A less conservative stability test for second-order linear
time-varying vector differential equations

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System stability and stability bounds play an essential role in control theory. This note is concerned with the exponential stability of a class of second-order linear time-varying vector differential equations with real piecewise continuous coefficient matrices. A less conservative explicit condition for stability of such a system is derived using the matrix measure theory and a more accurate upper bound for the decay exponent of its stable solution is established. Examples are included for illustration.

1. Introduction

Stability of stationary solutions of dynamic systems constitutes a very important topic in control theory and applications, which has recently attracted great research interest (Kaszkurewicz and Bhaya 1993, Aeyeles and Peuteman 1999, Liu and Molchanov 2002, Goeleven and Brogliato 2004, Gil’ 2004, Neresov and Haddad 2006). A fundamental method for stability analysis of continuous systems is the direct Lyapunov method and it has produced many strong results. However, to find a Lyapunov function often encounters serious mathematical difficulties. Because of the absence of complete solutions of linear time-varying systems, their stability analysis remains as one of the most difficult problems of control theory so far.

Let us consider the following linear time-varying system:

\[ \dot{x}(t) + 2A(t)x(t) + B(t)x = 0, \quad (1) \]

where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n \), \( A(t) \) and \( B(t) \) are real piecewise continuous \( n \times n \) matrices. Such system often arise in control theory (Harris and Miles 1980, Rugh 1996), for example, non-autonomous mechanical systems and flexible rotor system etc. System (1) is called exponentially stable at zero if the general exponent of the solution is negative. Note that the exponential stability is stronger than the asymptotic one and implies, in particular, system stability under persistent perturbations. It is noted that there are many explicit stability conditions for the second-order vector differential equations (see (Harris and Miles 1980, Rugh 1996, Tasso and Throumoulopoulos 2000, Inou et al. 2003, Inoue and Kato 2004) and references therein). Most of them however assume that the coefficient matrices are differentiable and slowly varying or dissipative. Recently, exponential stability criteria were obtained without the above assumption by using inequality techniques Zevin and Pinsky (2003) and Gil’ (2005). In particular, the explicit stability conditions were established in Gil’ (2005). In this note, the results of Gil’ (2005) are improved with help of the matrix measure theory. A less conservative explicit condition for exponential stability of (1) is derived, and a more accurate upper bound for the decay exponent of its solution obtained.

2. Explicit stability criteria

Consider a real square matrix, \( P = (p_{ij}(t))_{n \times n} \). Let \( \|P\|_o \) denote a matrix norm which is the operator norm.
induced by the corresponding vector norm of $|x|_\omega$. The matrix measure induced from a given matrix norm $\|P\|_\omega$ is defined (Desoer and Vidyasagar 1975, Fang and Kinkaid 1996) as

$$\mu_\omega(P) = \lim_{h \to 0^+} \frac{\|I + hP(t)\|_\omega - 1}{h},$$

where $I$ is the identity matrix. Properties and calculations on matrix measure can be found in Desoer and Vidyasagar (1975) and Fang and Kinkaid (1996), from which the matrix measure corresponding to the commonly-used matrix norms are collected and listed in table 1. Note the last case in table 1, where $|\cdot|_\omega = \text{diag}\{|\omega_1, \omega_2, \ldots, \omega_n| > 0\}$ is assumed, and with a slight abuse of notation, the vector norm $|x|_\omega$ is defined as $|x|_\omega = |x|^{-1}_1$ (Fang and Kinkaid 1996). Matrix measure is applicable to any matrix, either time-invariant or time-varying, deterministic or stochastic. This is why matrix measure can be used to study the stability of linear time varying systems or stochastic systems. The key idea is that the estimation of the solution of linear systems can be obtained using the matrix measure technique.

**Theorem 1:** The system in (1) is exponentially stable if there exists a positive constant, $m$, such that

$$I + \sqrt{\dot{P} + 4\dot{c} - 2m} \leq 0,$$

where $I = \text{Sup}_{\omega \geq 0} \max[0, 2\mu_\omega(mI - A(t))]$, $\dot{c} = \text{Sup}_{t \geq 0} \|2mA(t) - m^2I - B(t)\|_\omega$. If this is the case, the solution $x(t)$ of system (1) satisfies

$$\|x(t)\|_\omega \leq M \exp(\lambda t),$$

where $\lambda$ is real with $\lambda \leq (I + \sqrt{\dot{P} + 4\dot{c} - 2m})/2 - m < 0$.

**Proof:** Let

$$x(t) = e^{-mt}y(t),$$

and substitute it to (1), then

$$\dot{y}(t) - 2(mI - A(t))\dot{y} - C(t)y = 0,$$

where $C(t) = 2mA(t) - m^2I - B(t)$. This equation can be re-written as

$$\begin{align*}
\dot{y}_1(t) &= 2(mI - A(t))y_1 + C(t)y_2, \\
\dot{y}_2(t) &= y_1.
\end{align*}$$

(3) 

(4)

Let $|\cdot|_\omega$ be a vector norm and $\|\cdot\|_\omega$ the matrix norm induced by this vector norm, and denote $(y^2_1(t), y^2_2(t))^T$ as a solution of (3) and (4). For $t \geq 0$, we have

$$\begin{align*}
\frac{d}{dt}|y_1(t)|_\omega - 2\mu_\omega(mI - A(t))|y_1(t)|_\omega - ||C(t)||_\omega|y_2(t)|_\omega
\end{align*}$$

$$= \frac{d}{dt}|y_1(t)|_\omega - \mu_\omega(2(mI - A(t))|y_1(t)|_\omega - ||C(t)||_\omega|y_2(t)|_\omega$$

$$= \lim_{h \to 0^+} \frac{|y_1(t+h)|_\omega - |y_1(t)|_\omega - ||I + 2h(mI - A(t))||_\omega|y_1(t)|_\omega}{h}$$

$$\leq \lim_{h \to 0^+} \frac{|y_1(t+h)|_\omega - |I + 2h(mI - A(t))|y_1(t)|_\omega}{h}$$

$$\leq \lim_{h \to 0^+} \frac{|y_1(t+h)|_\omega - |C(t)||y_2(t)|_\omega}{h}$$

$$\leq \lim_{h \to 0^+} \frac{|y_1(t+h) - |I + 2h(mI - A(t))|y_1(t)|_\omega}{h}$$

$$\leq \lim_{h \to 0^+} \frac{|y_1(t+h)|_\omega - |C(t)||y_2(t)|_\omega}{h}$$

$$\leq \lim_{h \to 0^+} \frac{|y_1(t+h) - |I + 2h(mI - A(t))|y_1(t)|_\omega}{h}$$

$$\leq \lim_{h \to 0^+} \frac{|y_1(t+h)|_\omega - |C(t)||y_2(t)|_\omega}{h}$$

$$= \lim_{h \to 0^+} \frac{|y_1(t+h) - |y_1(t)|_\omega - h[2(mI - A(t))y_1(t) + C(t)y_2(t)]}{h}.$$
which, after using (3), becomes
\[
\frac{d|y_1(t)|_0}{dt} \leq 2\mu_0(mI - A(t))|y_1(t)|_0 + \|C(t)|_0|y_2(t)|_0 \\
\leq \text{Sup}_{t \geq 0} \max[0, 2\mu_0(mI - A(t))]|y_1(t)|_0 \\
+ \|C(t)|_0|y_2(t)|_0 = l|y_1(t)|_0 + \|C(t)|_0|y_2(t)|_0.
\]
Similarly, we can obtain
\[
\frac{d|y_2(t)|_0}{dt} \leq |y_1(t)|_0.
\]
Hence, one sees that
\[
|y_i(t)|_0 \leq z_i(t), \quad i = 1, 2,
\]
where \((z_1(t), z_2(t))\) is the solution of the following comparison system,
\[
\begin{align*}
\frac{dz_1(t)}{dt} &= l z_1(t) + c z_2(t), \\ 
\frac{dz_2(t)}{dt} &= z_1(t),
\end{align*}
\]
with \(\|C(t)|_0 \leq c\). Simple calculations yield
\[
z_i(t) \leq M \exp \left( l + \frac{\sqrt{l^2 + 4c}}{2} t \right),
\]
for some \(M > 0\). The result follows from (2), (5) and the above equation. This completes the proof.

In comparison with the results in Gil’ (2005), Theorem 1 does not require the assumption of \(A_R \geq m I\), where \(A_R = \frac{1}{2}(A + A^T)\), \(m R\) is a positive constant. Furthermore, if \(mI - A_R(t) = mI - (A(t) + A(t)/2) \leq 0\), then there holds \(\mu_0(mI - A(t)) \leq 0\) and the results in Gil’ (2005) follow from Theorem 1. Theorem 1 may also give more accurate upper bounds for the decay exponent of the solution, see the examples below for more details.

3. Examples

Example 1: Consider
\[
\ddot{x}(t) + 2A(t)\dot{x}(t) + B(t)x(t) = 0,
\]
where
\[
A(t) = \begin{pmatrix} 20.4 & -25 \\ -16 & 20.4 \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} 4.074 & -4.998 \\ -3.203 & 4.068 \end{pmatrix}.
\]
It follows that
\[
A_R(t) = \frac{A^T(t) + A(t)}{2} = \begin{pmatrix} 20.4 & -20.5 \\ -20.5 & 20.4 \end{pmatrix},
\]
and there exists no \(m > 0\) such that \(A_R(t) \geq m I\). Hence the stability criteria in Gil’ (2005) are not applicable. Taking \(m = 0.1\), \(\omega_1 = 4\) and \(\omega_2 = 5\), we have \(\mu_0(mI - A(t)) = \mu_0(0.1I - A(t)) = -0.3\), \(l = 0\) and \(c = \text{Sup}_{t \geq 0}\|C(t)|_0\| = \text{Sup}_{t \geq 0}\|0.2A(t) - 0.01I - B(t)|_0\| = 0.0076 < 0.01 = m^2\). By Theorem 1, Theorem 1 is exponentially stable.

Example 2: Let
\[
\ddot{x}(t) + 2a(t)\dot{x}(t) + b(t)x(t) = 0,
\]
where \(a(t) = 2 + \frac{1}{2}\sin^2 t\), \(b(t) = 4 + \cos t\). Take \(m = \inf_{t \geq 0}\omega(t) = 2\). Then, we have \(2ma(t) - b(t) = 8 + 2\sin^2 t - 4 - \cos t = 4 + 2\sin^2 t - \cos t < 8 = 2m^2\). But it does not hold that \(4 + 2\sin^2 t - \cos t > 4 = m^2\) for \(t \geq 0\), and the stability criteria in Gil’ (2005) is not applicable. However, we have \(l = \text{Sup}_{t \geq 0}\max[0, 2\mu_2 \times (mI - A(t))] = 0\) and \(c = \text{Sup}_{t \geq 0}\|C(t)|_0 = \text{Sup}_{t \geq 0}\|2ma(t) - m^2I - B(t)|_0 = \text{Sup}_{t \geq 0}\|2(\sin^2 t - \cos t) < 4 = m^2\). By Theorem 1, (7) is exponentially stable.

Example 3: Consider
\[
\ddot{x}(t) + 2A(t)\dot{x}(t) + B(t)x(t) = 0,
\]
where
\[
A(t) = \begin{pmatrix} 2 & \frac{1}{2}\sin^2 t \\ \frac{1}{2}\cos^2 t & 3 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 2.7 & \sin^2 t - 0.1 \\ \frac{7}{2}\cos^2 t & 4.8 \end{pmatrix}.
\]
Let \(m = 1\). Then,
\[
\frac{1}{2}(A(t) + A^T(t)) \geq mI = I,
\]
\[
2mA(t) - B(t) = \begin{pmatrix} 1.3 & 0.1 \\ 0 & 1.2 \end{pmatrix} \geq m^2I = I,
\]
\[
\text{Sup}_{t \geq 0}\|2mA(t) - B(t)|_2 \approx \sqrt{1.746918} \approx 1.32171 < 2 = 2m^2.
\]
From the stability criteria in Gil’ (2005), (8) is exponentially stable with a decay exponent of 0.4328. Now, we use Theorem 1 to estimate this decay exponent again. One has
\[
\mu_1(mI - A(t)) = \mu_1 \begin{pmatrix} -1 & -\frac{1}{2}\sin^2 t \\ -\frac{1}{2}\cos^2 t & -2 \end{pmatrix} < 0, \quad l = 0
\]
and
\[
c = \text{Sup}_{t \geq 0}\|C(t)|_1 = \text{Sup}_{t \geq 0}\|2ma(t) - m^2I - b(t)|_1
\]
\[
= \left\| \begin{pmatrix} 0.3 & 0.1 \\ 0 & 0.2 \end{pmatrix} \right\| = 0.3 < 1 = m^2.
\]
By Theorem 1, equation (8) is exponentially stable with decay exponent of 0.4523. We can see that a more
accurate upper bound for the decay exponent of the system (8) is obtained by using Theorem 1.

4. Conclusions

In this note, a less conservative explicit condition of exponential stability for the second-order linear time-varying vector differential systems has been derived using the matrix measure, and a more accurate upper bound for the general decay exponent of the system is established. A possible further research is to extend this method to high-order system.

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References
