Robust predictive control

*Optimization over state feedback policies*

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*Joint work with Paul Goulart and Jan Maciejowski*
Problem definition
- LTI system with bounded state disturbance
- Constraints on state and input
- Minimize expected value of a quadratic cost function

Optimization over affine state feedback policies
- Optimization problem is non-convex

Optimization over affine disturbance feedback policies
- Equivalent to affine state feedback
- Optimization problem is convex

Guaranteeing stability for receding horizon control (RHC)
Optimization problem we would like to solve

At each time instant, solve

$$\inf_{\pi := \{\mu_0(\cdot), \mu_1(\cdot), \ldots\}} \mathbb{E} \left[ \sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i \right]$$

where the measurement of the current state $x = x_0$,

$$x_{i+1} = A x_i + B u_i + w_i, \quad i = 0, 1, \ldots$$

$$u_i = \mu_i(x_0, \ldots, x_i), \quad i = 0, 1, \ldots$$

Additionally, the feedback policy $\pi := \{\mu_0(\cdot), \mu_1(\cdot), \ldots\}$, where each $\mu_i(\cdot)$ is a control law, should ensure the constraints

$$h_i(x_i, u_i) \leq 0, \quad i = 0, 1, \ldots$$

are satisfied for any disturbance sequence $\{w_0, w_1, \ldots\}$, where each $w_i \in W$. 
Optimization over feedback policies

Difficulties with solving the problem:

- Optimization is over functions: $\mu_i : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^m$
- Infinite number of decision variables: $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \ldots\}$
- Infinite number of constraints: $\forall w \in \mathcal{W}, h_i(x, \pi, w) \leq 0$
Optimization over feedback policies

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- Infinite number of constraints: $\forall w \in \mathcal{W}, h_i(x, \pi, w) \leq 0$

Towards a possible solution:

- Finite control horizon, i.e. $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$
- Parameterize $\pi$ in terms of a finite number of decision variables, e.g.

$$u_i = \mu_i(x_0, \ldots, x_i) = g_i + \sum_{j=0}^{i} L_{i,j} x_j$$

- Remove the universal quantifier from the description of constraints:

$$\forall w \in \mathcal{W}, h_i(x, \pi, w) \leq 0 \iff h_i^*(x, \pi) := \sup \{h_i(x, \pi, w) \mid w \in \mathcal{W}\} \leq 0$$
At each time instant, solve the optimization problem

\[ \mathbb{P}_N(x) : \inf_{\{g_i\},\{L_{i,j}\}} \mathbb{E} \left[ x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \right] \]

such that for all \( w := \left( w_0^T \cdots w_{N-1}^T \right)^T \in \mathcal{W} \),

\[
\begin{align*}
x_{i+1} &= A x_i + B u_i + w_i, & i &= 0, 1, \ldots, N - 1 \\
u_i &= g_i + \sum_{j=0}^{i} L_{i,j} x_j, & i &= 0, 1, \ldots, N - 1 \\
(x_i, u_i) &\in Z := \{ (x, u) \mid C x + D u \leq b \}, & i &= 0, 1, \ldots, N - 1 \\
x_N &\in X_f := \{ x \mid Y x \leq z \} 
\end{align*}
\]

where the measurement of the current state is \( x = x_0 \) and \( \mathcal{W} := W \times \cdots \times W \).
Write feedback policy in matrix form:

\[
\begin{pmatrix}
    u_0 \\
    \vdots \\
    u_{N-1}
\end{pmatrix}
= 
\begin{pmatrix}
    L_{0,0} & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    L_{N-1,0} & \cdots & L_{N-1,N-1} & 0
\end{pmatrix}
\begin{pmatrix}
    x_0 \\
    x_1 \\
    \vdots \\
    x_N
\end{pmatrix} 
+ 
\begin{pmatrix}
    g_0 \\
    \vdots \\
    g_{N-1}
\end{pmatrix}
\]

so that

\[
u = Lx + g
\]

and let \( A, B \) and \( E \) be matrices such that

\[
x = Ax + Bu + Ew = (I - BL)^{-1}(Ax + Bg + Ew),
\]

where the current measurement of the state \( x = x_0 \).
For a suitably-defined $Q$, $R$, $C$, $D$ and $c$, problem $\mathbb{P}_N(x)$ is equivalent to:

$$(L^*(x), g^*(x)) := \arg \inf_{(L,g)} \mathbb{E} \left[ x^T Q x + u^T R u \right]$$

subject to $L$ block lower triangular and

$$\forall w \in \mathcal{W}, \quad Cu + Dx \leq c, \quad u = Lx + g, \quad x = Ax + Bu + Ew$$

The receding horizon control (RHC) law is defined as:

$$u = \kappa_N(x) := L_{0,0}^*(x)x + g_0^*(x)$$
Optimization problem rewritten

For a suitably-defined $Q$, $R$, $C$, $D$ and $c$, problem $P_N(x)$ is equivalent to:

$$(L^*(x), g^*(x)) := \underset{(L,g)}{\operatorname{arg\ inf}} \mathbb{E} [x^T Q x + u^T R u]$$

subject to $L$ block lower triangular and

$$\forall w \in \mathcal{W}, \quad Cu + Dx \leq c, \quad u = Lx + g, \quad x = Ax + Bu + Ew$$

The receding horizon control (RHC) law is defined as:

$$u = \kappa_N(x) := L_{0,0}^*(x)x + g_0^*(x)$$

**Punchline:** Direct computation of $(L_{0,0}^*(x), g_0^*(x))$ is difficult, because the cost and constraint functions are non-convex in $(L, g)$:

$$x = (I - BL)^{-1}(Ax +Bg +Ew), \quad u = L(I - BL)^{-1}(Ax + Bg + Ew) + g$$
Affine disturbance feedback policies

Note that $w_i = x_{i+1} - Ax_i - Bu_i$. Consider parameterizing the control as an affine function of prior disturbances:

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j, \quad i = 0, \ldots, N - 1$$

Write this in matrix form:

$$
\begin{pmatrix}
    u_0 \\
    u_1 \\
    \vdots \\
    u_{N-1}
\end{pmatrix}
= 
\begin{pmatrix}
    0 & \cdots & \cdots & 0 \\
    M_{1,0} & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    M_{N-1,0} & \cdots & M_{N-1,N-2} & 0
\end{pmatrix}
\begin{pmatrix}
    w_0 \\
    w_1 \\
    \vdots \\
    w_{N-1}
\end{pmatrix}
+ 
\begin{pmatrix}
    v_0 \\
    v_1 \\
    \vdots \\
    v_{N-1}
\end{pmatrix}
$$

or

$$u = Mw + v$$
Affine state feedback and affine disturbance feedback are equivalent:

- Given any initial state $x$, the input and state trajectories due to state and disturbance feedback can be made to be equal for all allowable disturbance sequences:
  - Given any $(L, g)$, there exists a pair $(M, v)$ such that $u = Mw + v = Lx + g$ for all $w \in W$
  - Given any $(M, v)$, there exists a pair $(L, g)$ such that $u = Mw + v = Lx + g$ for all $w \in W$

- The sets of initial states for which feasible affine state and disturbance feedback policies exists, are equal
Equivalence between feedback policies

Affine state feedback and affine disturbance feedback are equivalent:

- Given any initial state $x$, the input and state trajectories due to state and disturbance feedback can be made to be equal for all allowable disturbance sequences:
  - Given any $(L, g)$, there exists a pair $(M, v)$ such that $u = Mw + v = Lx + g$ for all $w \in \mathcal{W}$
  - Given any $(M, v)$, there exists a pair $(L, g)$ such that $u = Mw + v = Lx + g$ for all $w \in \mathcal{W}$

- The sets of initial states for which feasible affine state and disturbance feedback policies exists, are equal

Sketch of proof:

$$M = L(I - BL)^{-1}E$$
$$v = L(I - BL)^{-1}(Ax + Bg) + g$$
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- Given any initial state $x$, the input and state trajectories due to state and disturbance feedback can be made to be equal for all allowable disturbance sequences:
  
  - Given any $(L, g)$, there exists a pair $(M, v)$ such that
    
    $$u = Mw + v = Lx + g \text{ for all } w \in \mathcal{W}$$
  
  - Given any $(M, v)$, there exists a pair $(L, g)$ such that
    
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- The sets of initial states for which feasible affine state and disturbance feedback policies exists, are equal

Sketch of proof:

$$L = (I - ME^\dagger B)^{-1} ME^\dagger$$

$$g = (I - ME^\dagger B)^{-1}(v - M(A^\dagger)^T A x)$$
Equivalent, but convex optimization problem

For a suitably-defined $Q$, $R$, $C$, $D$ and $c$, problem $P_N(x)$ is equivalent to:

$$(M^*(x), v^*(x)) := \arg \inf_{(M,v)} \mathbb{E} [x^T Q x + u^T R u]$$

subject to $M$ strictly block lower triangular and

$$\forall w \in \mathcal{W}, \quad Cu + Dx \leq c, \quad u = Mw + v, \quad x = Ax + Bu + Ew$$

The receding horizon control (RHC) law is now given by

$$u = \kappa_N(x) := L_{0,0}^*(x)x + g_0^*(x) = v_0^*(x)$$

**Punchline:** Direct computation of $v_0^*(x)$ is easy if $\mathcal{W}$ is convex, because the cost and constraint functions are convex in $(M, v)$!
Convexity of constraints

For a given initial state $x$, one can rewrite the input and state constraints as:

$$Fv + (FM + G)w \leq c + Hx, \quad \forall w \in \mathcal{W},$$

where $F$, $G$, $c$ and $H$ are appropriately defined.

Remove the universal quantifier to get a set of finite and tractable constraints:

$$Fv + \sup_{w \in \mathcal{W}} (FM + G)w \leq c + Hx,$$

where the supremum is taken row-wise.

As an example, note that

$$W := \{w \mid \|w\|_{\infty} \leq \eta\} \Rightarrow \sup_{w \in \mathcal{W}} a^T w = \eta \|a\|_1$$
Convexity of constraints

For a given initial state $x$, one can rewrite the input and state constraints as:

$$Fv + (FM + G)w \leq c + Hx, \quad \forall w \in W,$$

where $F$, $G$, $c$ and $H$ are appropriately defined.

Remove the universal quantifier to get a set of finite and tractable constraints:

$$Fv + \sup_{w \in W} (FM + G)w \leq c + Hx,$$

where the supremum is taken row-wise.

As an example, note that

$$W := \{w \mid \|w\|_\infty \leq \eta\} \Rightarrow \sup_{w \in W} (FM + G)w = \eta \text{abs}(FM + G)1$$

Hence, the set $\{(M, v) \mid Fv + \eta \text{abs}(FM + G)1 \leq c + Hx\}$ is convex.
Convexity of optimization problem

Without loss of generality, assume that

\[ \mathbb{E}[w] = 0 \text{ and } W := \{ w \mid \|w\|_\infty \leq \eta \} \]

\( \mathbb{P}_N(x) \) is equivalent to a **tractable** and **convex** optimization problem:

\[
(M^*(x), v^*(x)) = \arg \inf_{(M,v)} (Ax + Bv)^T Q(Ax + Bv) + v^T Rv
\]

where the optimization is subject to \( M \) strictly block lower triangular and

\[
Fv + \eta \text{abs}(FM + G)1 \leq c + Hx
\]

This is because:

- \( u \) and \( x \) are linear in \( w \)
- We have perfect state information
- The cost is quadratic and the expectation operator is linear
Conditions for stability

Choose a terminal control law \( u = Kx \) such that \( A + BK \) stable

Let \( X_K := \{ x \mid Cx + DKx \leq b \} \)

Compute a terminal constraint \( X_f \subseteq X_K \) such that \( X_f \) is robust positively invariant for \( x_{k+1} = (A + BK)x_k + w_k \), i.e.

\[
(A + BK)x + w \in X_f, \quad \forall x \in X_f, w \in W
\]

Choose \( Q \succ 0, R \succ 0 \) and \( P \succ 0 \) such that the terminal cost \( V_f(x) := x^T P x \) is a Lyapunov function for the undisturbed system \( x_{k+1} = (A + BK)x_k \) in the sense that

\[
V_f((A + BK)x) - V_f(x) \leq -x^T (Q + K^T RK)x, \quad \forall x \in X_f
\]

NB: Similar assumptions as in conventional RHC, but proof of stability is more involved
Guaranteeing stability

If \( W \) is a polytope, then the closed-loop RHC system

\[
x_{k+1} = Ax_k + B\kappa_N(x_k) + w_k, \quad w_k \in W
\]

is input-to-state stable (ISS):

\[
\|x_{k+1}\| \leq \beta(\|x_0\|, k) + \gamma\left(\sup_{i\in\{0,\ldots,k\}} \|w_i\|\right)
\]

where \( \beta(\cdot) \) is a \( \mathcal{KL} \)-function (continuous, non-negative and increasing in first argument, decreasing to zero in second argument) and \( \gamma(\cdot) \) is a \( \mathcal{K} \)-function (continuous, non-negative and increasing)

The input and state constraints are satisfied for all time, given any disturbance sequence \( \{w_0, w_1, \ldots\} \), where each \( w_i \in W \).
Conclusions and remarks

- Affine **state** feedback policies
  - Optimization problem is **non-convex**
- Affine **disturbance** feedback policies
  - Equivalent to affine state feedback
  - Optimization problem is **convex** and **tractable**
  - Solve a QP if all sets are polytopic
  - Number of decision variables and constraints is $O(N^2)$
- Receding horizon control
  - Choose appropriate terminal control law, cost and constraint
  - Input-to-state stable (ISS)
  - Guarantee constraint satisfaction for all time
- Can exploit structure for efficient computation
  - Sparse interior point code guarantees solution in time $O(N^3)$
- Can use other cost functions, such as $H_\infty$ control / $\ell_2$ gain minimization