Abstract: Most approaches of chance-constrained robust MPC utilise an open-loop model for uncertainty prediction; this results in conservative control. To avoid this some approaches propose robust MPC (RMPC) based on closed-loop prediction, and these often give improved dynamic performance at a small expense to constraint handling performance. The aim of this paper is to extend the methodology of (Warren, 2005), which is a chance-constrained RMPC based on closed-loop prediction, through the use of state space models. Moreover, we adopt dual mode MPC in the constrained optimisation, which allows us to guarantee stability for the infinite horizon. The use of the closed-loop paradigm for dual mode MPC has advantages in numerical conditioning and robustness analysis. A case study, looper-tension control, is used to evaluate the robustness of the proposed RMPC strategy on a realistic process as well as extending previous studies of MPC for rolling mills (Choi et al., 2004) to the robust case.

Keywords: Robust model predictive control, Disturbance uncertainty, Closed-loop prediction, Hot rolling mills

1. INTRODUCTION

There exist several types of disturbances in process industries which affect stability, performance and robustness. Various assumptions have been made to deal with these disturbances appropriately in MPC design. The simplest approach is to ignore the disturbance or to assume a constant disturbance over the horizon. However, these assumptions may result in poor process performance and allow constraint violations due to the ignored effects of changes in the future disturbances.

Therefore, one of the most common assumptions made about disturbances is that they are unknown but bounded. One approach using this assumption has been called 'Min–max MPC' [(Lee and Yu, 1997),(Scokaert and Mayne, 2005)]. However, this approach has the major drawbacks of conservative control performance and expensive computational cost in the on-line optimization (Badgwell, 1997). Another approach (Chisci et al., 2001) avoided the online computational issues, but remains conservative in performance due to the unrealistic assumption on future disturbances. Therefore, both these approaches for disturbance handling have the common drawback of overly conservative controls.

Another sensible approach for disturbances has been developed using the assumption of stochastic disturbances. In this approach it is assumed that it is unrealistic to control process outputs absolutely within constraint bands, and so deterministic constraints are replaced with probabilistic (chance) constraints. Moreover, one optimises the expected performance of the system instead of worst case performance; this avoids the conservative nature of the worst case approaches. The optimisation problem is formulated naturally as
Recently several robust MPC (RMPC) techniques have been proposed to deal with stochastic uncertainty [(Schwarm and Nikolaou, 1999), (Li et al., 2002), (Xie et al., 2005)]. However, most of them utilise an open–loop prediction model, which results in conservative controls because it does not accurately model the uncertainty in closed–loop system. Therefore, the state–of–the art technology proposes RMPC based on closed–loop prediction [(Warren, 2005), (Arellano-Garcia et al., 2004), (Batina, 2004)], resulting in improved dynamic performance at a small expense to constraint handling performance. Among them one approach (Warren, 2005) proposes a RMPC for disturbance uncertainty. It uses a step–weight model and formulates uncertainty propagation in the closed–loop, thus resulting in improved constraint handling performance and less conservative control. However, this approach does not give any guarantee of stability. Moreover, it adopts the open–loop paradigm (OLP) for constrained optimisation while it uses a closed–loop model for uncertainty prediction.

In this paper we extend the methodology of (Warren, 2005) to the state space case. Moreover, we adopt the closed loop paradigm (CLP) of dual mode MPC for the constrained optimisation; this gives several advantages: (i) the use of dual mode MPC allows us to guarantee stability for the infinite horizon; (ii) the use of CLP gives better numerical conditioning as well as making robustness analysis more straightforward compared to the use of OLP (Rossiter, 2003). A case study, looper–tension control, demonstrates the efficacy of a proposed RMPC strategy. Moreover, it evaluates robustness of the strategy on a realistic process as well as extending previous studies of MPC for rolling mills (Choi et al., 2004) to the robust case.

The paper is organised as follows. Section 2 details the problem formulation, a chance–constrained robust MPC based on closed–loop prediction is discussed in section 3, a case study of RMPC to looper–tension control is presented in section 4, and conclusions are given in section 5.

2. PROBLEM FORMULATION

In this section we formulate a chance–constrained MPC problem for a state space model under stochastic disturbances. Consider the following linear time invariant (LTI) discrete system:

\[
x_{k+1} = Ax_k + Bu_k + Eu_k, \quad k = 0, \ldots, \infty
\]
\[
y_k = Cx_k + Du_k, \quad k = 0, \ldots, \infty
\]

\[
st. \quad y_{\text{min}} \leq y_k \leq y_{\text{max}}, \quad k = 0, \ldots, \infty
\]
\[
u_{\text{min}} \leq u_k \leq u_{\text{max}}, \quad k = 0, \ldots, \infty
\]

where \(D = 0\), and \(w_k\) is normally distributed random disturbances with known probability density function. Performance will be assessed by the cost

\[
J = \sum_{k=0}^{\infty} (r_k - y_k)^T Q (r_k - y_k) + u_k^T R u_k
\]

Due to the assumption of random disturbances, deterministic constraints (2) will be replaced by the following chance–constraints

\[
Pr\{y_k \leq y_{\text{max}}\} \geq \alpha_y, \quad k = 0, \ldots, \infty,
\]
\[
Pr\{y_k \geq y_{\text{min}}\} \geq \alpha_y, \quad k = 0, \ldots, \infty,
\]
\[
Pr\{u_k \leq u_{\text{max}}\} \geq \alpha_u, \quad k = 0, \ldots, \infty,
\]
\[
Pr\{u_k \geq u_{\text{min}}\} \geq \alpha_u, \quad k = 0, \ldots, \infty
\]

where \(Pr\{\text{inequality}\}\), \(Pr\{\text{inequality}\}\) is the probability that output \(y\) and input \(u\) satisfy the respective constraints, and \(\alpha_y, \alpha_u\) is the specified probability, or confidence level. Moreover, in the cost function (3) we use expected values \(\bar{y}_k, \bar{u}_k\) in place of \(y_k, u_k\) respectively. Thus

\[
J = \sum_{k=0}^{\infty} (r_k - \bar{y}_k)^T Q (r_k - \bar{y}_k) + \bar{u}_k^T R \bar{u}_k
\]

In summary, the control problem is transformed to find an MPC control minimizing expected cost (5) subject to chance–constraints (4) for given system (1). This is the formulation of the chance–constrained optimisation used hereafter.

3. A CHANCE–CONSTRAINED RMPC BASED ON CLOSED–LOOP PREDICTION

In constraints (4), \(y_k\) and \(u_k\) can be predicted over the horizon by either a closed–loop or an open–loop prediction model. Here, to avoid conservative control, we use a closed–loop model in the MPC formulation.

3.1 Closed–loop prediction for a state space model

First, we formulate the closed–loop prediction of inputs and outputs over the horizon using future control actions. An unconstrained MPC control
law (Rossiter, 2003) for a nominal model can be given by:

\[ u_k = P_r r_k - K \dot{x}_k - K_2 \dot{w}_k \]  

(6)

where, \( P_r, K \) and \( K_2 \) represent feed–forward and feedback control gains, \( \dot{x}, \dot{w} \) are the state and disturbance estimates respectively.

**Remark 1** This control law contains an implicit integral action which results in zero steady–state disturbance violations of the actual outputs even disturbances. These prediction errors may imply prediction errors between predicted outputs and nominal model and constant disturbances.

Prediction equation (7) is based on a distribution evaluated at distribution function of the standard normal distribution.

\[ \begin{align*}
\hat{y}_k &= P_{x_1} \hat{x}_k + P_{r_1} \tau_k + P_{w_1} \hat{w}_k \\
\bar{y}_k &= P_{x_2} \hat{x}_k + P_{r_2} \tau_k + P_{w_2} \bar{w}_k
\end{align*} \]

(7)

where, \( P_{x_1} = \left( \begin{array}{ccc} C\phi & -K \phi & -K \phi^2 \\
C\phi^2 & -K & -K \phi \\
\vdots & \vdots & \vdots \end{array} \right), P_{x_2} = \left( \begin{array}{ccc} C\phi B P_r & 0 & 0 \\
C\phi^2 B P_r & C\phi B P_r & \vdots \\
\vdots & \vdots & \vdots \end{array} \right), P_{r_1} = \left( \begin{array}{ccc} P_r & 0 & 0 \\
-K B P_r & P_r & \vdots \\
-K \phi B P_r & -K B P_r & P_r \end{array} \right), P_{r_2} = \left( \begin{array}{ccc} P_r & 0 & 0 \\
-K B P_r & P_r & \vdots \\
-K \phi B P_r & -K B P_r & P_r \end{array} \right), P_{w_1} = \left( \begin{array}{ccc} C(E - B K_2) & C\phi( E - B K_2) + C(E - B K_2) \\
C\phi^2 (E - B K_2) + C\phi( E - B K_2) + C(E - B K_2) & \vdots \end{array} \right), \\
P_{w_2} = \left( \begin{array}{ccc} -K_2 & -K(E - B K_2) - K_2 \\
-K\phi(E - B K_2) - K_2 & \vdots \end{array} \right), \\
\end{align*} \]

where, \( \phi = (A - BK) \), \( \hat{y}_k = [\hat{y}_{k+1} \hat{y}_{k+2} \cdots]^T \), and \( \bar{y}_k = [\bar{y}_{k+1} \bar{y}_{k+2} \cdots]^T \).

**Remark 2** Prediction equation (7) is based on a nominal model and constant disturbances.

The prediction for the nominal model can induce prediction errors between predicted outputs and actual outputs due to the uncertainty of future disturbances. These prediction errors may imply constraint violations of the actual outputs even though the predicted outputs are feasible. From hereon we omit the formulation of the input constraints to save space. The prediction errors in the \( k \)-th horizon can be computed by

\[ y_{err,k} = y_{act,n+k-1} - \hat{y}_{n+k-1|n-1} \]  

(8)

where, \( n = 1, 2, \cdots \infty \). Therefore, process outputs under disturbance uncertainty can be estimated from eqn.(7) and eqn.(8),

\[ \hat{y}_k = \bar{y}_k + y_{err,k} \]  

(9)

From eqn.(9) the uncertainty over the horizon due to stochastic disturbances can be estimated by

\[ \sigma_{\hat{y}_k}^2 = \sigma_{y_{err,k}}^2 \]  

(10)

where, \( \sigma_{\hat{y}_k}^2, \sigma_{y_{err,k}}^2 \) represent variance of process outputs and prediction errors respectively. These variance can be computed by the Monte Carlo simulations through various disturbance realisations (Warren, 2005).

### 3.2 A Robust MPC formulation

Chance constraints (4) can be reformulated to linear constraints by using eqns.(9),(10). First, we transform these equations into the following standard normal forms.

\[ P_{\alpha_1} \frac{\hat{y} - \bar{y}}{\sigma_{\hat{y}}} \leq \frac{y_{max} - \bar{y}}{\sigma_{\hat{y}}} \geq \alpha_y, \]

\[ P_{\alpha_2} \frac{\hat{y} - \bar{y}}{\sigma_{\hat{y}}} \geq \frac{y_{min} - \bar{y}}{\sigma_{\hat{y}}} \geq \alpha_y \]

(11)

Then the above equations become:

\[ \frac{y_{max} - \bar{y}}{\sigma_{\hat{y}}} \geq K_{\alpha_y}, \quad \frac{y_{min} - \bar{y}}{\sigma_{\hat{y}}} \leq -K_{\alpha_y} \]

(12)

where, \( K_{\alpha_y} \) is the value of the inverse cumulative distribution function of the standard normal distribution evaluated at \( \alpha_y \). These equations can be rewritten as follows:

\[ \hat{y}_k \leq y_{max} - K_{\alpha_y} \sigma_{\hat{y}}, \quad -\hat{y}_k \leq -y_{min} - K_{\alpha_y} \sigma_{\hat{y}} \]

(13)

where, \( K_{\alpha_y}, \sigma_{\hat{y}} \) can be calculated prior to the optimisation. The resultant equations become linear constraints without a probability and give the original constraints appropriate restrictions by taking account of the variance of the future uncertainty. Conversely, if we simply replace \( y \) by \( \hat{y} \) in eqn. (4) then chance–constraints become:

\[ \hat{y}_k \leq y_{max}, \quad -\hat{y}_k \leq -y_{min} \]

(14)

**Definition 1** (Robust MPC) A constrained MPC control law to minimize expected cost function (5) subject to constraints (13) is RMPC.

**Definition 2** (Nominal MPC) A constrained MPC control law to minimize expected cost function (5) subject to constraints (14) is nominal
MPC (NMPC). This may allow constraint violations more often than RMPC.

Thus a RMPC formulation with chance constraints is:

\[ \min_u J = \sum_{k=0}^{\infty} [r_k - \bar{y}_k]^T Q [r_k - \bar{y}_k] + \bar{u}_k^T R \bar{u}_k \]  

(15)

s.t. \[ \bar{y}_k \leq y_{\max} - K_{\alpha_s} \sigma_{\bar{y},k} \]
\[ -\bar{y}_k \leq -y_{\min} - K_{\alpha_s} \sigma_{\bar{y},k} \]  

(16)

This is a deterministic optimisation problem and can be solved by a quadratic programming solver. However, it is an optimisation over the infinite horizon, and therefore may be intractable. Therefore, we adopt a dual model MPC strategy to solve this constrained optimisation with guaranteed stability, and implement the strategy by utilising the CLP.

3.3 LQ optimal RMPC

3.3.1. RMPC Cost Function with CLP  Let the constrained MPC control law \([R ossiter \ et \ al., \ 1998, \ S colkaert \ and \ R awlings, \ 1998]\) be

\[ u_k = P_c r_k - K \bar{x}_k - K_2 \bar{w}_k + c_k, \quad k = 0, \cdots, n_c - 1 \]  

(17)

\[ u_k = P_c r_k - K \bar{x}_k - K_2 \bar{w}_k, \quad k \geq n_c \]

where \( n_c \) is the control horizon, \( c_k \) are d.o.f. available for constraint handling. The future predictions using constrained control (17) are:

\[ \bar{y}_k = P_c \bar{x}_k + P_{c_1} c_k + P_{w_1} \bar{w}_k \]  

(18)

\[ P_{c_1} = \begin{pmatrix} \ \ CB & 0 & 0 \cdots \\
\ C\phi B & CB & 0 \cdots \\
\ C\phi^2 B & C\phi B & CB \cdots \end{pmatrix} \]

The outputs after \( n_c \) steps will be denoted as

\[ \bar{y}_{k+n_c|k} = P_{x_{n_c}} \bar{x}_k + P_{c_{n_c}} c_k + P_{r_{n_c}} \bar{r}_k + P_{w_{n_c}} \bar{w}_k \]  

(19)

By substituting eqn.(18) into eqn.(15), the cost function for dual mode prediction is given by

\[ J = c_{n_c}^T S_{c_{n_c}} c_{n_c} + 2 c_{x_{n_c}}^T S_{c_{x}} x + k \]  

(20)

where,

\[ S_c = P_{c_1}^T \text{diag}(Q) P_{c_1} + P_{c_{n_c}}^T \text{diag}(R) P_{c_{n_c}} \]
\[ + P_{w_{n_c}}^T P_{w_{n_c}} \]  

(21)

\[ S_{c_{x}} = P_{c_1}^T \text{diag}(Q) P_{x_1} + P_{c_{n_c}}^T \text{diag}(R) P_{x_{n_c}} \]
\[ + P_{w_{n_c}}^T P_{w_{n_c}} \]

and \( k \) does not depend upon \( c_{n_c} \) and hence can be ignored. \( P \) is the solution of the Lyapunov equation: \( P^T P\phi = P - \phi^T Q \phi - K^T R K \). One logical choice for linear control (17) in mode 2 is in fact that which minimizes the infinite horizon cost in the constraint free case. Such an algorithm is called linear quadratic optimal MPC (LQMPMC) (Rossiter, 2003). For MIMO system, the optimisation of (20) can be replaced with

\[ \min_{c_{k+i}, \ i=0,1,\cdots} J = \sum_{i=0}^{n_c-1} c_{k+i}^T \hat{Q} c_{k+i} \]

(22)

where, \( \hat{Q} \) is computed from

\[ \hat{Q} = B^T \Gamma B + R, \ \Sigma - \phi^T \Gamma \phi = Q + K^T R K \]

(23)

3.3.2. Constraint Handling for infinite horizon For constraint handling, substituting eqn.(18) into eqn.(16) gives

\[ P_{c_1} c_k + P_{d_1} - y_{\max} - K_{\alpha_s} \sigma_{\bar{y},k} \leq 0 \]
\[ -P_{c_1} c_k - P_{d_1} - y_{\min} - K_{\alpha_s} \sigma_{\bar{y},k} \leq 0 \]  

(24)

where, \( k = 0, \cdots, n_c - 1, P_{d_1} = P_{x_1} \bar{x}_k + P_{r_1} \bar{r}_k + P_{w_1} \bar{w}_k \). The constraint satisfaction in mode 2 can be guaranteed by

\[ x_{k+n_c|k} \in S_{\max} = \{x : C_{\max} x - d_{\max} \leq 0\} \]  

(25)

Substituting (19) into (25) gives

\[ C_{\max} [P_{x_{n_c}} \bar{x}_k + P_{c_{n_c}} c_k + P_{r_{n_c}} \bar{r}_k + P_{w_{n_c}} \bar{w}_k] -d_{\max} \leq 0 \]  

(26)

The problem results in the constrained convex optimisation and can be solved by quadratic programming. The first control perturbation \( c_k \) in \( c_{k+i}, \ i = 0, \cdots, n_c - 1 \) is applied to the constrained control law (17).

The proposed RMPC has several advantages compared to other chance–constrained RMPC: (1) It formulates a constrained optimisation problem for a state space model, and therefore can deal with MIMO cases straightforwardly; (2) It utilises a closed–loop model to predict future uncertainty, which results in a less conservative control compared to the use of an open–loop prediction model; (3) The use of dual mode MPC in the constrained optimisation allows us to guarantee stability for the infinite horizon and moreover; (4) The use of CLP gives better numerical conditioning of the optimisation as well as better robustness compared to the use of OLP.
4. A CASE STUDY OF THE RMPC TO LOOPER–TENSION CONTROL

This section demonstrates the efficacy of the RMPC proposed in section 3 on looper–tension control of hot rolling mills. An earlier work (Choi et al., 2004) of MPC for this control problem shows infeasibility due to unknown disturbances with the use of NMPC. Thus, through the case study, we investigate the effectiveness of RMPC by evaluating robust performance in comparison to NMPC.

4.1 Process Model and Constraints

A looper–tension system can be modelled by the following state space model (details in (Choi et al., 2004)):

\[ \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*} \] (27)

where, \( y_{1,max} = 2\, [N/mm^2] \), \( y_{1,min} = -2\, [N/mm^2] \), \( y_{2,max} = 0.02\, [rad] \), \( y_{2,min} = -0.02\, [rad] \), and constraints should be changed by the strip parameters such as thickness, width, rolling force, temperature etc. Also, \( D = 0 \), \( A = \)

\[ \begin{pmatrix}
K_{11}\phi_l & K_{1}\sigma & K_{1}\theta & 0 & \phi_l & 0 \\
-J_i & EK_{i\theta} & -J_i & 0 & J_i & 0 \\
-LG_l & 0 & E(1 + f) & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & K_{m3}\sigma & 0 & K_{m3}\phi_mK_{m1} & 0 & K_{m3}\phi_m \\
-J_m & 0 & -J_mR_m & 0 & 0 & 0 \\
-K_l2 & 0 & 0 & 0 & -K_l2 & 0
\end{pmatrix} \]

\[ B = \begin{pmatrix}
0 & K_{11}\phi_l & J_i \\
0 & 0 & 0 \\
0 & K_{m1}\phi_mK_{m3} & 0 \\
0 & R_mJ_m & 0 \\
0 & K_{l2} & 0
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}, \quad E = \begin{pmatrix}
0 & -EK_{i\theta} \\
0 & LG_l \\
0 & 0
\end{pmatrix} \]

In this system, the major disturbances are mass flow imbalances from actions of thickness controllers including AGC (Automatic Gauge Control) and REC (Roll Eccentricity Control). Here we assume periodic control signals, caused by responses to skid marks and roll eccentricity, may not be large, and therefore these may not induce constraint violations. However, control actions arising from: (i) entry thickness deviations; (ii) temperature deviations and (iii) set-up mismatch, may be large to cause constraint violations as well as being stochastic. The violations may induce severe quality defects in a strip and, in the worst case, process instability. We represent these disturbances by \( w(t) \) in eqn.(27).

4.2 Simulation Results

For the controller design, cost weighting matrices \( Q = C^TC \), \( R = eyc(2) \) are chosen, and \( n_c = 3, n_y = 50 \) are used for the control and output horizons respectively. Figure 1 depicts simulation results for NMPC and RMPC with a 95% confidence level. Both simulations are performed under the same stochastic disturbances for convenience of comparison. It is clear from the figure, that utilisation of RMPC improves significantly the constraint handling performance of the tension output, when compared to NMPC.

Table 1 summarises key performance measures for various controllers. One criterion of robust performance is integral of absolute value of constraint violation (IAVCV), represents constraint handling
Table 1. Comparison of robust performance for various MPC.

<table>
<thead>
<tr>
<th>Controller</th>
<th>IAVCV</th>
<th>J</th>
<th>std(y1)</th>
<th>std(y2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMPC</td>
<td>15.3</td>
<td>15742.4</td>
<td>0.6205</td>
<td>0.0018</td>
</tr>
<tr>
<td>RMPC1 (90%)</td>
<td>2.7</td>
<td>6132.2</td>
<td>0.6292</td>
<td>0.0012</td>
</tr>
<tr>
<td>RMPC2 (95%)</td>
<td>0.6</td>
<td>7199.3</td>
<td>0.5930</td>
<td>0.0011</td>
</tr>
</tbody>
</table>

It is computed from \( \int y_{err, c} \ dt \) where \( y_{err, c} = |y - y_{constraint}| \) for \( |y| > |y_{constraint}| \), and \( y_{err, c} = 0 \) otherwise. Performance index \( J = \sum c^T Q c \) of eqn.(22) is a valid measure for robust performance in that for LQ MPC the size of the perturbations \( c \) is a direct measure of the distance from the unconstrained optimum. Standard deviations of \( y_1, y_2 \) are indices which represent how tension and looper angle deviate from their steady state values and thus also indicate dynamic performance.

The summaries in the table show: (i) the use of RMPC improves constraint handling performance as well as reducing output deviations under disturbances compared to NMPC; (ii) RMPC with a higher confidence level respects constraints more strictly than that of a lower confidence level and (iii) there is a limitation to increases in the confidence level in order to avoid problems such as infeasibility, large cost function and violent outputs.

From this case study it is obvious that RMPC can be a useful design strategy to improve robust performance for disturbance uncertainty. Moreover, the use of RMPC gives looper–tension control improved constraint handling performance, which contributes to improvement of product quality and stabilisation of the process in hot rolling mills.

5. CONCLUSIONS

This paper gives two main contributions. First we extend a previous methodology (Warren, 2005) of a chance–constrained RMPC through the use of state space models. Moreover, the use of dual mode MPC in the constrained optimisation allow us to guarantee stability for the infinite horizon. Secondly, the efficacy of proposed RMPC is demonstrated through the case study of looper–tension control; RMPC shows improved constraint handling performance compared to NMPC. Therefore, it can be a useful design strategy for looper–tension control as an extension of a previous MPC study (Choi et al., 2004) for hot rolling mills.

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