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In this paper, the stability in the sense of boundedness of inverters operated in parallel is proven and the fail-safe operation is achieved for generic linear or nonlinear loads described in the generalised dissipative Hamiltonian form. The robust droop controller (RDC), recently proposed in the literature for achieving accurate proportional power sharing and tight voltage regulation, is implemented in a nonlinear matrix form to achieve a bounded inverter voltage and inherit a fail-safe capability. Using nonlinear Lyapunov methods, it is proven that the RDC with fail-safe approximates the original RDC, generates a bounded inverter output voltage within the given technical limits and guarantees nonlinear closed-loop system stability in the sense of boundedness. When the prescribed limits are violated, e.g. due to sensor failure, the proposed method rapidly shuts down in a continuous-time manner, thus disconnecting the inverter to prevent a complete system failure even when the protection circuit fails. Extensive simulation results are presented to demonstrate this approach for two single-phase paralleled inverters feeding a linear and a nonlinear load under a sensor failure scenario.

Keywords: parallel operation of inverters; proportional load sharing; nonlinear systems; stability; fail-safe operation

1. Introduction

Microgrids consist of renewable energy sources, energy storage devices and local loads, and function as small power systems that can be operated autonomously or connected to a wide power network (Guerrero, Chandorkar, Lee, & Loh, 2013; Guerrero, Loh, Lee, & Chandorkar, 2013; Weiss, Zhong, Green, & Liang, 2004; Zhong & Hornik, 2013). The integration of renewable energy sources is accomplished via power electronic inverters which are often required to operate in parallel. Droop control techniques (Barklund, Pogaku, Prodanovic, Hernandez-Aramburo, & Green, 2008; Guerrero, Chandorkar, et al., 2013; Guerrero, Loh, et al., 2013) are often used to avoid circulating currents among the inverters and their operation is based on a decentralised manner without communication among the inverters (Chandorkar, Divan, & Adapa, 1993; Tuladhar, Jin, Unger, & Mauch, 1997).

It becomes obvious that accurate and proportional power sharing among the inverters operated in parallel is of major importance in a microgrid both in grid-connected and stand-alone modes. Particularly, in stand-alone systems, load sharing is a challenging task (Sao & Lehn, 2005) and inverters operated with the conventional droop control technique are able to share accurately and proportionally the load only in the case where all inverters introduce the same per-unit output impedance (Zhong, 2013; Zhong & Hornik, 2013). This is an inherent limitation of the conventional droop control, which has a static structure. To overcome this problem, several control designs have been proposed to achieve both real and reactive power sharing for parallel inverters (Guerrero et al., 2006; Guerrero, Matas, de Vicuna, Castilla, & Miret, 2007; Majumder, Chaudhuri, Ghosh, Ledwich, & Zare, 2010; Zhong, 2013). Among these techniques, the robust droop controller (RDC) proposed in Zhong (2013) introduces a dynamic structure and has been proven to achieve accurate load sharing and tight voltage regulation even if numerical computational errors, disturbances, noises, parameter drifts and/or component mismatches occur.

Although extensive research has been done in the field of power sharing providing powerful solutions, the stability properties of the proposed techniques have not been adequately exploited. Local stability results are often obtained using small-signal modelling and linearisation approaches (Coelho, Cortizo, & Garcia, 2002; Majumder et al., 2010; Marwali, Jung, & Keyhani, 2007; Mohamed & El-Saadany, 2008). Several conditions for achieving local exponential stability for frequency droop control have been exploited in Simpson-Porco, Dörfler, and Bullo (2013) assuming a purely inductive network and fixed or bounded voltage magnitudes for the conventional droop control. Due to the nonlinearities of the droop controller, which arise from the
calculation of the real and the reactive power, it becomes obvious that the nonlinear stability analysis is essential for investigating the behaviour of paralleled inverters. In this frame, \( L_\infty \) stability of the conventional droop control has been proven with additional asymptotic stability for the lossless microgrid case (Schiffer, Ortega, Astolfi, Raisch, & Sezi, 2014). The system modelling was based on the Kron-reduced network approach with the assumption of linear load description.

To the best of the authors’ knowledge, the nonlinear stability of the robust droop control, which achieves accurate load sharing independently from the type of the load (linear or non-linear), has not been proven yet. In this paper, the stability in the sense of boundedness of two single-phase parallel-operated inverters feeding a generic linear or non-linear load and operating under the RDC, as proposed in Zhong (2013), is investigated. Although only the case of two inverters is considered, the proposed method can be generalised for a system with a large number of paralleled inverters. The RDC is also proven to achieve output voltage regulation near the rated value, based on an integral function in the voltage droop technique, while the frequency droop method is based on the conventional scheme. As a result, the conditions for frequency synchronisation among the paralleled inverters still hold (Simpson-Porco et al., 2013) and, as noted in the present paper, the stability of the system in the sense of boundedness is directly related to the voltage droop. Due to the integral control (IC) structure of the RDC, which has a dynamic structure opposed to the static conventional voltage droop control, as well as the nonlinear functions involved, closed-loop stability in the sense of boundedness is very difficult to be proven.

It is widely known that the IC techniques are used to achieve regulation in cases of external disturbances and parameter variations, but they can guarantee stability under necessary and sufficient conditions of the plant and under low integral gain (Fliegner, Logemann, & Ryan, 2003, 2006; Morari, 1985; Mustafa, 1994). The application of the IC has been extended to non-minimum phase, nonlinear systems using output feedback control and high-gain observers (Isidori, 1997; Kay & Khalil, 2004; Khalil, 2000; Mahmoud & Khalil, 1996) for achieving semi-global results. These results were further extended in Jiang and Mareels (2001) where a robust integral controller was designed according to the relative degree of the nonlinear plant, while recently, conditional integrators were proposed in Singh and Khalil (2004) and Li and Khalil (2012), which provide the integral action inside a boundary layer and act as a stable system outside of it. However, the implementation of these methods requires knowledge of the plant (parameters, relative degree, etc.) that is not always available in practice. Saturation units are also used in order to guarantee the boundedness of the controller output, but this fact can be devastating for the system when external disturbances or actuator and sensor malfunctions occur (Morari, 1985), since they will lead to undesired oscillations. Therefore, it is desired to have the dynamic function of the IC with a bounded structure to guarantee stability and also operate in a way to ensure fail-safe operation of the system. That is, in the event of a failure, no harm is caused to the system.

In this paper, the closed-loop system stability in the sense of boundedness of two inverters operated in parallel is analysed by implementing the RDC dynamics in a nonlinear matrix form to approximate the behaviour of the original RDC and generate a bounded controller output (inverter output voltage) within the given technical limits. Moreover, the fail-safe operation is achieved. The two inverters are considered to feed a common load that is described in the generalised dissipative Hamiltonian form, which has been proven to represent the general case of a power-electronic driven dynamic system (Konstantopoulos & Alexandridis, 2013; Ortega, Loria, Nicklasson, & Sira-Ramirez, 1998). The control output stays within the required limits, guaranteeing closed-loop nonlinear stability in the sense of boundedness. In the event of a failure, when the inverter output voltage tries to violate the given requirements, the proposed method forces the controller states to suitably shut down and stop the inverter’s operation, thus achieving a fail-safe operation. This transition is proven in a continuous-time and smooth manner without requiring additional saturation units or switches. The proposed structure also prohibits any oscillating response leading to a possible convergence to the desired equilibrium or to the origin. Therefore, even if the protection circuit fails, the controller design can guarantee that the system will not lead to instability. According to the authors’ knowledge, this is the first time that closed-loop stability is proven while achieving accurate power sharing for inverters operated in parallel with a general nonlinear load, and at the same time that the fail-safe operation is achieved. This brings a significant step forward in microgrid operation, since fail-safe operation is achieved without any monitoring devices and no external communication is required, thus maintaining the decentralised structure of the droop control. In order to demonstrate this, simulation results are presented for two single-phase parallel-operated inverters when the voltage sensor of one inverter becomes faulty.

The paper is organised as follows. In Section 2, the problem is formulated by obtaining the dynamic model of the system consisting of two single-phase inverters and a load along with its properties and an overview of the RDC is presented. In Section 3, some preliminaries, including input-to-state (practical) stability and the generalised small-gain theorem are briefly presented. In Section 4, the nonlinear implementation of the RDC is proposed and applied to the parallel operation of inverters and its performance is investigated. In Section 5, the stability of the closed-loop system is proven using nonlinear analysis. In Section 6, extensive simulation results are provided to demonstrate the effectiveness of the proposed method with respect to the
original RDC and, finally, in Section 7, some conclusions are drawn.

2. Problem description

2.1 System modelling

Figure 1 represents the system under consideration consisting of two single-phase inverters connected in parallel to feed a common load. An LC filter is connected at the output of each inverter with $L_1$, $L_2$, and $C_1$, $C_2$ being the filter inductances and capacitances, respectively, for each inverter. In practice, each inductor and capacitor introduce parasitic resistances represented as $R_1$ and $R_2$ in series with the inductors (typically very small) and $r_{C1}$ and $r_{C2}$ in parallel with the capacitors (typically very large). Variables $v_{r1}$, $v_{r2}$ and $i_1$, $i_2$ are the inverter output voltages and currents, respectively, while $v_o$ and $i_L$ are the load voltage and current, respectively. It is often considered that the filter capacitors along with the parasitic resistances can be regarded as a part of the load, and therefore, $C_1$, $C_2$, $r_{C1}$ and $r_{C2}$ can represent some of the load characteristics as well (Zhong & Hornik, 2013).

The dynamic equations of the system are

$$L_1 \frac{di_1}{dt} = -R_1i_1 - v_o + v_{r1},$$

$$L_2 \frac{di_2}{dt} = -R_2i_2 - v_o + v_{r2},$$

$$(C_1 + C_2) \frac{dv_o}{dt} = i_1 + i_2 - \frac{r_{C1} + r_{C2}}{r_{C1}r_{C2}}v_o - i_L.$$  \hspace{1cm} (1)

In order to investigate a generic load case that includes both linear and nonlinear loads, the load can be described in the generalised dissipative Hamiltonian form (Konstantopoulos & Alexandridis, 2013; Ortega et al., 1998), which uses average analysis to represent loads that are fed by power electronic devices (power converters) in a nonlinear continuous-time structure. This representation covers the linear load case as well. Therefore, the load dynamics are given as

$$M\dot{q} = (J(q, \mu) - R)q + Gv_o,$$  \hspace{1cm} (2)

where $q = [i_L, q_1, q_2, ... q_{(m-1)}]^T \in \mathbb{R}^m$ represents the states of the load and $\mu$ is a bounded vector in a closed set which describes the duty-ratio signals of the converters. Matrix $M$ is constant and positive definite, $J$ is skew-symmetric, $R$ is constant and positive definite and $G = [1 \, 0_{1 \times (m-1)}]^T$. For the load equation (2), the load voltage $v_o$ can be considered as an input to the load system (in fact, this is usually the case when, for example, a voltage source device is connected at the inverter’s output). It should be also noted that all nonlinearities of the load and the bounded duty-ratio signals $\mu$ are restricted into the skew-symmetric matrix $J$. This is a common issue in power systems, especially for power-converter-fed loads (Karagiannis, Mendes, Astolfi, & Ortega, 2003; Konstantopoulos & Alexandridis, 2013; Ortega et al., 1998). As a result, combining (1) and (2), the complete plant system can be written into the generalised dissipative Hamiltonian form:

$$\hat{M}\ddot{x} = (\hat{J}(\dot{x}, \mu) - \hat{R})\dot{x} + \hat{G}u,$$  \hspace{1cm} (3)

where the state vector is $\dot{x} = [i_1 \, i_2 \, v_o \, q]^T$, the input vector is $u = [v_{r1} \, v_{r2}]^T$ and matrices $\hat{M}$, $\hat{J}$ and $\hat{R}$, as defined below, retain the properties already mentioned for $M$, $J$ and $R$, respectively, i.e.

$$\hat{M} = \begin{bmatrix} L_1 & 0 & 0 & 0_{1 \times m} \\ 0 & L_2 & 0 & 0_{1 \times m} \\ 0 & 0 & C_1 + C_2 & 0_{1 \times m} \\ 0_{m \times 1} & 0_{m \times 1} & 0_{m \times 1} & M \end{bmatrix},$$

$$\hat{J} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0_{1 \times (m-1)} \\ 0 & 0 & -1 & 0 & 0_{1 \times (m-1)} \\ 1 & 1 & 0 & -1 & 0_{1 \times (m-1)} \\ 0 & 0 & 1 & 0 & J_{12} \\ 0_{(m-1) \times 1} & 0_{(m-1) \times 1} & 0_{(m-1) \times 1} & 0_{(m-1) \times 1} & -J_{12}^T \end{bmatrix},$$

$$\hat{R} = \begin{bmatrix} R_1 & 0 & 0 & 0_{1 \times m} \\ 0 & R_2 & 0 & 0_{1 \times m} \\ 0 & 0 & r_{C1} + r_{C2} & 0_{1 \times m} \\ 0_{m \times 1} & 0_{m \times 1} & r_{C1r_{C2}} & 0_{m \times 1} \end{bmatrix},$$

$$\hat{G} = \begin{bmatrix} 1 \, 0_{1 \times (m-1)} \\ 0 \, 0_{1 \times (m-1)} \end{bmatrix},$$

where $J = \begin{bmatrix} 0 & J_{12} \\ -J_{12} & -J_{22} \end{bmatrix}$ with $J_{12}$ and $J_{22}$ being $1 \times (m-1)$ and $(m-1) \times (m-1)$ matrices, respectively.

It should be noted that the plant system (3) is a nonlinear bounded-input bounded-output (BIBO) stable system, where the output vector can be the whole state vector or a part of it.

2.2 Overview of the RDC

Although droop control is the most widely used technique for power sharing, conventional droop control techniques fail to achieve accurate power sharing when the inverters do not have the same per-unit output impedances (Zhong & Hornik, 2013). This issue can be addressed using the RDC technique proposed in Zhong (2013), which includes a dynamic voltage droop controller opposed to the
conventional one. The RDC for each inverter $i \in \{1, 2\}$, as shown in Figure 2, takes the following form:

\[
\dot{E}_i = K_e (E^* - V_o) - n_i Q_i \\
\dot{\theta}_i = \omega^* - m_i P_i,
\]

where $E_i$ and $\theta_i$ are the RMS value and the phase angle of the $i$th inverter output voltage, respectively; $E^*$ and $\omega^*$ are the rated voltage and angular frequency, respectively; $V_o$ represents the RMS voltage of the load and $P_i, Q_i$ are the real and reactive power delivered to the load by the $i$th inverter. Control parameters $K_e, n_i$ and $m_i$ are suitably selected by the desired voltage and frequency droop ratio (Zhong, 2013). Thus, the control input of the plant (inverter output voltage) is given in the following form:

\[
v_{oi} = \sqrt{2} E_i \sin(\theta_i).
\]

The controller and the dynamics are nonlinear, while the RMS load voltage is a nonlinear function of $v_o$, i.e.

\[
V_o (v_o) = \sqrt{\frac{1}{T} \int_{t}^{t+T} v_o^2(\tau) d\tau},
\]

with $T$ being the period of the periodic signal $v_o$, and the real and reactive powers are also nonlinear functions of $v_o$ and $i_i$, i.e. $P_i (v_o, i_i) = \frac{1}{T} \int_{t}^{t+T} v_o(\tau) i_i(\tau) d\tau$, $Q_i (v_o, i_i) = \frac{1}{T} \int_{t}^{t+T} v_o(\tau) i_i^*(\tau) d\tau$, with $v_o$ being the signal $v_o$ delayed by $\frac{T}{2}$. This makes it very difficult to directly investigate the stability of the closed-loop system. Several researchers have recently proved the stability of the inverter-based systems, but only when the conventional droop controller is used and under the assumption of a linear load (Schiffer et al., 2014; Simpson-Porco et al., 2013). When the controller introduces a dynamic structure, the difficulty increases. To the best of the authors’ knowledge, the stability analysis using the RDC, which achieves accurate power sharing for a general type of load, has not yet been exploited.

3. Preliminaries

3.1 Input-to-state (practical) stability

Consider the nonlinear system

\[
\dot{x} = f(t, x, u),
\]

where $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is piecewise continuous in $t$ and locally Lipschitz in $x$ and $u$.

Definition 1 (Khalil, 2001): The system (7) is said to be input-to-state stable (ISS) if there exist a class $KL$ function $\beta$ and a class $K$ function $\gamma$ such that for any initial state $x(0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq 0$ and satisfies

\[
\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma\left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|\right).
\]

2.3 Problem formulation

Stability analysis of inverters operated in parallel with accurate load sharing control techniques is a problem that several researchers are trying to solve nowadays. From the system modelling, it can be realised that since the plant system is BIBO, if the controller outputs $v_o$ remain bounded, then a bounded response is guaranteed. Particularly, according to (6), the frequency dynamics (5) do not affect the boundedness of the inverter output voltage since they are introduced in the sinusoidal function $\sin(\theta_i)$. Although the frequency dynamics can be omitted from the stability analysis in the sense of boundedness, they play a key role in the existence of the desired equilibrium point, as it is presented in Simpson-Porco et al. (2013). As a result, the voltage dynamics (4) are mainly responsible for guaranteeing a bounded controller output and are represented by a nonlinear function in an IC structure. It should be also noted that, in practice, the inverter output voltage should not exceed a maximum limit $E_{\text{max}}$, corresponding to a percentage over the rated voltage $E^*$, for safety reasons. Therefore, the question is how to guarantee a bounded IC structure of the form of (4) with specific limits $E_i < E_{\text{max}}$ and prove the closed-loop system stability.
Suppose that stable (ISpS) input-to-state practically then the system (7) is said to be input-to-state practically stable (ISpS) (Jiang, Teel, & Praly, 1994).

3.2 Small-gain theorem
Consider the composite feedback interconnection form $\Sigma$ of two subsystems $\Sigma_1$ and $\Sigma_2$ shown in Figure 3, where

$$\dot{x} = f(t, x, z, u_1)$$

$$\dot{z} = h(t, z, x, u_2).$$

The stability of the interconnected system $\Sigma$ can be characterised by the small-gain theorem (Jiang & Mareels, 1997; Jiang et al., 1994; Vidyasagar, 2002). Although several versions of the small-gain theorem have been proposed in the literature (Jiang et al., 1994; Khalil, 2001; Nesić & Teel, 2004), the following theorem introduces the generalised ISpS small-gain theorem, which proves ISpS of $\Sigma$ by verifying the ISpS of $\Sigma_1$ and $\Sigma_2$ and checking a condition about their ISpS gains.

**Theorem 1** (Jiang & Mareels, 1997; Jiang et al., 1994): Suppose that

1. $\Sigma_1$ is ISpS (resp. ISS) with respect to $v_1 = (z, u_1)$, with gain $\gamma_1$ from $z$ to $x$, i.e.

   $$\|z(t)\| \leq \beta_1 (\|z(0)\|, t) + \gamma_1 \left( \sup_{0 \leq \tau \leq t} \|z(\tau)\| \right)$$

   $$+ \bar{\gamma}_1 \left( \sup_{0 \leq \tau \leq t} \|u_1(\tau)\| \right) + d_1$$

   for some $\beta_1 \in K\mathcal{L}$, $\gamma_1, \bar{\gamma}_1 \in \mathcal{K}_\infty$ and $d_1 \geq 0$.

2. $\Sigma_2$ is ISpS (resp. ISS) with respect to $v_2 = (x, u_2)$, with gain $\gamma_2$ from $x$ to $z$, i.e.

   $$\|z(t)\| \leq \beta_2 (\|z(0)\|, t) + \gamma_2 \left( \sup_{0 \leq \tau \leq t} \|x(\tau)\| \right)$$

   $$+ \bar{\gamma}_2 \left( \sup_{0 \leq \tau \leq t} \|u_2(\tau)\| \right) + d_2$$

   for some $\beta_2 \in K\mathcal{L}$, $\gamma_2, \bar{\gamma}_2 \in \mathcal{K}_\infty$ and $d_2 \geq 0$.

3. There exist two functions $\rho_1, \rho_2 \in \mathcal{K}_\infty$ and a non-negative real number $s_1$ such that

   $$(Id + \rho_2) \circ \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(s) \leq s$$

   $$(Id + \rho_1) \circ \gamma_1 \circ (Id + \rho_2) \circ \gamma_2(s) \leq s$$

   for all $s \geq s_1$.

4. Nonlinear implementation of the RDC for parallel operation of inverters
Since the main task of the paper is the proof of stability and fail-safe operation of inverters operated in parallel while maintaining the advantages of the RDC, in this section, the RDC voltage dynamics (4) are implemented in a nonlinear form to generate a bounded output signal. Since each inverter should fulfil a fail-safe operation, a maximum inverter output voltage $E_{\text{max}}$ is provided for the technical requirements and the controller should shut down when the inverter output voltage tries to exceed this value. In this frame, the voltage dynamics of the proposed RDC with fail-safe are given as follows:

$$\dot{E}_i = \begin{bmatrix} -k \left( \frac{E_{i, \text{max}}^2}{E_{\text{max}}^2} + \frac{(E_{\text{max}} - V_o)^2}{\epsilon^2} \right) - 1 \\ -\frac{e^2}{E_{\text{max}}^2} (K_s (E^* - V_o) - n_i Q_i) \end{bmatrix} \begin{bmatrix} E_i \\ E_{qi} \end{bmatrix},$$

where $E_{qi}$ represents an extra controller state variable, $k$ is a large positive constant gain and $\epsilon$ is a small positive constant.

One can consider the following Lyapunov function candidate for the controller states:

$$V_i = \frac{E_i^2}{E_{\text{max}}^2} + \frac{E_{qi}^2}{\epsilon^2},$$

Figure 3. Composite feedback interconnection.
which has an ellipsoid form on the $E_i - E_{qi}$ plane at the origin with time derivative

$$\dot{V}_i = -2k \left( \frac{E_i^2}{E_{max}^2} + \frac{(E_{qi} - 1)^2}{\epsilon^2} - 1 \right) V_i. \quad (15)$$

As a result, the sign of $\dot{V}_i$ is related to the ellipse at the point $(0, 1)$ defined by

$$C_i = \left\{ E_i, E_{qi} \in \mathcal{R} : \frac{E_i^2}{E_{max}^2} + \frac{(E_{qi} - 1)^2}{\epsilon^2} = 1 \right\}. \quad (16)$$

As a result, for a sufficiently large $k$ and for positive (blue line) or negative (red line) values of the function $K_e(E^* - V_o) - n_iQ_i$, the phase portrait of the controller states $E_i$ and $E_{qi}$ is shown in Figure 4. Positive (blue line) and negative (red line) values of $K_e(E^* - V_o) - n_iQ_i$ describe clockwise and counterclockwise movements of $E_i$ and $E_{qi}$ on the upper semi-ellipse of $C_i$, respectively, in Figure 4.

Therefore, for initial conditions inside $C_i$ or above $C_i$ with $E_i(0) \in [-E_{max}, E_{max}]$, the controller states are quickly attracted on the upper semi-ellipse of $C_i$ on which $\frac{E_i^2}{E_{max}^2} + \frac{(E_{qi} - 1)^2}{\epsilon^2} = 1$. As soon as the controller states $E_i$ and $E_{qi}$ are attracted on the upper semi-ellipse of $C_i$, then they can converge to a point $(E_i^*, E_{qi}^*)$ on this curve where $K_e(E^* - V_o) - n_iQ_i = 0$ holds true in order to achieve accurate power sharing. If $|E_i|$ tries to violate $E_{max}$, then both $E_i$ and $E_{qi}$ are quickly attracted at the origin leading the system to shut down. In this case, the pulse-width modulation (PWM) operation shuts down and the $i$th inverter that violates this technical requirement automatically stops operating. This enhances the operation of the microgrid since no monitoring devices are needed, and applies a backup solution for a fail-safe operation if the protection circuits fail.

As a result, once the controller states are attracted onto the upper semi-ellipse of $C_i$, they can exclusively move on this curve as long as $|E_i| < E_{max}$ resulting in

$$\dot{E}_i = (K_e(E^* - V_o) - n_iQ_i) E_{qi}, \quad (17)$$

from the first dynamic equation of (13), where $E_{qi}$ takes a value in the interval $[1, 1 + \epsilon]$, because $\frac{E_i^2}{E_{max}^2} + \frac{(E_{qi} - 1)^2}{\epsilon^2} - 1 = 0$ on the ellipse. By choosing a sufficiently small $\epsilon$, it is clear that while $E_i$ and $E_{qi}$ travel on the upper semi-ellipse of $C_i$, it holds true that $E_{qi} \approx 1$ and approximately

$$\dot{E}_i \approx K_e(E^* - V_o) - n_iQ_i, \quad (18)$$

which implies that (13) approximates the RDC (4).

The following properties lead to the desired control operation:

1. Initial conditions $E_{q0}$ and $E_{q0}$ are chosen inside $C_i$ or above $C_i$ with $E_{q0} \in [-E_{max}, E_{max}]$.
2. Parameter $\epsilon$ is chosen sufficiently small.
3. Parameter $k$ is chosen sufficiently large.

Since before starting each inverter, the inverter output voltage is initially zero, the initial conditions could be chosen as $E_{q0} = 0$ and $E_{q0} = 1 + \epsilon$ in order to start from a point on the upper semi-ellipse of $C_i$. If mismatches or disturbances occur to the initial conditions, the diagonal elements of (13) will attract the states onto the desired curve, thus proving that the proposed method is robust to external disturbances.

As already mentioned, the parameter $\epsilon$ should be sufficiently small in order to well approximate the RDC. In a real application (simulation or experiment), it becomes obvious that it should be chosen larger than the relative tolerance.

The third property is critical in order to have quick and direct (straight line) convergence onto the curve $C_i$ or at the origin, along the line towards the origin. Therefore, $k$ should be chosen sufficiently large (as large as possible). It should be noted that $k$ describes the attractiveness towards the curve $C_i$ and, consequently, the origin, while the function $K_e(E^* - V_o) - n_iQ_i$ forms the rotation (clockwise if positive or counter-clockwise if negative) of the states on the upper semi-ellipse of $C_i$. If $k$ is sufficiently large, then in cases where $|E_i|$ tries to exceed $E_{max}$, the controller states are directly attracted at the origin, since the large $k$ monopolises the movement towards the origin with respect to the rotation caused by $K_e(E^* - V_o) - n_iQ_i$ (Figure 4). Therefore, the controller output $E_i$ will always stay within the given limits $[-E_{max}, E_{max}]$. For a sufficiently large $k$, this is always guaranteed.

The proposed method is depicted in Figure 5, where the RMS voltage dynamics are implemented in Figure 6.
5. Stability of the closed-loop system

After incorporating the voltage dynamics (13) into the system, the resulting closed-loop system can be observed as a feedback interconnection form of the nonlinear plant given from (3), with input the vector of the inverter voltages \(v_{ri}\) as given from (6) and output the state vector \(\tilde{x}\), and the nonlinear controller (13) with input \(\tilde{x}\) and output \(E_i\). Then, the following proposition guarantees a stable closed-loop system in the sense of boundedness.

**Proposition 1:** The closed-loop system consisting of the parallel inverters feeding a common load (3) under the control (6)–(13)–(5) is stable in the sense of boundedness, i.e. the closed-loop system solution is bounded for all \(t \geq 0\).

**Proof:** Beginning from the plant system (3), consider the following Lyapunov function candidate:

\[
W(\tilde{x}) = \frac{1}{2} \tilde{x}^T \tilde{M} \tilde{x}
\]

for which it holds true that

\[
\frac{1}{2} \lambda_{\min}(\tilde{M}) \|\tilde{x}\|^2 \leq W(\tilde{x}) \leq \frac{1}{2} \lambda_{\max}(\tilde{M}) \|\tilde{x}\|^2,
\]

where \(\lambda_{\min}(\tilde{M})\) and \(\lambda_{\max}(\tilde{M})\) denote the minimum and maximum eigenvalues of \(\tilde{M}\).

Then, the time derivative of \(W\) is calculated as follows:

\[
\dot{W} = -\tilde{x}^T \tilde{R} \tilde{x} + \tilde{x}^T \tilde{G} u \leq -\lambda_{\min}(\tilde{R}) \|\tilde{x}\|^2 + \|\tilde{x}\|_2 \|u\|_2
\]

using the norm properties and the Cauchy–Schwarz inequality, where \(\lambda_{\min}(\tilde{R})\) is the smallest eigenvalue of \(\tilde{R}\).

Therefore, there exists \(0 < \theta < 1\), such that

\[
\dot{W} \leq -(1 - \theta)\lambda_{\min}(\tilde{R}) \|\tilde{x}\|^2 - \theta \lambda_{\min}(\tilde{R}) \|\tilde{x}\|^2_2 + \|\tilde{x}\|_2 \|u\|_2 \\
\leq -(1 - \theta)\lambda_{\min}(\tilde{R}) \|\tilde{x}\|^2_2, \quad \forall \|\tilde{x}\|_2 \geq \frac{1}{\theta \lambda_{\min}(\tilde{R})} \|u\|_2.
\]

Since the Lyapunov function \(W\) is radially unbounded, then the inequality (20) implies that the plant system is ISS (Khalil, 2001) providing

\[
\|\tilde{x}(t)\|_2 \leq \beta_1(\|\tilde{x}(0)\|_2) + \frac{\lambda_{\max}(\tilde{M})}{\lambda_{\min}(\tilde{M})} \frac{1}{\theta \lambda_{\min}(\tilde{R})} \sup_{0 \leq \tau \leq t} \|u(\tau)\|_2,
\]

where \(\beta_1\) is a class \(K\mathcal{L}\) function. Taking into account that \(u\) is the input vector containing the inverter voltages \(v_{ri}\), i.e. \(u = [v_{r1} \ v_{r2}]^T\) with \(v_{ri}\) given from (6) and applying the norm properties, (21) results in

\[
\|\tilde{x}(t)\|_2 \leq \beta_1(\|\tilde{x}(0)\|_2) + \frac{\lambda_{\max}(\tilde{M}) \sqrt{2}}{\lambda_{\min}(\tilde{M})} \frac{1}{\theta \lambda_{\min}(\tilde{R})} \sup_{0 \leq \tau \leq t} \left\|E_i(\tau)\right\|_2,
\]

i.e. the plant system is ISS and introduces a finite-gain \(\gamma_{\text{plant}} = \frac{\lambda_{\min}(\tilde{M}) \sqrt{2}}{\lambda_{\min}(\tilde{M})} \lambda_{\max}(\tilde{R})\).

In the same frame, by considering as Lyapunov function candidate for the controller system (13), the function \(V_i\) as given from (14), then according to the analysis presented in Section 4, it is proven that the derivative of \(V_i\) is negative outside \(C_i\). It can be easily obtained that there exists a closed-set in the form of the Lyapunov function (14), outside of which \(\dot{V}_i < 0\) holds true. This is represented by ellipse \(S_i\) in Figure 7, which is in the form

\[
\frac{E_i^2}{E_{i,\text{max}}^2} + \frac{E_{q,i}^2}{e^2} = n^2
\]

for different values of \(n\). Since \(S_i\) has the same form of \(C_i\) and they intersect at point \((0, 1 + \epsilon)\), then it holds true that
shows that the controller states $E_i$ and $E_{qi}$ introduce an ultimate bound. Particularly, since $V_i$ is given from (14), it holds true that

$$n = \frac{1+n}{\epsilon}. \quad \text{As a result,}$$

$$S_i = \left\{ E_i, E_{qi} \in \mathcal{R} : \frac{E_i^2}{(1+\epsilon)E_{max}} + \frac{E_{qi}^2}{(1+\epsilon)^2} = 1 \right\}$$

From (15), it yields that there exists a positive constant $a > 0$ such that $^1$

$$\dot{V}_i \leq -a V_i, \quad \forall \left\| E_i \right\|_2 \geq \min \left( \frac{1+\epsilon}{\epsilon} E_{max}, 1+\epsilon \right) > 0.$$

Therefore, according to the Lyapunov-like theorem for ultimate boundedness (Khalil, 2001), it is proven that for any initial conditions $E_{q0}$ and $E_{qi0}$, there exists a class $\mathcal{K}$ function $\beta_2$ and a future time instant $T \geq 0$ such that

$$\left\| E_i \right\|_2 \leq \beta_2 \left( \left\| E_{q0} \right\|_2, t \right) \forall t \leq T \quad (25)$$

and

$$\left\| E_{qi} \right\|_2 \leq \sqrt{\max \left( E_{max}^2, \epsilon^2 \right)} \times \min \left( \frac{1+\epsilon}{\epsilon} E_{max}, 1+\epsilon \right) \forall t \geq T,$$

$$\Rightarrow \left\| E_i \right\|_2 \leq \max \left( E_{max}, \epsilon \right) \times \min \left( \frac{(1+\epsilon)E_{max}}{\epsilon}, 1+\epsilon \right) \forall t \geq T. \quad (26)$$

Therefore, the control states solution can be written in the following form:

$$\left\| E_i \right\|_2 \leq \beta_2 \left( \left\| E_{q0} \right\|_2, t \right) + d. \quad (27)$$

where $d = \max \left( E_{max}, \epsilon \right) \min \left( \frac{(1+\epsilon)E_{max}}{\epsilon}, 1+\epsilon \right)$ is a positive constant. Since inequality (27) is satisfied independently from any bounded input $K_e(E^* - V_i) - n_iQ_i$ of the controller, the controller states can be written in the ISP form of (9) with zero gain, i.e. $\gamma_{control} = 0$, regardless of the selection of $E_{q0}$, $E_{qi0}$, $E_{max}$, $k$ and $\epsilon$. This becomes clear from the fact that the function $K_e(E^* - V_i) - n_iQ_i$ does not affect the stability analysis of the controller dynamics, as presented in Section 4, using the Lyapunov method.

Since the closed-loop system is given in the composite feedback interconnection form with the plant given in the general form of (10) and the controller given in the general form of (11), the small-gain theorem given in Theorem 1 can be applied. Particularly, the plant system is ISS with gain $\gamma_{plant}$, as proven from (22), and the controller is ISP with zero gain $\gamma_{control} = 0$, as proven from (27). As a result, the condition (12) is obviously satisfied. Therefore, the closed-loop system is ISP and since no external input exists, the closed-loop system solution is bounded for all $t \geq 0$. 

As it is observed from the proof of Proposition 1, the proposed method generates a bounded inverter voltage $v_{qi} = \sqrt{2}E_i \sin(\theta)$, since $E_i$ is bounded, independently from the conditions mentioned in Section 4. This guarantees global stability in the sense of boundedness for parallel inverters operating under this method. However, if the conditions of Section 4 additionally hold, then closed-loop stability is maintained with the inverter RMS voltage satisfying the technical limits, i.e. $E_i < E_{max}$. As a result, closed-loop system stability is proven and fail-safe operation is guaranteed in the sense that if one inverter’s RMS voltage tries to violate the maximum value $E_{max}$, then according to the dynamics of (13), the controller states will quickly converge to the origin suitably disconnecting the inverter from the system.

Furthermore, according to the analysis presented in Section 4, the controller states will either converge to the desired equilibrium or to the origin without exhibiting an oscillatory behaviour, assuming that a stabilising solution exists. Particularly, assume initially that $K_e(E^* - V_i) - n_iQ_i$ is constant. Then, since a closed trajectory with $V_i = 0$ exists on the $E_i - E_{qi}$ plane, i.e. the ellipse $C_i$, there is a possibility of the existence of a limit cycle. However, according to the critical point criterion (Khalil, 2001), since there is no critical point in the closed set $C_i$, then there are no limit cycles on the $E_i - E_{qi}$ plane. Additionally, according to the Poincare–Bendixon theorem (Khalil, 2001), the controller (13) is a second-order dynamical system, and therefore, no chaotic behaviour exists on the $E_i - E_{qi}$ plane.
a result, the controller states $E_i$ and $E_{qi}$ can only converge to the origin after travelling on the upper semi-ellipse of $C_i$. Therefore, the controller dynamics by themselves cannot exhibit an oscillatory response. Additionally, using different time-scale theory between the plant and the controller (Khalil, 2001), the same analysis can prove that the controller states will either converge to a point on the upper semi-ellipse of $C_i$, where $K_e(E^* - V_o) - n_i Q_i = 0$ or to the origin. Therefore, if the voltage dynamics operate exclusively on the upper semi-ellipse of $C_i$, then the closed-loop system will converge to the desired equilibrium.

6. Simulation results

To investigate the system stability under normal and abnormal conditions, two single-phase inverters operating in parallel and feeding a common load are considered using the proposed RDC with fail-safe and the original RDC, as proposed in Zhong (2013). Each inverter is powered by a 400 V DC voltage source and the power ratings are $S_1 = 0.5$ kVA and $S_2 = 1$ kVA for inverters 1 and 2, respectively. It is expected that $P_2 = 2P_1$ and $Q_2 = 2Q_1$. Both inverters operate with a switching frequency of 15 kHz and the line frequency of the system is 50 Hz. The rated voltage of the inverters is $E^* = 230$ V and $K_e = 10$. The filter inductors are $L_1 = 2.2$ mH, $L_2 = 2$ mH with parasitic resistances $R_1 = 0.3 \Omega$, $R_2 = 0.2 \Omega$ and the filter capacitors are $C_1 = C_2 = 10 \mu F$ with parasitic resistances $r_{c1} = r_{c2} = 100 \Omega$. According to National Grid Electricity Transmission PLC (2010), the desired voltage drop ratio is chosen as $n_i S_i^* / K_e E^* = 0.25\%$ and the frequency drop ratio is chosen $m_i S_i^* / \omega^* = 0.1\%$. Therefore, the droop coefficients are calculated as $n_1 = 0.0115$, $n_2 = 0.0057$, $m_1 = 6.2832 \times 10^{-4}$ and $m_2 = 3.1416 \times 10^{-4}$. Assuming a technical requirement that permits the inverter voltage not to exceed the rated voltage by more than 10% at any time, $E_{\text{max}}$ is
Figure 10. Simulation results of the RDC and the RDC with fail-safe (nonlinear load).

chosen as $E_{\text{max}} = 1.1E^* = 253$ V. For the operation of the RDC with fail-safe, parameters $k$ and $\epsilon$ are chosen as $k = 1000$ and $\epsilon = 0.01$ to satisfy the conditions mentioned in Section 4.

Two different load cases are considered. Initially, both inverters are connected to a linear load with resistance $R_L = 57\Omega$, while in the second case, they feed a nonlinear load, as shown in Figure 8, with $C_L = 800\mu F$, $R_L = 100\Omega$, $L_L = 2.2\ mH$ and a parasitic resistance $r_L = 0.3\Omega$. This nonlinear load is a special case of a controllable rectifier as noted in Karagiannis et al. (2003), where the dynamic load equations satisfy (2). In both cases, a $-30\%$ permanent error occurs at the load voltage sensor of inverter 1 (the measured RMS load voltage $V_o$ becomes $30\%$ less than the actual for inverter 1) at time instant $t = 1\ s$, in order to investigate the behaviour of the parallel-operated inverters under a system failure scenario.

The results of the parallel operation with a linear load are shown in Figure 9. It is observed that before the malfunction of the voltage sensor of inverter 1, both the RDC and the proposed RDC with fail-safe operate in exactly the same way, verifying the theoretical analysis of the paper that the proposed method approximates the performance of the RDC. As a result, both the real and reactive powers are shared accurately and proportionally to the inverters ratings, as shown in Figure 9(a)–(d), while the load voltage is regulated close to the rated value, maintaining all the advantages of the RDC underlined in Zhong (2013), as verified in Figure 9(e) and 9(f). After the system failure at time instant $t = 1\ s$, the voltage of inverter 1 increases and exceeds the maximum allowed value $E_{\text{max}}$, as shown in Figure 9(i), while the voltage of inverter 2 drops (Figure 9(j)). This causes both inverter currents $i_1$ and $i_2$ to increase to higher values as it is observed in Figure 9(g) and 9(h). The RDC operation forces both currents to continuously increase leading the system to instability and eventually to the destruction of both inverters. On the other hand, as it is shown in Figure 9(i), when the voltage of inverter 1 ($E_1$) tries to exceed the maximum limit $E_{\text{max}}$, the RDC with fail-safe quickly shuts down inverter 1 in order to guarantee a fail-safe operation. The second inverter does not violate the technical requirements, since the inverter voltage $E_2$ stays below $E_{\text{max}}$, and continues feeding the load by itself after the disconnection of inverter 1. Therefore, both the inverter currents avoid increasing to high values, thus protecting the inverters, while simultaneously the load voltage stays close to the rated voltage as shown in Figure 9(f). To verify the voltage dynamics operation, the phase portraits of the controller states $E_1$, $E_{q1}$ and $E_2$, $E_{q2}$ are plotted in Figure 9(k) and 9(l), respectively. Before the system failure,
the controller states operate on the upper semi-ellipse $C_1$ (and $C_2$ respectively), and after $t = 1$ s, when $E_1$ tries to exceed $E_{\text{max}}$, both states $E_1$ and $E_{\text{q1}}$ are quickly attracted at the origin (Figure 9(k)) leading to a shutdown operation of inverter 1. On the other hand, as it is clear from Figure 9(l), since $E_2$ stays below $E_{\text{max}}$, both states $E_2$ and $E_{\text{q2}}$ stay on the upper semi-ellipse of $C_2$ for all time.

The results of the parallel operation with a nonlinear load are shown in Figure 10. As in the case of the linear load, the RDC with fail-safe suitably approximates the RDC in normal operation guaranteeing its main advantages. Additionally, when the system failure occurs, the RDC with fail-safe forces inverter 1 to shut down, since the inverter voltage $E_1$ tries to violate the maximum technical limit $E_{\text{max}}$, while inverter 2 continues feeding the load solely by always staying inside the technical operating limits. This verifies the desired operation of accurate power sharing with guaranteed closed-loop stability for the general nonlinear system. The fail-safe operation of the system with the proposed method is also proven, since it shuts down the inverter which tries to violate the technical requirements and cause an unstable response. This approach can be used for multiple inverters operating in parallel where the RDC with fail-safe can achieve accurate load sharing and handle system failures by suitably shutting down each inverter that is trying to harm the system. This presents a new generation of control design in microgrids, thus maintaining the decentralised structure of the droop control.

7. Conclusions

In this paper, stability in the sense of boundedness and fail-safe operation is investigated for inverters operated in parallel and feeding a common load. By implementing the RDC voltage dynamics in a nonlinear matrix form, it is analytically proven that the proposed method approximates the performance of the RDC by incorporating all its advantages for accurate proportional load sharing and voltage regulation near the rated value when the inverter output voltage stays within the given limits. If these limits are violated due to system failures, then the proposed method forces the controller states to quickly converge to the origin. The transition between the two operating modes is accomplished without external switches and the output voltage remains a continuous-time smooth signal. To this end, the proposed method introduces a nonlinear structure based on the idea of the RDC, and by using nonlinear Lyapunov methods and the generalised small-gain theorem, it is proven that the closed-loop system solution remains bounded. The analysis is conducted for a generic system with parallel-operated inverters feeding a generic load (linear or nonlinear) presented in the generalised dissipative Hamiltonian form. Extensive simulation results of two single-phase inverters operated in parallel and feeding a linear and a nonlinear load were presented to verify the proposed method under a system failure scenario.

It should be noted that a different selection of the controller parameters can lead to a continuous operation of the inverter with the faulty sensor, i.e., not to shut down but to operate in an open-loop manner where the states do not converge to the origin but to the rated voltage $E^*$ or to the maximum voltage $E_{\text{max}}$.

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Note

1. For details, see Khalil (2001, Section 4.8).

References


