Robust Model Predictive Control

Colloquium on Predictive Control
University of Sheffield, April 4, 2005

David Mayne
(with Maria Seron and Sasa Raković)

Imperial College London
Contents

I. Robust control problem
II. Conventional model predictive control
III. Disturbance/uncertainty
IV. Feedback model predictive control
V. Tube model predictive control
VI. Novel robust model predictive control
VII. Conclusions
Robust control problem

- Uncertain System

\[ x^+ = f(x, u, w) = Ax + Bu + w \]

- Constraints:

\[ x \in \mathbb{X}, \quad u \in \mathbb{U}, \quad w \in \mathbb{W} \]

- \( \phi(k; x, u, w) \), solution of \( x^+ = f(x, u, w) \) at time \( k \)

- \( u \triangleq \{u_0, u_1, \ldots, u_{N-1}\}; \ also \ w. \)

- Control objectives: stabilization and performance
Conventional RHC

- **System:** $z^+ = \bar{f}(x, v) = Az + Bv$
- **Constraints:** $z \in \mathbb{Z}$, $v \in \mathbb{V}$.
- **FH OC Pb is:** $P_N(z)$:
  \[
  \min_{\mathbf{u}} V(z, \mathbf{v}) \triangleq \sum_{i=0}^{N-1} \ell(z_i, v_i) + V_f(z_N)
  \]
  - $z_i \in \mathbb{Z}$, $v_i \in \mathbb{V}$, $z_N \in \mathbb{Z}_f$
  - $z_i = \bar{\phi}(i; z, v)$, solution of $z^+ = Az + Bv$ at time $k$. 
Conventional RHC

- Solution: \( v^0(z) = \{ v^0_0(z), v^0_1(z), \ldots, v^0_{N-1}(z) \} \).
- Value f’n: \( V^0_N(z) = V_N(z, v^0(z)) \), domain \( Z_N \).
- RH control law: \( \kappa_N(z) \triangleq v^0(0; z) \).
- IF A: \( V_f(\cdot) \) is local CLF in sense:
  - \( \min_{u \in V} \{ V_f(\bar{f}(z, u)) + \ell(z, u) \mid \bar{f}(z, u) \in Z_f \} \leq V_f(z) \) for all \( z \in Z_f \), THEN:
    - \( V^0_N(f(z, \kappa_N(z)) + \ell(z, \kappa_N(z)) \leq V^0_N(z) \) \( \forall z \in Z_N \)
    - and \( V^0_N(z) \leq V_f(z) \) \( \forall z \in Z_f \) \( \Rightarrow \) \( V^0_N(\cdot) \) is a Lyapunov fn for controlled system.
Value function $V^0_N(\cdot)$

**Proposition 1** Assume that $\ell(\cdot)$ and $V_f(\cdot)$ are quadratic and pos. def. and that (stabilizing) assumption A is satisfied. Then

$$V^0_N(z) \geq c_1|z|^2, \quad \forall z \in Z_N$$

$$V^0_N(f(z, \kappa_N(z))) \leq V^0_N(z) - c_1|z|^2, \quad \forall z \in Z_N$$

$$V^0_N(z) \leq c_2|z|^2, \quad \forall z \in Z_f$$

**Theorem 1** The origin is exp. stable for nominal system with MPC law $\kappa_N(\cdot)$ (if $Z_N$ is bounded). Solution same as that obtained with DP.
Nominal robustness

- Can use nominal controller \( u = \kappa_N(x) \) to control uncertain system \( x^+ = Ax + Bu + w \).
- Under some conditions, get ‘ultimate boundedness’.

**Diagram:**
- \( \mathcal{L}_V(c_2) \)
- \( Z_N \)
- \( \mathcal{L}_V(c_1) \)
- \( Z_f \)
‘Feedback’ RHC

- Conservative. State constraints may render $x^+ = \bar{f}(x, \kappa_N(x))$ non-robust (Teel)

- better to design controller to be robust.

- predicting effect of uncertainty

- hence, use feedback RHC (in which decision variable is policy $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ (sequence of control laws).

- rather than $u = \{u_0, u_1, \ldots, u_{N-1}\}$ (sequence of control actions).
Control policy

Figure 1: $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$
Whether control is ol ($u$) or fb ($\pi$):
- obtain tube of predicted trajectories
- Tube more ‘compact’ with fb policy $\pi$
- Satisfaction of constraints easier (possible)
- Optimizing over control sequences yields worse result than with DP
- Optimizing over control policies yields same result as with DP
Tube \( \{X_0, X_1, \ldots, X_N\} \)
Simplification

- Exact tube $\{X_0, X_1, \ldots, X_N\}$ complex: $a^N$.
- Policy $\pi$ complex ...inf dimensional
- Outer approx’n: $X_i = z_i \oplus S$.
- $z_i = \bar{\phi}(i; z, v(\cdot))$, solution at time $i$ of:
- Nominal system: $\dot{z} = \hat{f}(z, v) = Az + Bv$.
- $S$ robust pos. invariant for $\dot{x} = A_K x + w$
- $A_K S \oplus W \subset S$, $A_K \triangleq A + BK$ stable.
- Policy $\pi = \{\mu_i(\cdot)\}$,
- $\mu_i(x; z, v(\cdot)) = v_i + K(x - z_i)$ (K & R).
$x \in X_0$ implies $x_i = \phi(i; x, \pi, w) \in X_i \forall i, w$. 

$(X_i \rightarrow \{z_i\} \text{ as } W \rightarrow \{0\})$
Nominal OC Pb

- Nominal system is $z^+ = Az + Bv$.
- $\phi(i; z, v)$ is sol’n at $i$ of nominal system.
- Nominal OC Pb: same as before with tighter constraints:

\[
v_i \in \mathbb{V} \triangleq \mathbb{U} \ominus KS \text{ (W small enough)}
\]

\[
z_i \in \mathbb{Z} \triangleq \mathbb{X} \ominus S \text{ (W small enough)}
\]

\[
z_N \in \mathbb{Z}_f \subset \mathbb{X} \ominus S, \text{ where}
\]

\[
\mathcal{U}_N(z) = \{v \text{ satisfying all constraints, } z(0) = z\}
\]
Control policy

Consider policy $\pi^0 = \pi^0(z)$ constructed from solution to modified nominal OC Pb:

- $\pi^0(z) \triangleq \{ \mu_0(\cdot; z), \mu_1(\cdot; z), \ldots, \mu_{N-1}(\cdot; z) \}$
- $\mu_i(x; z) \triangleq v_i^0(z) + K(x - z_i^0(z))$

**Proposition 2** Policy $\pi^0(z)$ steers any any initial state of $x^+ = Ax + Bu + w$ lying in $X_0(x) = z \oplus S$ to $X_f = Z_f \oplus S \subseteq X$ in $N$ steps satisfying all constraints for all admissible disturbance sequences. (C&R&Z, M&L)
Control policy

Figure 3: Tube

\[ X_i = z_i + S \]
Receding horizon

- Above ... single shot
- Receding horizon ... two factors:
- Controlling tube ... not a trajectory
- Control to set $S$ (not origin) ... $S$ is \textit{robust} ‘origin’
- Prob. 1: Assumption equivalent to A for case when terminal ‘state’ is a set $X$ rather than a point $x$?
- Prob. 2: Value function that is zero in $S$ (the origin)?
Terminal Set

\[ X_f = Z_f \oplus S \]
Let $\mathcal{X}_f \triangleq \{ z \oplus S \mid z \in Z_f \}$.

Suppose (i) $S$ is robust pos. invariant for $x^+ = A_K x + w$, and, (ii) $Z_f$ is positively invariant for $z^+ = A_K z$ (don’t need same $K$’s).

**IF:**

$X \in \mathcal{X}_f$

**THEN:**

$X^+ = A_K X \oplus W \in \mathcal{X}_f$
New Optimal Control Problem

- To get Lyapunov function zero in $S$ (origin)
- propose new finite horizon O.C Pb:

**OC Pb:** $\mathbb{P}^*_N(x)$:

$$V^*_N(x) = \min_{z,v} \{ V_N(z, v) \mid v \in U_N(z), \ x \in z \oplus S \}$$

$$= \min_{z} \{ V^0_N(z) \mid x \in z \oplus S \}$$

**Solution:**

$$(z^*(x), \ v^*(x))$$

$$v^*(x) = \{ v^*_0(x), v^*_1(x), \ldots, v^*_{N-1}(x) \}$$
Model predictive control law

Implicit model predictive control law:

\[ \kappa_N^*(x) = v_0^*(x) + K(x - z^*(x)) \]
Exponential stability

$$z_i^{*} \oplus S$$

$$z_i, x_i, z_{i+1}, z_{i+1}^{*}$$

$$i \quad i + 1$$
Exponential stability

- $z_i^* \rightarrow z_{i+1} \rightarrow z_{i+1}^*$,

- $V_N^0(z_{i+1}) \leq V_N^0(z_i^*) - \ell(z_i^*, \kappa_N(z_i^*))$.

- $V_N^0(z_{i+1}^*) \leq V_N^0(z_{i+1})$

- $\implies z_i \rightarrow 0$ exponentially

- But $x_i \in z_i \oplus S$

- $\implies x_i \rightarrow S$ (rob) exponentially

**Theorem 2** The set $S$ is rob. exponentially stable (if $X_N^*$ is bounded) for $x^+ = Ax + B\kappa_N^*(x) + w$.
The domain of attraction is $X_N = Z_N \oplus S$. 
Exponential stability
Conclusions

- Have presented novel version of model predictive control
- Uses feedback mpc and bounding tube
- And initial state as a decision variable
- Simple online optimization problem (QP)
- $V_N^*(\cdot)$ zero in set $\mathcal{S}$ (origin)
- Set $\mathcal{S}$ is exponentially stable
- Moral: control tube, not individual trajectories.