be obtained in a direct way using (58) and the basis \( \{\varphi_j^* : j = 1, 2, \ldots, d_i\} \). This ends the proof.
D. A Rank Condition for Delay Systems With
Finite-Dimensional State (When Delay-Free) and Spaces

Consider the delay system

\[ \dot{x}(t) = Ax(t) + A_1x(t-r) + Pu(t) + P_1u(t-r) \]  

(60)

where \( A \) and \( A_1 \) are \( n \times n \) real matrices, \( P = (p_{ij})_{k,j} \) and \( P_1 = (p_{ij}^1)_{i,j} \) are \( n \times m \) matrices, and \( r > 0 \) is the delay. Here, \( L\varphi = A_1\varphi(-r) \) and \( B\psi = P\psi(0) + P_1\psi(-r) : W^{1,\varphi}([-r,0],\mathbb{R}^n) \to \mathbb{R}^n \). Let the control function \( u(t) = (u_1(t), u_2(t), \ldots, u_m(t))^T \) for \( t \geq -r \). We have

\[ Bu_t = \sum_{j=1}^m B_j(\psi_j(0)) \]

with

\[ B_j\psi = \left( \begin{array}{c} p_{1j}\psi(0) + p_{i1}^1\psi(-r) \\ p_{2j}\psi(0) + p_{i2}^1\psi(-r) \\ \vdots \\ p_{nj}\psi(0) + p_{ni}^1\psi(-r) \end{array} \right) \]

We note that the operator \( L \) and \( B \) satisfy the conditions (A1) and (A2), see [13, Remark 2]. Since \( A \) is a finite dimensional matrix, \( T(t) = e^{tA} \) is compact for \( t > 0 \). In this case,

\[ \Delta(\lambda) = \lambda I - A - e^{-\tau \lambda}A_1, \quad \lambda \in \mathbb{C} \]

with \( \sigma^+ = \{ \lambda_1, \lambda_2, \ldots, \lambda_l \} = \{ \lambda \in \mathbb{C} : \det \Delta(\lambda) = 0 \text{ and } \Re \lambda \geq 0 \} \). The dimension of \( \text{Ker} \Delta(\lambda)^* \) is \( d_i \) for \( i = 1, 2, \ldots, l \) and the basis of \( \text{Ker} \Delta(\lambda)^* \) is \( \varphi_1^i, \varphi_2^i, \ldots, \varphi_{d_i}^i \). Denote the \( n \times d_i \) matrix formed by the basis as

\[ \varphi^i = (\varphi_1^i, \varphi_2^i, \ldots, \varphi_{d_i}^i), \quad i = 1, 2, \ldots, l. \]

According to Theorem 10, we have the following:

**Corollary 1:** The system (60) is feedback stabilizable if and only if

\[ \text{Rank}[(P + e^{-\tau \lambda}P_1)^* \cdot \varphi^i] = d_i, \quad i = 1, 2, \ldots, l. \]  

(61)

As this condition is obtained from (42), which is an extension of (1), it should be equivalent to the condition (1) proposed in [26]. This is indeed true. In fact, we have

\[ \left( \begin{array}{c} \Delta(\lambda)^* \\ (P + e^{-\tau \lambda}P_1)^* \end{array} \right) \cdot \varphi^i = \left( \begin{array}{c} 0 \\ (P + e^{-\tau \lambda}P_1)^* \end{array} \right) \cdot \varphi^i. \]  

(62)

Assume that for each \( i = 1, 2, \ldots, l \),

\[ \text{Rank} \left[ \Delta(\lambda)^* P + e^{-\tau \lambda}P_1 \right] = n \]

(63)

which implies that \( \left( \begin{array}{c} \Delta(\lambda)^* \\ (P + e^{-\tau \lambda}P_1)^* \end{array} \right) \) has full column rank and

\[ \text{Rank}[(P + e^{-\tau \lambda}P_1)^* \cdot \varphi^i] = \text{Rank}[\varphi^i] = d_i. \]

On the other hand, without loss of generality (otherwise carrying out some elementary operations), assume that

\[ \Delta(\lambda)^* = \left( \begin{array}{cc} I & \lambda_i \\ 0 & 0_d \end{array} \right) \]

which gives the following \( \varphi^i \) having rank \( d_i \):

\[ \varphi^i = \left( \begin{array}{c} -\lambda_i \\ I_{d_i} \end{array} \right). \]

Now with the partition of \( (P + e^{-\tau \lambda}P_1)^* = (E_1 \quad E_2) \) compatible with \( \Delta(\lambda)^* \), we have (62) as follows:

\[ \left( \begin{array}{cc} I & \lambda_i \\ 0 & 0_d \end{array} \right) \left( \begin{array}{c} -\lambda_i \\ I_{d_i} \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ E_2 - E_1\lambda_i \end{array} \right). \]

Since (61) holds, \( E_2 - E_1\lambda_i \) has full column rank \( d_i \). Multiply the above with \( \left( \begin{array}{cc} 1 & 0 \\ 0 & I \end{array} \right) \) from the left, then

\[ \left( \begin{array}{cc} I & \lambda_i \\ 0 & 0_d \end{array} \right) \left( \begin{array}{c} -\lambda_i \\ I_{d_i} \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ E_2 - E_1\lambda_i \end{array} \right). \]

Since \( E_2 - E_1\lambda_i \) has full column rank \( d_i \), the leftmost big matrix in the above identity has full column rank as well. This means that \( (\Delta(\lambda)^* (P + e^{-\tau \lambda}P_1)^*)^* \) has full column rank. In other words, the condition (63) holds.

V. EXAMPLES

We start from giving two examples which illustrate the result of Corollary 1. In the third example we treat the case of systems with multiple delays. We end this section by an example of distributed delay systems governed by an uniformly elliptic differential operator of second order on a bounded domain of the Euclidean space.

**Example 1:** Consider the system (60) with

\[ A = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad A_1 = 0, \quad P = \left( \begin{array}{c} p_{11} \\ p_{21} \end{array} \right), \quad P_1 = \left( \begin{array}{c} p_{11}^1 \\ p_{21}^1 \end{array} \right). \]

Then, \( \Delta(\lambda) = \lambda I - A, \sigma(A) = \{-1, 1\} \) and \( \sigma^+ = \{1\} \). Hence

\[ \text{Ker} \Delta(1)^* = \text{span}\{ \varphi_1^1 \} \quad \text{with} \quad \varphi_1^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \]

which implies that \( d_1 = \text{dim} \text{Ker} \Delta(1)^* = 1 \). Now (61) is reduced to

\[ \text{Rank} \left( \begin{array}{c} p_{11} + e^{-\tau}p_{11}^1 \\ p_{21} + e^{-\tau}p_{21}^1 \end{array} \right) * \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \text{Rank}(2p_{11} + p_{21} + e^{-\tau}(2p_{11}^1 + p_{21}^1)) = 1. \]

Hence, the delay system is feedback stabilizable if and only if \( 2p_{11} + p_{21} + e^{-\tau}(2p_{11}^1 + p_{21}^1) \neq 0 \). In other words, the system is not stabilizable if

\[ r = \ln \left( -2p_{11}^1 + p_{21}^1 \right) \left( 2p_{11} + p_{21} \right). \]
Example 2: Consider the system (60) with
\[
A = \begin{pmatrix} 0 & -r \\ e^{-r} & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P_1 = 0.
\]
This was studied in [26] (with a typo in A). Here
\[
\Delta(\lambda) = \begin{pmatrix} \lambda & 0 \\ e^{-r} + e^{-\lambda r} & \lambda - 1 \end{pmatrix}, \quad \sigma^+ = \{0, 1\}, \quad \text{and}
\]
\[
\ker \Delta(0)^* = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \quad \ker \Delta(1)^* = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.
\]
Now we have
\[
\text{Rank} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 1 \quad \text{and} \quad \text{Rank} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 0.
\]
Thus, \( \lambda_1 = 0 \) is stabilizable but \( \lambda_2 = 1 \) is not stabilizable.

Example 3: Consider the state-input delay system
\[
\dot{x}(t) = Ax(t) + \sum_{k=1}^s L_k x(t - r_k) + \sum_{k=1}^s B_k u(t - r_k), \quad t \geq 0
\]
\[
x(0) = z, \quad x(t) = \varphi(t), \quad u(t) = \psi(t), \quad \text{a.e.} \ t \in [-r, 0]
\]
for \( r = \{r_1, r_2, \ldots, r_s\} \). Here the operator \( A \) generates an immediately compact semigroup on a reflexive Banach space \( X \), the family \( \{L_k\}_{1 \leq k \leq s} \subset \mathcal{L}(X) \), and for each \( k = 1, 2, \ldots, s \),
\[
B_k(u_1, u_2, \ldots, u_m)^\top := (B_k^1(u_1), B_k^2(u_2), \ldots, B_k^m(u_m))^\top
\]
where \( B_k^i : C \to X \) is linear and continuous for any \( i = 1, 2, \ldots, m \). As a result, \( B_k \in \mathcal{L}(C^m, X) \). We assume that \( 0 < r_1 < r_2 < \cdots < r_s = r \) and the control \( u \) takes values in \( C^m \). Here,
\[
L \varphi = \sum_{k=1}^s L_k \varphi(-r_k) \quad \text{and} \quad B \psi = \sum_{k=1}^s B_k \psi(-r_k)
\]
for all \( \varphi \in W^{1,p}([-r, 0], X) \) and \( \psi \in W^{1,p}([-r, 0], C^m) \), which satisfy the conditions (A1) and (A2), respectively, by [13, Theorem 3, Remark 2]. In this example,
\[
\Delta(\lambda) = \lambda - A - \sum_{k=1}^s e^{-\lambda r_k} L_k, \quad \lambda \in \mathbb{C}
\]
with \( \sigma^* = \{0, r_1, r_2, \ldots, r_s\} = \{0, r_1, r_2, \ldots, r_s\} \). The dimension of \( \ker \Delta(\lambda) \) is \( d_i \) for each \( i = 1, 2, \ldots, l \) and the basis of \( \ker \Delta(\lambda)^* \) is \( \varphi_1^*, \varphi_2^*, \ldots, \varphi_{d_i}^* \).

Example 4: Let \( \Omega \) be a bounded open set of \( \mathbb{R}^d \) with \( C^2 \) boundary \( \partial \Omega \), and with unit normal \( n(x) \). We consider the elliptic operator
\[
A(\partial, x) := \text{div}(a(x) \nabla \cdot) + (b(x), \nabla \cdot) + c(x), \quad x \in \Omega
\]
where \( a(x) = [a_{ij}(x)] \in \mathbb{R}^{d \times d} \) with \( a_{ij} = a_{ji} \in C^1(\mathbb{R}^d, \mathbb{R}^{d \times d}) \) and \( a(x) \geq \kappa I \) for some \( \kappa > 0 \) and all \( x \in \Omega \), \( b(x) = (b_1(x), b_2(x), \ldots, b_d(x))^\top \) with \( b_i \in C^1(\Omega) \). Then \( \kappa \) is essentially bounded on \( \Omega \), i.e., \( c \in L^\infty(\Omega) \). We consider the partial differential equation with source and input delays
\[
\partial_t u(t, x) - \Delta \varphi(t, x) + b(x) \partial_t u(t, x) + \int_0^t d\mu(\theta, x) u(t + \theta, x)
\]
\[
= m \int_0^t d\nu(\theta, x) u(t + \theta, x), \quad t > 0, \quad x \in \Omega
\]
Here we assume that \( \mu \) is a bounded variation function on \( \theta \) and for all \( f \in L^2(\Omega) \) it satisfies
\[
\sup_{\theta \in \mathbb{R}, \theta \neq 0} \int_\Omega \| \mu(\theta, y) f(y) \|^2 \, dy \leq |b| \|
\]
for some constant \( \beta > 0 \). Similarly, we assume that \( \nu_k \) is the bounded variations function of \( \theta \) for each \( k = 1, 2, \ldots, m \), and satisfies
\[
\sup_{\theta \in \mathbb{R}, \theta \neq 0} \int_\Omega \| \nu_k(\theta, y, z) \|^2 \, dy \leq |b| \|
\]
for any \( z \in \mathbb{C} \) and a constant \( \kappa_k > 0 \). On the other hand, we assume that the initial state \( g \in L^2(\partial \Omega) \) and the history functions \( \varphi \in L^2([\lambda_1, 0], L^2(\Omega)) \) and \( \psi \in L^2([\lambda_i, 0], C^m) \), for \( k = 1, 2, \ldots, m \), the control function \( u = (u_1, u_2, \ldots, u_m)^\top \in L^2(\mathbb{R}^d) \). We now introduce the following realization operator
\[
A_g := \text{div}(\partial, \cdot) g
\]
\[
D(A_g) := \{ g \in W^{2,2}(\Omega) : \partial \partial_n g(x) + \xi(x) g(x) = 0, \ x \in \partial \Omega \}. \]
and \( Bu_t = (B_1 B_2 \cdots B_m) \begin{pmatrix} (u_{1k})_t \\ (u_{2k})_t \\ \vdots \\ (u_{mk})_t \end{pmatrix} \)

where \( B_{i\psi} = \int_{-r}^{0} dp_k(\theta, \psi) \)

for \( \psi \in W_{1,2}([-r, 0], \mathbb{C}) \) and \( i = 1, 2, \ldots, m \). By [13, Theorem 3] the operators \( L \) and \( B \) satisfy the conditions (A1) and (A2). In this example,

\[ \Delta(\lambda) = \lambda - A - \int_{-r}^{0} d\mu(\theta, \psi)e^{\lambda \theta}, \quad \lambda \in \mathbb{C} \]

with \( \sigma^+ = \{\lambda_1, \lambda_2, \ldots, \lambda_l\} = \{\lambda \in \mathbb{C} : \Delta(\lambda) \) is not invertible and \( \Re \lambda \geq 0\}. The dimension of \( \text{Ker} \Delta(\lambda) \) is \( d_i \) for \( i = 1, 2, \ldots, l \) and the basis of \( \text{Ker} \Delta(\lambda) \) is \( \{\varphi_1, \varphi_2, \ldots, \varphi_k\} \).

According to Theorem 10, the system (65) is feedback stabilizable if and only if, for each \( i = 1, 2, \ldots, l \), see the equation shown at the bottom of the page.

VI. CONCLUSION

In this paper, we have presented an analytic approach to address the feedback stabilizability of state-input delay systems in Banach spaces. Based on the transformation of the delay system into a delay-free open loop system, we have introduced a large class of feedback operators that stabilize the delay system. Making use of the recent results on infinite dimensional well-posed and regular linear systems, we have extended the well-known necessary condition for the feedback stabilizability of state-input delay systems when the delay-free state space is finite dimensional to the infinite dimensional case. To make it sufficient we have introduced extra conditions that the initial delay-free system is governed by an immediately compact semigroup and that the control space is finite dimensional. This shows that the state-delay equation is governed by an immediately compact semigroup on a product space, which allows us to use the well-known results on feedback stabilizability of distributed parameter linear systems. Moreover, when the state space is reflexive, we have presented a rank condition for the feedback stabilizability of the state-input delay system in terms of control operators and eigenvectors. Some examples are given to show the applications.

APPENDIX A

ADMISSIBILITY OF OBSERVATION OPERATORS

FOR PERTURBED SEMIGROUPS

Here, we recall some results on admissibility of observation operators for perturbed semigroups. Let \( Z, Y \) be two Banach spaces, \( (V(t))_{t \geq 0} \) be a \( C_0 \)-semigroup on \( Z \) with generator \( G : \mathcal{D}(G) \subset Z \rightarrow Z \), and \( p > 1 \) be a real number.

An operator \( P \in \mathcal{L}(\mathcal{D}(G), Z) \) is called a Miyadera-Voigt perturbation for \( G \) if

\[ \int_{0}^{\alpha} \| P \mathcal{V}(\tau)z \| d\tau \leq \gamma \| z \| \quad (66) \]

for all \( z \in \mathcal{D}(G) \) and some constants \( \alpha > 0, 0 < \gamma < 1 \). It is known that if \( P \) satisfies the estimate (66) then the operator \( G + P \) with domain \( \mathcal{D}(G + P) := \mathcal{D}(G) \)

generates a \( C_0 \)-semigroup on \( Z \), see [8, p.196]. Now if we use the notation (9) then by Hölder inequality it is clear that every \( P \in \mathcal{O}_2^p(G) \) satisfies (66), and then \( G + P \) is a generator on \( Z \).

The following theorem, taken from [12], shows the invariance of admissibility of observation operators.

**Theorem 11**: Let \( P \in \mathcal{O}_2^p(G) \) then \( G + P \) generates a \( C_0 \)-semigroup on \( Z \) and

\[ \mathcal{O}_2^p(G) = \mathcal{O}_2^p(G + P). \quad (67) \]

APPENDIX B

GENERALIZED HAUTUS CRITERION

Here, we recall from [38] the generalized Hautus criterion. Let \( G \) be the generator of a strongly continuous semigroup on a Banach space \( Z, U \) be another Banach space, and \( B \in \mathcal{L}(U, \mathbb{C}_{-1}) \). The open-loop system defined by \( G \) and \( B \) is denoted by \( (G, B) \). Let \( \hat{C} \) be the Yosida extension of \( C \in \mathcal{L}(\mathcal{D}(G), U) \). The operator \( G_{-1} + B \hat{C} \) is endowed with its natural domain defined by \( \mathcal{D}(G_{-1} + B \hat{C}) = \{ z \in \mathcal{D}(\hat{C}) : (G_{-1} + B \hat{C})z \in Z \} \).

The following definition is taken from [38, Def.2.1].

**Definition 12**: \((G, B, \hat{C})\) is feedback stabilizable if there exists \( C \in \mathcal{L}(\mathcal{D}(G), U) \) such that

i) \((G, B, \hat{C})\) generates a regular system \( \Sigma \) on \( Z, U, U \);

ii) the identity operator \( I_U : U \rightarrow U \) is an admissible feedback operator for \( \Sigma \);
iii) \( G_{-1} + B \mathcal{C} \) generates an exponentially stable semigroup on \( Z \).

In this case, we say that \( \mathcal{C} \) stabilizes \((G,B)\).

We denote by \( \text{Ran}[\lambda - G_{-1} - B] \) the subspace of \( Z_{-1} \) which consists of all vectors of the form \((\lambda - G_{-1})z + Bu\), where \( z \in Z \) and \( u \in U \).

The following proposition, taken from [38, Prop.3.5] (combined with [38, Remark 1]), is an extension of the Hautus criterion for stabilizability of infinite-dimensional systems.

**Proposition 4:** If \((G,B)\) is feedback stabilizable, then there exists a \( \delta > 0 \) such that

\[
\text{Ran}[\lambda - G_{-1} - B] \supset Z
\]

for all \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) < -\delta \).

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