Output frequency response function of nonlinear Volterra systems

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Abstract

An expression for the output frequency response function (OFRF), which defines the explicit analytical relationship between the output spectrum and the system parameters, is derived for nonlinear systems which can be described by a polynomial form differential equation model. An effective algorithm is developed to determine the OFRF directly from system simulation or experimental data. Simulation studies demonstrate the significance of the OFRF concept, and verify the effectiveness of the algorithm which evaluates the OFRF numerically. These new results provide an important basis for the analytical study and design of a wide class of nonlinear systems in the frequency domain.

Keywords: Nonlinear systems; Volterra series theory; Differential equation model; Output frequency response

1. Introduction

The concept of the frequency response function (FRF) has been widely used in many fields of engineering to investigate and study system behaviours. If a system is linear the relationship between the system output frequency response and the input is well known; the output spectrum \( Y(j\omega) \) is equal to the input spectrum \( U(j\omega) \) multiplied by the system FRF \( H(j\omega) \). The simple linear frequency domain relationship analytically describes the effect of system properties on the output frequency response. This analytical relationship has been applied in control engineering for systems analysis and controller design, in electronics and communications for the synthesis of analogue and digital filters, and in mechanical and civil engineering for the analysis of vibrations.

Nonlinear systems have been widely studied by many authors and significant progress towards understanding these systems has been made. Many of these studies have been based in the time domain with results relating to the Volterra series (Boyd & Chua, 1985; Rugh, 1981), NARMAX (Nonlinear AutoRegressive Moving Average with eXogenous input) models (Billings & Chen, 1989), neural networks and fuzzy systems, and classical nonlinear models such as the Duffing equation and the Van der Pol oscillator. The study of nonlinear systems in the frequency domain is based on the concept of generalised frequency response functions (GFRFs) (George, 1959) that extend the linear FRF concept to the nonlinear case for a wide class of nonlinear systems which possess the property of fading memory and can therefore be described by the Volterra series model (Boyd & Chua, 1985). Many studies in the frequency domain have focused on system modelling which involve the determination of the GFRFs from system input/output data (Adams, 2002; Billings & Peyton-Jones, 1990; Nam & Powers, 1994). The output frequency response of nonlinear systems was recently studied by Lang and Billings (1996, 1997). These studies extended the basic linear relationship between the input and output spectra and derived an explicit relationship between the input and output frequencies of nonlinear systems. Based on these results, Billings and Lang (2002) proposed the concept of energy transfer filters and developed a general procedure for the design of energy transfer filters which can be implemented using the NARX model with input nonlinearities.

Unlike linear systems, the relationship between the input and output spectra of nonlinear systems is much more complicated. The relationship involves complex multi-dimensional
Integration known as the association of variables and a summation with a possibly infinite number of terms (Rugh, 1981). This complicates the effect of the system parameters on the output frequency response. Consequently, the linear system frequency domain analysis and design approaches cannot easily be extended to the nonlinear case.

In this paper, the output frequency response of nonlinear systems is investigated in order to circumvent some of the difficulties of analysis and design of nonlinear systems in the frequency domain. The focus is to derive an explicit analytical relationship between the output spectrum and the coefficients of a polynomial form nonlinear differential equation model, and to develop an effective method for the determination of this relationship directly from system simulation or experimental test data. This analytical relationship will be referred to as the output frequency response function (OFRF). The OFRF and associated results reveal how the output frequency response of a wide class of nonlinear systems depend on the model coefficients which define the system nonlinearity, and provide a practically significant extension of the widely applied linear frequency domain relationship $Y(j\omega) = H(j\omega)U(j\omega)$ to the nonlinear case. Simulation studies are conducted. The results verify the theoretical derivations and demonstrate the effectiveness of the algorithm proposed for the determination of the OFRF from simulation or experimental data. This research should provide an important basis for the analytical study and design of nonlinear systems in the frequency domain.

2. Basic input/output relationships of nonlinear systems in the frequency domain

It is well known that the output frequency response of a stable time-invariant linear system can be expressed by

$$Y(j\omega) = H(j\omega)U(j\omega)$$

when the system is subject to an input where the Fourier Transform exists. In (1), $Y(j\omega)$ and $U(j\omega)$ are the system input and output spectra which are the Fourier Transforms of the system time domain input $u(t)$ and output $y(t)$, respectively.

Extending Eq. (1) to the nonlinear case is achieved by considering the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$y(t) = \sum_{n=1}^{N} \left[ \prod_{i=1}^{n} \int_{-\infty}^{\infty} h_{n}(\tau_{1}, \ldots, \tau_{n}) \prod_{i=1}^{n} u(t - \tau_{i}) \, d\tau_{i} \right],$$

where $h_{n}(\tau_{1}, \ldots, \tau_{n})$ is the $n$th-order Volterra kernel, and $N$ denotes the maximum order of the system nonlinearity.

In Lang and Billings (1996), an expression for the output frequency response of this class of nonlinear systems to a general input was derived. The result is given by

$$Y(j\omega) = \sum_{n=1}^{N} Y_{n}(j\omega) \quad \text{for } \forall \omega,$$

$$Y_{n}(j\omega) = \frac{1}{(2\pi)^{n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_{n}(j\omega_{1}, \ldots, j\omega_{n}) \times \prod_{i=1}^{n} U(j\omega_{i}) \, d\sigma_{n\omega}.$$  \hspace{1cm} (3)

This expression reveals how the nonlinear mechanisms operate on the input spectrum to produce the system output frequency response, where $Y_{n}(j\omega)$ represents the $n$th-order output frequency response of the system,

$$H_{n}(j\omega_{1}, \ldots, j\omega_{n}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{n}(\tau_{1}, \ldots, \tau_{n}) \times e^{-j(\omega_{1}\tau_{1} + \cdots + \omega_{n}\tau_{n})} \, d\tau_{1} \cdots d\tau_{n}$$

is the definition of the $n$th-order GFRF, and

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_{n}(j\omega_{1}, \ldots, j\omega_{n}) \prod_{i=1}^{n} U(j\omega_{i}) \, d\sigma_{n\omega}$$

denotes the integration of $H_{n}(j\omega_{1}, \ldots, j\omega_{n}) \prod_{i=1}^{n} U(j\omega_{i})$ over the $n$-dimensional hyperplane $\omega_{1} + \cdots + \omega_{n} = \omega$.

For linear systems, Eq. (1) shows that the possible output frequencies are the same as the frequencies in the input. For nonlinear systems described by Eq. (2), however, the relationship between the input and output frequencies at the steady state is generally given by

$$f_{Y} = \bigcup_{n=1}^{N} f_{Y_{n}},$$

where $f_{Y}$ denotes the non-negative frequency range of the system output, and $f_{Y_{n}}$ represents the non-negative frequency range produced by the $n$th-order system nonlinearity. This is much more complicated than in the linear system case.

In Lang and Billings (1997), an explicit expression for the output frequency range $f_{Y}$ was derived for the nonlinear systems subjected to a general input with a spectrum given by

$$U(j\omega) = \begin{cases} U(j\omega) & \text{when } \omega \in (a, b), \\ 0 & \text{otherwise}. \end{cases}$$  \hspace{1cm} (6)

where $b > a \geq 0$. The result obtained is

$$f_{Y} = f_{Y_{n}} \bigcup_{i=0}^{i^{*}} I_{k},$$

$$f_{Y_{n}} = \begin{cases} \bigcup_{k=0}^{i^{*}-1} I_{k} & \text{when } nb \frac{na}{(a+b)} \leq 1, \\ \bigcup_{k=0}^{i^{*}} I_{k} & \text{when } nb \frac{na}{(a+b)} \geq 1, \end{cases}$$

$$i^{*} = \frac{na}{(a+b)} + 1 \quad \text{[.] means to take the integer part},$$

$$I_{k} = (na - k(a+b), \quad \text{for } k = 0, \ldots, i^{*} - 1,$$

$$nb - k(a+b))$$

$$I_{i^{*}} = (0, nb - i^{*}(a+b)).$$
In (7) $p^*$ could be taken as $1, 2, \ldots, \lfloor N/2 \rfloor$, the specific value of which depends on the system nonlinearities. If the system GFRFs $H_{N-(2l-1)}(\cdot) = 0$, for $i = 1, \ldots, q - 1$, and $H_{N-(2q-1)}(\cdot) \neq 0$, then $p^* = q$. This is a significant analytical description for the output frequencies of nonlinear systems, which extends the well-known relationship between the input and output frequencies of linear systems to the nonlinear case.

Compared with the time domain methods, the frequency domain approaches often provide a much more physically meaningful insight into the behaviour of systems under investigation. Probably because of this, linear frequency domain methods have historically dominated the theory of control and signal processing for many years, and are still the most widely used approaches in engineering. Eq. (1) and the well-known concept that the output frequencies are the same as the input frequencies for linear systems are the very basic relationships behind these linear methods. An important reason for the wide applications of linear frequency domain approaches is the simplicity of these basic relationships. Differential or difference equation model parameters of linear systems can directly be mapped to the parameters of the system FRF $H(j\omega)$. Consequently, Eq. (1) explicitly shows how the parameters of a linear system time domain model affects the output response at any frequency.

Eqs. (2)–(7) are the direct extension of the basic linear frequency domain relationships to the nonlinear case. However, because of the complexity of the nonlinear frequency domain relationships, as far as we are aware of, little research has been reported which explicitly relates the output frequency response of a general class of nonlinear systems to the parameters of the system time domain model so as to facilitate the extension of the linear frequency domain analysis and design methods to the nonlinear case.

Motivated by the direct map of the linear system time domain model parameters to the parameters of $H(j\omega)$ and the consequent straightforward relationship between the system output spectrum $Y(j\omega)$ and these model parameters, in the following section, an explicit analytical relationship between the output spectrum and the coefficients of a polynomial form nonlinear differential equation model will be derived from Eqs. (2)–(7). The relationship is referred to as the OFRF. The OFRF provides a practically significant extension of the linear relationship $Y(j\omega) = H(j\omega)U(j\omega)$ to the nonlinear case and can be used to considerably facilitate nonlinear system analysis and design in the frequency domain.

3. OFRF of nonlinear Volterra systems

3.1. The differential equation model

Consider nonlinear systems which can be described by a differential equation of a polynomial form such that

$$
\sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{l_1, \ldots, l_{p+q} = 0}^{L} c_{pq}(l_1, \ldots, l_{p+q}) \prod_{i=1}^{p+q} D^{l_i} y(t) \prod_{i=p+1}^{L} D^{l_i} u(t) = 0,
$$

where the operator $D$ is defined by

$$
D^{l_i} y(t) = d^l y(t)/d t^l.
$$

In (8), the $M$ and $L$ are the maximum degree of nonlinearity in terms of $y(t)$ and $u(t)$, and the maximum order of derivative, respectively. Because (8) is used to present a valid input/output map, assume the coefficient $c_{1,0}(0) \neq 0$, and rearrange this equation to give

$$
-c_{1,0}(0)y(t) = \sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{l_1, \ldots, l_{p+q} = 0}^{L} c_{pq}(l_1, \ldots, l_{p+q}) 
\times \prod_{i=1}^{p+q} D^{l_i} y(t) \prod_{i=p+1}^{L} D^{l_i} u(t).
$$

The nonlinear differential equation model (10) has been widely used in physical system modelling. For example, the differential equation model of a widely used mechanical system (Caffery, Giacomini, & Worden, 1993) is given by

$$
\frac{m}{2} \frac{d^2 y(t)}{d t^2} + ky(t) + a_1 \frac{d y(t)}{d t} + a_2 \left( \frac{d y(t)}{d t} \right)^2 + a_3 \left( \frac{d y(t)}{d t} \right)^3 = u(t),
$$

which is a specific instance of Eq. (10) and may be obtained from (10) with

$$
c_{0,1}(0) = -1, \quad c_{1,0}(2) = m, \quad c_{1,0}(0) = k, \quad c_{1,0}(1) = a_1, \quad \text{else } c_{p,q}(\cdot) = 0.
$$

$$
c_{2,0}(1, 1) = a_2, \quad c_{3,0}(1, 1, 1) = a_3
$$

The main objective of the present study is to directly relate the output frequency response of nonlinear systems described by (10) to the coefficients $c_{p,q}(\cdot)$ to derive the OFRF of the systems. The first step will be to map the system time domain model to the frequency domain to investigate how the GFRFs of the systems can be described as explicit functions of these model parameters.

3.2. Generalised frequency response functions

Consider nonlinear systems which can be described by (10) and which satisfy the following assumptions:

(i) The systems are stable at zero equilibrium.

(ii) The systems can equivalently be described by the Volterra series model (2) with $N \geq M$ over a regime around the equilibrium.

It is known from Section 2 that the output frequency response of the nonlinear systems is given by Eq. (3), and the relationship between the system input and output frequencies are described by Eqs. (5)–(7).
It should be noted that Eqs. (3)–(7) are the steady state representation of system (10). Therefore the results which will be achieved based on these equations are valid for the system steady state behaviours. In addition, Assumptions (i) and (ii) imply that nonlinear systems which exhibit subharmonics, chaos, and the phenomena of bifurcations are not considered in the present study. However, because all nonlinear systems with the characteristics of fading memory (for dynamic systems fading memory is related to the concept of a unique steady state: intuitively a system has fading memory if two input signals which are close in the recent past, but not necessarily close in the remote past, yield present outputs which are close) over a region of operation can be described by the Volterra series model (Boyd & Chua, 1985), the results achieved under the assumptions will be significant for a wide class of nonlinear systems or useful over practical operating regions of these nonlinear systems.

The GFRFs in Eq. (4) for this class of nonlinear systems can be determined from the system differential equation model (10) by using the harmonic probing method (Bedrosian & Rice, 1971). The basic idea of this method is to apply an input \( u(t) \) which is a combination of exponentials such that

\[
u(t) = \sum_{i=1}^{R} e^{j\omega_i t}, \quad 1 \leq R \leq N
\]

to excite the system under study, and to substitute both the exponential input and the output response of the Volterra series model (2) to this input into a parametric model of the system. The GFRF \( H_R(\omega_1, \ldots, \omega_R) \) of the system can then be obtained by extracting the coefficient of \( e^{j(\omega_1 + \cdots + \omega_R)} \) from the resulting expression.

Following this procedure, an effective algorithm was derived by Billings and Peyton-Jones (1990), which can be used to recursively determine the GFRFs of the nonlinear differential equation model (10). This result is summarised in the following proposition.

**Proposition 1.** Under Assumptions (i) and (ii), the GFRFs of the nonlinear differential equation model (10) can be recursively determined from the model parameters as follows:

\[
\begin{align*}
H_n(\omega_1, \ldots, \omega_N) & = \sum_{l_1=0}^{L} c_{l_1}(l_1)(\omega_1 + \cdots + \omega_N)^{l_1} \times H_n(\omega_1, \ldots, \omega_N) \\
& = \sum_{l_1=0}^{L} c_{0\omega}(l_1, \ldots, l_n)(\omega_1)^{l_1} \cdots (\omega_N)^{l_n} \\
& \quad + \sum_{q=0}^{n-1} \sum_{l_1=0}^{L} \sum_{l_n=0}^{L} c_{pq}(l_1, \ldots, l_n)(\omega_1)^{l_1} \cdots (\omega_{n-q})^{l_{n-q}} \times H_{n-q,p}(\omega_1, \ldots, \omega_{n-q}) \\
& \quad + \sum_{p=0}^{n} \sum_{l_1=0}^{L} \sum_{l_p=0}^{L} c_{p0}(l_1, \ldots, l_p)H_{np}(\omega_1, \ldots, \omega_N),
\end{align*}
\]

where

\[
H_{np}(\cdot) = \sum_{i=1}^{n-p+1} H_i(\omega_1, \ldots, \omega_n)H_{n-i,p-1}(\omega_{i+1}, \ldots, \omega_n) \times (\omega_1 + \cdots + \omega_n)^{l_p}
\]

with

\[
H_{n1}(\omega_1, \ldots, \omega_n) = H_n(\omega_1, \ldots, \omega_n)(\omega_1 + \cdots + \omega_n)^{l_1}.
\]

**Proof of Proposition 1.** See Billings and Peyton-Jones (1990).

This recursive algorithm can considerably facilitate the numerical evaluation of the GFRFs of model (10) and can determine the GFRFs up to any order of interest. For the specific mechanical system described by (11), for example, the GFRFs up to 3rd order can be calculated recursively using this algorithm to produce the results below.

\[
\begin{align*}
H_1(\omega_1) & = \frac{1}{m(\omega_1)^2 + a_1(\omega_1) + k}, \\
H_2(\omega_1, \omega_2) & = \frac{a_2 H_1(\omega_1)H_1(\omega_2)(\omega_1)(\omega_2)}{m(\omega_1 + \omega_2)^2 + a_1(\omega_1 + \omega_2) + k}, \\
H_3(\omega_1, \omega_2, \omega_3) & = \frac{a_2 \left[ H_1(\omega_1)H_2(\omega_2, \omega_3)(\omega_1)^2(\omega_2)^2 + H_1(\omega_2)H_2(\omega_1, \omega_2)(\omega_1)^2(\omega_2)^2 \right]}{m(\omega_1 + \omega_2 + \omega_3)^2 + a_1(\omega_1 + \omega_2 + \omega_3) + k}.
\end{align*}
\]

However, this recursive algorithm cannot be readily used to explicitly reveal how the coefficients of the nonlinear differential equation (10) affect the system GFRFs. In order to solve this problem, a general result on an explicit analytical relationship between the coefficients of model (10) and the system GFRFs is derived, the result is given in the following proposition.

**Proposition 2.** Denote \( C_n, n \geq 2 \), as a set of the parameters of model (10) as below:

\[
C_n = \left\{ \begin{array}{lcl}
c_{0\omega}(l_1, \ldots, l_n), & l_1 = 0, \ldots, L; & i = 1, \ldots, n, \\
c_{pq}(l_1, \ldots, l_{p+q}), & l_1 = 0, \ldots, L; & i = 1, \ldots, p + q, \\
c_{p0}(l_1, \ldots, l_p), & l_1 = 0, \ldots, L; & i = 1, \ldots, p, \\
& & p = 2, \ldots, n.
\end{array} \right.
\]

Then, given the parameters of \( H_1(\cdot) \) and frequency variables \( \omega_1, \ldots, \omega_n \), the \( n \)-th-order GFRF of nonlinear systems, which are described by differential equation model (10) and satisfy Assumptions (i) and (ii), can be expressed as a polynomial
function of the system parameters in a set given by

\[ C_n^N = C_2 \cup C_3 \cup \ldots \cup C_n. \]  

(19)

### Proof of Proposition 2

See Appendix.

**Remark 2.1.** Proposition 2 reveals that the GFRF \( H_n(\omega) \) for model (10) can be explicitly related to the model parameters in \( C_2^n \) by a polynomial function. Although this conclusion cannot directly be used to obtain the polynomial relationship, it is an important basis for the derivation of the OFRF of nonlinear systems.

**Remark 2.2.** The explicit relationship between \( H_n(\omega) \) and the model parameters in \( C_2^n \) can be determined via a symbolic computation. For example, to determine the relationship between \( H_3(\omega_1, \omega_2, \omega_3) \) and the model coefficients in \( C_2^3 \), a symbolic computation can be implemented by substituting the expression for \( H_2(\omega_1, \omega_2) \) in terms of \( H_1(\cdot) \) and the parameters in \( C_2^2 \) into the recursive computation expression for \( H_3(\omega_1, \omega_2, \omega_3) \) given by Proposition 1, and performing some necessary algebraic manipulations.

In order to demonstrate the analytical relationship revealed by Proposition 2, consider system (11) with \( m = 240 \text{ kg}, k = 16000 \text{ N/m}, \) and \( a_1 = 290 \text{ N/m} \). The explicit analytical relationship between the system GFRF \( H_3(\omega_1, \omega_2, \omega_3) \) and the parameters in \( C_2^3 \), which are \( a_2 \) and \( a_3 \) in this specific case, has been obtained as

\[
H_3(\omega_1, \omega_2, \omega_3) = a_2^2 \frac{\omega_3\omega_2(\omega_2 + \omega_3)\omega_1}{\beta(\omega_1)\beta(\omega_2)\beta(\omega_3)(\omega_2 + \omega_3)} \frac{(\omega_3\omega_2(\omega_1 + \omega_2)\omega_1)}{\beta(\omega_1 + \omega_2 + \omega_3)\beta(\omega_1 + \omega_2 + \omega_3)} + a_3 \frac{\omega_3\omega_2\omega_1}{\beta(\omega_1)\beta(\omega_2)\beta(\omega_3)(\omega_2 + \omega_3)}.
\]  

(20)

where \( \beta(\cdot) = 240(\cdot)^2 + 296(\cdot) + 16000 \). Obviously, given \( \omega_1, \omega_2, \omega_3 \), the GFRF \( H_3(\omega_1, \omega_2, \omega_3) \) has been expressed as an explicit polynomial function of parameters \( a_2 \) and \( a_3 \).

### 3.3. Output frequency response function

The general expression for the output frequency response of nonlinear systems has been given by Eq. (3). For nonlinear systems which can be described by model (10) and which satisfy Assumptions (i) and (ii), it can be shown that given a particular input there exists an explicit analytical relationship between the output frequency response and the model parameters. This conclusion is described by the following proposition.

**Proposition 3.** Under Assumptions (i) and (ii), given the parameters \( c_{10}(l_1) \) and \( c_{01}(l_1) \), \( l_1 = 0, \ldots, L \) of \( H_1(\cdot) \) and an input with spectrum \( U(\omega) \), the output spectrum \( Y(\omega) \) of nonlinear systems described by model (10) at any frequency \( \omega \) of interest can be expressed by a polynomial function of the model parameters in \( C_2^N \) as

\[
Y(\omega) = \sum_{(j_1, \ldots, j_N) \in J} \gamma_{j_1, \ldots, j_N}(\omega)x_1^{j_1} \ldots x_N^{j_N},
\]  

(21)

where \( x_1, \ldots, x_N \) are the elements in \( C_2^N \), \( J \) is a set of \( s_N \)-dimensional nonnegative integer vectors which contain the exponents of those monomials \( x_1^{j_1} \ldots x_N^{j_N} \) which are present in the polynomial representation (21). \( \gamma_{j_1, \ldots, j_N}(\omega) \) represents the coefficient of the term \( x_1^{j_1} \ldots x_N^{j_N} \), which is a function of frequency variable \( \omega \) and depends on the parameters in \( H_1(\cdot) \) of the system.

### Proof of Proposition 3

See Appendix.

**Remark 3.1.** If \( N > M \) or some coefficients in the general differential equation model (10) are known constants, many terms in (21) are in fact zeros or involve no unknown parameter variables. This will produce a much simpler polynomial representation for \( Y(\omega) \), and this is actually the situation in many practical cases.

**Remark 3.2.** Eq. (21) which is different from the general expression for \( Y(\omega) \) given by Eq. (3) describes the output frequency response of system (10) as a function of the system parameters \( x_1, \ldots, x_N \) and is therefore referred to as the OFRF of the system. This is an important extension of the linear frequency domain relationship (1) to the nonlinear case. In contrast to general perspectives on the effects of system parameters on nonlinear system output frequency responses, this result reveals that given a specific input and the system parameters which define the system linear characteristics, there exists a simple polynomial relationship between the output spectrum and the other system parameters for a wide class of nonlinear systems. The dependence of \( \gamma_{j_1, \ldots, j_N}(\omega) \) in (21) on the parameters in \( H_1(\cdot) \) implies a considerable importance of the system linear characteristics for system output behaviours. However, when these linear characteristic parameters are fixed, the system output frequency response can be determined using the OFRF, a polynomial function of the other system parameters.

**Remark 3.3.** The determination of Eq. (21) for a specific system under a given input will be discussed in the next section. But the structure of Eq. (21) can be worked out using a straightforward symbolic computation procedure. This procedure basically substitutes the system GFRFs determined as indicated in Remark 2.2 on Proposition 2 into Eq. (3), and then conducts the necessary manipulations to express the system output spectrum as an explicit function of all the system parameters apart from those in \( H_1(\cdot) \).
In order to demonstrate the results which can be obtained following Remark 3.3, consider the mechanical system (11) again. The symbolic computation procedure is applied to this system. After sophisticated symbolic manipulations, a simple expression for the system output spectrum is obtained as

\[ Y(\omega) = a_1^2 P_{42}(\omega) + a_2 a_3 P_{41}(\omega) + a_2^2 P_{31}(\omega) + a_2 P_{21}(\omega) + P_{11}(\omega). \]  

(22)

Eq. (22) is the specific form of the OFRF (21) for the case of system (11) with \( N = 4 \), that is, the system nonlinearity up to 4th order is taken into account. \( P_{ij}(\omega) \), \( i_1 = 1, 2, 3, 4 \); \( i_2 = 1, 2 \), which are the functions of \( \omega \) and the system parameters \( \overline{a} \), \( a_1 \), and \( k \), can exactly be determined by the symbolic computation procedure but the results will not be presented here due to space limitations. What is important is that Eq. (22) provides a specific structure for the OFRF, a polynomial representation of the output spectrum \( Y(\omega) \) of system (11) in terms of parameters \( a_2 \) and \( a_3 \). Similar results can be achieved using symbolic computation for any specific nonlinear system which can be described by differential equation model (10). The results can then be used to conduct the system analysis and design in the frequency domain.

4. Determination of the OFRF

Proposition 3 indicates that the output spectrum of system (10) can be described as a polynomial function of the system parameters which characterise the system nonlinearities, while Remark 3.3 of this proposition shows that a symbolic computation procedure can be used to determine the structure of this function. The significance of these results is that they reveal, for the first time, how the output frequency response of nonlinear systems can be analytically related to the system parameters, and show how to determine the basic structure of this relationship.

The polynomial form OFRF given by Proposition 3 involves a series of functions of frequency \( \omega \) such as, for example, \( P_{111}(\omega) \), \( i_1 = 1, 2, 3, 4 \); \( i_2 = 1, 2 \), for system (11). These functions are defined by the specific structure of the system model, the input spectrum of interest, and the system linear characteristic parameters such as, for example, \( \overline{a} \), \( a_1 \), and \( k \) in system (11). In order to use this OFRF for system analysis and design, these functions have to be determined. A straightforward method is to determine these functions directly using symbolic computations and then evaluate the values of these functions at all frequencies of interest. However, the complicated symbolic manipulations and numerical integrations involved in these operations imply that the method is not applicable in most practical situations. To solve this problem and ensure that the significant OFRF generally described in Proposition 3 can be used practically to conduct nonlinear system analysis and design, an algorithm is proposed to evaluate the values of these functions at frequencies of interest directly from system simulation or experimental test data. The algorithm is summarised in Proposition 4 in the following.

Proposition 4. Under the same conditions of Proposition 3 and assuming that Eq. (21) can be written as

\[ Y(\omega) = \sum_{(j_1, ..., j_N) \in J} \gamma_{j_1, ..., j_N}(\omega)x_{j_1}^1 \cdots x_{j_N}^N \]

where \( m_i \) are the maximum power of \( x_i \), \( i = 1, ..., S_N \), in the polynomial expression for the output spectrum \( Y(\omega) \) of system (10), the functions \( \gamma_{j_1, ..., j_N}(\omega) \), \( j_i = 0, ..., m_i \), \( i = 1, ..., S_N \), in (23) can be determined as

\[
\begin{bmatrix}
    \gamma(\omega) \\
    \vdots \\
    \gamma_{m_1 \cdots m_S}(\omega)
\end{bmatrix}
= \begin{bmatrix}
    X^{-1}M \cdot \\
    \vdots \\
    X^{-1}M
\end{bmatrix} \begin{bmatrix}
    Y(\omega) \\
    \vdots \\
    Y(\omega)
\end{bmatrix},
\]

(24)

where \( \overline{M} = (m_1 + 1)(m_2 + 1) \cdots (m_S + 1) \),

\[ X_{\overline{M}} = \begin{bmatrix}
    (x_{11}^0 \cdots x_{1N}^0) & \cdots & (x_{1m_1 \cdots m_N}^m) \\
    \vdots & \ddots & \vdots \\
    (x_{1m_1 \cdots m_N}^m) & \cdots & (x_{1m_1 \cdots m_N}^m)
\end{bmatrix},
\]

(25)

and \( Y_{\overline{M}}(\omega) \) is the output spectrum of system (10) when the parameters \( x_i \), \( i = 1, ..., S_N \) of the system are taken as \( x_{i1}^0, ..., x_{im_i}^m \) with \( m_1 = 1, ..., \overline{M} \), \( x_{im_i} \in \{ x_i(1), ..., x_i(m_i + 1) \} \), \( i = 1, ..., S_N \), \( m_1 = 1, ..., \overline{M} \), \( x_{i1} \neq x_{i1}, ..., x_{iS_N} \), \( i \neq v \), \( x_i(1), ..., x_i(m_i + 1) \) are \( m_i + 1 \) different values that can be taken by the system parameter \( x_i \), \( i = 1, ..., S_N \).

Proof of Proposition 4. This can be achieved following a procedure in Section 9.6 in Stetter (2004). The details are omitted due to space limitations.

Remark 4.1. Proposition 4 is described and proved assuming that all the \( (m_1 + 1)(m_2 + 1) \cdots (m_S + 1) \) terms are available in the polynomial form OFRF (23). In practice, however, given \( m_i \), \( i = 1, ..., S_N \), the number of terms in (23) is often less than the maximum number of \( (m_1 + 1)(m_2 + 1) \cdots (m_S + 1) \). For these cases, the functions of \( \omega \) on the right-hand side of (23) can be determined as described in a corollary to Proposition 4 as follows.

Corollary 4.1. Under the same conditions as Proposition 4, consider the case where some terms in (23) are missing...
so that

\[ Y(j) = \sum_{j_1 \in \{0,\ldots,m_1\}} \cdots \sum_{j_{SN} \in \{0,\ldots,m_{SN}\}} \gamma_{j_1\ldots j_{SN}}(o)x_{j_1}^{0}\ldots x_{j_{SN}}^{0}. \]

(26)

The functions \( \gamma_{j_1\ldots j_{SN}}(o) \) available in (26) can be determined as

\[
\begin{bmatrix}
\gamma_{j_1\ldots j_{SN}}(o) \\
\vdots \\
\gamma_{j_1\ldots j_{SN}}(o)
\end{bmatrix}
\begin{bmatrix}
\mathbf{Y}(j) \\
\vdots \\
\mathbf{Y}(j)
\end{bmatrix}
= (\mathbf{X}_M^T \mathbf{X}_M)^{-1} \mathbf{X}_M^T
\begin{bmatrix}
\mathbf{Y}(j) \\
\vdots \\
\mathbf{Y}(j)
\end{bmatrix},
\]

(27)

where \( \begin{bmatrix}
\gamma_{j_1\ldots j_{SN}}(o) \\
\vdots \\
\gamma_{j_1\ldots j_{SN}}(o)
\end{bmatrix} \) denotes a vector which is composed of all \( \gamma_{j_1\ldots j_{SN}}(o) \) available in (26), and

\[
\mathbf{X}_M =
\begin{bmatrix}
\text{SUBSET of } \{(x_{j_1}^0 \ldots x_{j_{SN}}^0)\ldots (x_{j_1}^{m_1} \ldots x_{j_{SN}}^{m_{SN}})\}
\end{bmatrix},
\]

(28)

SUBSET of \( \{(x_1^0 \ldots x_{SN}^0), \ldots, (x_{M}^0 \ldots x_{SN}^0)\} \) denotes a row vector, the components of which are those terms in \( \{(x_{j_1}^0 \ldots x_{j_{SN}}^0) : j_1 = 0, \ldots, m_1, i = 1, \ldots, SN \} \) which are available in (26). SUBSET of \( \{(x_{j_1}^0 \ldots x_{j_{SN}}^0)\ldots (x_{j_1}^{m_1} \ldots x_{j_{SN}}^{m_{SN}})\}, \bar{m} = 1, \ldots, \bar{M}, \) in (28) represents

SUBSET of \( \{(x_1^0 \ldots x_{SN}^0), \ldots, (x_{M}^0 \ldots x_{SN}^0)\} \)

evaluated at the \( n \)th set of the system parameters \( x_1, \ldots, x_{SN} \) as indicated in Proposition 4.

**Proof of Corollary 4.1.** The corollary can be proved following the same procedure for the proof of Proposition 4. □

**Remark 4.2.** Proposition 4 and Corollary 4.1 indicate that the OFRF (23) or (26) of system (10) can be determined directly from system simulation or practical test data. To achieve this objective, totally \( \bar{M} = (m_1 + 1)(m_2 + 1)\ldots(m_{SN} + 1) \) simulation studies or experimental tests are needed, and the system parameters \( x_1, \ldots, x_{SN} \) should take different sets of values of \( x_{1\bar{m}}, \ldots, x_{SN\bar{m}}, \bar{m} = 1, \ldots, \bar{M} \) during these simulations or experimental tests. Basically, the OFRF is determined from the output spectra of the system under study over the \( \bar{M} \) different sets of parameters \( x_1, \ldots, x_{SN} \). It is worth noting that \( \bar{M} \), which is the number of required simulation studies or experimental tests, should be determined by a predetermined structure of the system OFRF such as Eq. (22) for system (11).

### 5. Simulation studies and discussions

This section is concerned with a demonstration of the significance of the OFRF of nonlinear systems proposed in Proposition 3 and a verification of the effectiveness of the method described in Proposition 4 and Corollary 4.1 for the determination of the OFRF. To achieve this objective, simulation studies have been performed for system (11) to determine the relationship between the output spectrum and nonlinear characteristic parameters of the system, and to compare the output spectrum obtained via a FFT operation on the simulated system output data with the result evaluated using the determined OFRF. Two different cases of \( a_2 = 0, a_3 \neq 0 \) and \( a_2 \neq 0, a_3 = 0 \) are considered respectively, while the other system parameters were fixed as

\[ m = 240 \text{ kg}, \quad k = 16000 \text{ N/m}, \quad a_1 = 296 \text{ Ns/m}. \]

For the first case \((a_2 = 0, a_3 \neq 0)\), a specific realisation of a band limited random signal was used as the input \( u(t) \) of interest.

The OFRF of the system was derived by assuming \( N = 5 \), the result is a second-order polynomial function of parameter \( a_3 \) given by

\[ Y(j\omega) = a_3^2 P_{31}(j\omega) + a_3 P_{31}(j\omega) + P_{11}(j\omega), \]

(29)

where \( P_{31}(j\omega) \), \( P_{31}(j\omega) \), and \( P_{11}(j\omega) \) are complicated functions of the frequency \( \omega \) and are defined by the specific input, and the other system parameters \( m, k, \) and \( a_1 \). Eq. (29) is a specific case of the general expression (23) with \( S_N = S_5 = 1, m_{SN} = 2, x_1 = a_3, \) and \( \gamma_{j_1}(o) = P_{11}(j\omega), \gamma_{j_1}(o) = P_{31}(j\omega), \gamma_{j_2}(o) = P_{31}(j\omega), \) and other \( \gamma_{j_1\ldots j_{SN}}(o) = 0 \).

Proposition 4 was applied to compute \( P_{31}(j\omega), P_{31}(j\omega), \) and \( P_{11}(j\omega) \) in (29) to determine the OFRF of system (11) for this case. Three simulation studies were conducted where system (11) was excited by the same input \( u(t) \) but with parameter \( a_3 \) taking three different values of \( a_3(1) = 1000, a_3(2) = 3000, \) and \( a_3(3) = 5000 \), respectively. The output spectra of the system were evaluated from the three sets of simulated system outputs to yield the results \( Y^1(j\omega), Y^2(j\omega), \) and \( Y^3(j\omega) \). The estimates for \( P_{31}(j\omega), P_{31}(j\omega), \) and \( P_{11}(j\omega) \) were then obtained from Eq. (24) as

\[
\begin{bmatrix}
\hat{P}_{11}(j\omega) \\
\hat{P}_{31}(j\omega) \\
\hat{P}_{31}(j\omega)
\end{bmatrix} =
\begin{bmatrix}
1 & a_3(1) & a_3^2(1) \\
1 & a_3(2) & a_3^2(2) \\
1 & a_3(3) & a_3^2(3)
\end{bmatrix}^{-1}
\begin{bmatrix}
Y^1(j\omega) \\
Y^2(j\omega) \\
Y^3(j\omega)
\end{bmatrix}.
\]

(30)

Thus an OFRF of system (11) when subject to the specific input is obtained as

\[ \hat{Y}(j\omega) = a_3^2 \hat{P}_{31}(j\omega) + a_3 \hat{P}_{31}(j\omega) + \hat{P}_{11}(j\omega). \]

(31)

In order to show the significance of the obtained result and to verify the effectiveness of the method described in Proposition 4 from which the result is determined, (31) was used to evaluate the output spectrum of system (11) when \( a_3 \) takes values different from \( a_3(i), i = 1, 2, 3 \). Then \( \hat{Y}(j\omega) \) thus obtained was compared with \( Y(j\omega) \) determined from the simulated system.
The result is the polynomial function of parameters subjected to the input for the first case of simulation study. The output data under the same values of \( a_3 \) show the comparison of the amplitude and phase of \( \hat{Y}(j\omega) \) and \( Y(j\omega) \) when \( a_3 = 4000 \). Fig. 3 shows the comparison of the amplitude of \( \hat{Y}(j\omega) \) and \( Y(j\omega) \) when \( a_3 = 6000 \). An excellent agreement between \( \hat{Y}(j\omega) \) and \( Y(j\omega) \) can be observed from these results even in the case of \( a_3 = 6000 \) which is beyond the range of \( a_3 \)'s from which the OFRF (31) was determined.

For the second case \( a_2 \neq 0, a_3 \neq 0 \), the input signal of interest for the output frequency response analysis was

\[
u(t) = \frac{200}{\pi} \frac{[\sin(15t) - \sin(3t)]}{t}, \quad t \in [-40.955 \text{s}, 40.96 \text{s}].
\]

The OFRF of the system was derived by assuming \( N = 4 \) for this case. The result is the polynomial function of parameters \( a_2 \) and \( a_3 \) which has been given by Eq. (22). It is again a specific case of the general expression (23) but with \( S_N = S_4 = 2 \), \( m_1 = 3, m_2 = m_{S_N} = 1 \), \( x_1 = a_2, x_2 = a_3 \),

\[
\gamma_{30}(\omega) = P_{42}(j\omega), \quad \gamma_{11}(\omega) = P_{41}(j\omega),
\]

\[
\gamma_{20}(\omega) = P_{32}(j\omega), \quad \gamma_{01}(\omega) = P_{31}(j\omega),
\]

\[
\gamma_{10}(\omega) = P_{21}(j\omega), \quad \gamma_{00}(\omega) = P_{11}(j\omega),
\]

and other

\[
\gamma_{j_1...j_{S_N}}(\omega) = 0.
\]

Corollary 4.1 was applied to work out \( P_{42}(j\omega), P_{41}(j\omega), P_{32}(j\omega), P_{31}(j\omega), P_{21}(j\omega), P_{11}(j\omega) \) in (22) to determine the OFRF of system (11) for this case. \((m_1 + 1)(m_2 + 1) = 8\) simulation studies were conducted where system (11) was excited by the same input \( u(t) \) with parameters \( a_2, a_3 \) taking eight different sets of values

\[
a_2 = a_2(1) = 500, \quad a_3 = a_3(1) = 200;
\]

\[
a_2 = a_2(1) = 500, \quad a_3 = a_3(2) = 700;
\]

\[
a_2 = a_2(2) = 800, \quad a_3 = a_3(1) = 200;
\]

\[
a_2 = a_2(2) = 800, \quad a_3 = a_3(2) = 700;
\]

\[
a_2 = a_2(3) = 1000, \quad a_3 = a_3(1) = 200;
\]

\[
a_2 = a_2(3) = 1000, \quad a_3 = a_3(2) = 700;
\]

\[
a_2 = a_2(4) = 1500, \quad a_3 = a_3(1) = 200;
\]

\[
a_2 = a_2(4) = 1500, \quad a_3 = a_3(2) = 700.
\]

The output spectra of the system were evaluated from the eight simulated system outputs to yield the results of \( Y^1(j\omega),...,Y^8(j\omega) \). The estimates for \( P_{42}(j\omega), P_{41}(j\omega), \)
\[ P_{32}(j\omega), P_{31}(j\omega), P_{21}(j\omega), \text{ and } P_{11}(j\omega) \] were then obtained from Eq. (27) as
\[ \{\hat{P}_{11}(j\omega), \hat{P}_{21}(j\omega), \hat{P}_{31}(j\omega), \hat{P}_{41}(j\omega), \hat{P}_{32}(j\omega)\}^T \]
\[ = (A^T A)^{-1}A^T \{Y^1(j\omega), \ldots, Y^8(j\omega)\}^T, \tag{32} \]
where
\[ A = \begin{bmatrix} 1 & a_{21} & a_{31} & a_{21}^2 & a_{21}a_{31} & a_{21}^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{28} & a_{38} & a_{28}^2 & a_{28}a_{38} & a_{28}^3 \end{bmatrix}. \tag{33} \]
This produces the OFRF of system (11) when subject to the input as
\[ \hat{Y}(j\omega) = a_2^2 \hat{P}_{42}(j\omega) + a_2a_3 \hat{P}_{41}(j\omega) + a_2^2 \hat{P}_{32}(j\omega) + a_3 \hat{P}_{31}(j\omega) + a_2 \hat{P}_{21}(j\omega) + \hat{P}_{11}(j\omega). \tag{34} \]
To verify the effectiveness of (34), \( \hat{Y}(j\omega) \) was evaluated at \( a_2 = 700 \) and \( a_3 = 500 \), and \( a_2 = 2000 \) and \( a_3 = 800 \), respectively, and was compared with \( Y(j\omega) \) determined from the simulated output data of system (11) under the same input and for the same two sets of parameters \( a_2 \) and \( a_3 \). The results are shown in Figs. 4 and 5. Apart from the lower frequency range in Fig. 5, very good matches between the simulated output spectra and the output spectra evaluated using the determined output frequency response function have again been achieved.

The above simulation studies sufficiently demonstrate the significance of the proposed output frequency response function of nonlinear systems in the representation of the system output spectrum and the effectiveness of the method, which numerically determines this function as described in Proposition 4 or Corollary 4.1 directly from the system output data. The difference between the simulated output spectrum and the output spectrum evaluated using the output frequency response function (34) over the lower frequency band as seen in Fig. 5 is due to the truncation error with (34) which only takes the system nonlinearity up to 4th order into account.

Generally, when there is no a priori knowledge about the maximum order \( N \) of the system nonlinearity, under Assumptions (i) and (ii), the output spectrum of system (10) should be described as
\[ Y(j\omega) = \sum_{(j_1, \ldots, j_N) \in J} \gamma_{j_1, \ldots, j_N}(\omega)x_1^{j_1} \cdots x_N^{j_N} + \varepsilon_N(\omega), \tag{35} \]
where \( \varepsilon_N(\omega) \) is the truncation error due to an arbitrary choice of the maximum order \( N \) of system nonlinearities. \( \varepsilon_N(\omega) \) is not only determined by the same factors which define functions \( \gamma_{j_1, \ldots, j_N}(\omega) \), but is also a function of the system parameters \( x_1, \ldots, x_N \). When the values of \( x_1, \ldots, x_N \) are within a certain range around 0, ..., 0, the effect of \( \varepsilon_N(\omega) \) is negligible.

The determined output frequency response function
\[ \hat{Y}(j\omega) = \sum_{(j_1, \ldots, j_N) \in \hat{J}} \hat{\gamma}_{j_1, \ldots, j_N}(\omega)x_1^{j_1} \cdots x_N^{j_N} \]
can therefore be used to represent the system output spectrum for system analysis and design. Figs. 1–4 illustrate such an ideal situation where \( \hat{Y}(j\omega) \) describes \( Y(j\omega) \) very well. However, when the values of \( x_1, \ldots, x_N \) are beyond this range, the effect of \( \varepsilon_N(\omega) \) in (35) becomes significant. Consequently a difference between \( \hat{Y}(j\omega) \) and \( Y(j\omega) \) can be observed as illustrated by the results shown over the lower frequency range in Fig. 5. To solve this problem the assumed maximum order \( N \) of system nonlinearity should be increased to include higher order nonlinear terms in the output frequency response function description \( \hat{Y}(j\omega) \) for \( Y(j\omega) \) so as to reduce the effect of \( \varepsilon_N(\omega) \) in (35). This can be achieved by applying the symbolic computation procedure given in Remark 3.3 for a higher assumed maximum order \( N \) of system nonlinearity. Of course much more
involved symbolic evaluations have to be conducted to reach the final result. An alternative approach based on the orthogonal least square (OLS) method (Billings & Chen, 1989) is currently under study to determine an appropriate OFRF structure and, at the same time, estimate the functions $\gamma_{j_1...j_n}(\omega)$ in the general OFRF expression (21). The results obtained using these different approaches will be discussed in future publications.

The ultimate objective of the study of the OFRF of nonlinear Volterra systems is to use this novel concept to perform analysis and design of nonlinear systems in the frequency domain. This will be the focus of further research studies by the authors in this area. For analysis, the OFRF can be directly used to investigate how the system parameters, which define system nonlinearities, affect the system output frequency responses. For design, the basic idea is to use the analytical polynomial representation of the output spectrum defined by the OFRF to determine the values of the system parameters to achieve a desired output frequency response. It is, however, worth noting that in order to implement a design thus achieved, the uncertainty in the values of some design parameters may need to be considered in practice to avoid possible error amplification due to the complicated nonlinear relationship between the output spectrum and the design parameters. One solution could be to use the information from an OFRF based sensitivity analysis to find the design parameters which have sensitive impact on the output spectrum and ensure the accuracy of these parameters under control, or to only use parameters a slight variation of which does not cause a considerable difference in the system output spectrum for the design.

6. Conclusions

In this paper, an explicit analytical relationship between the output frequency response and the parameters of nonlinear systems which can be described by the differential equation model (10) has been derived. This analytical relationship is referred to as the output frequency response function (OFRF) of nonlinear systems. An algorithm has also been developed to determine the OFRF directly for the system simulation or experimental test data. Simulation studies have demonstrated the significance of the OFRF concept and verified the effectiveness of the algorithm which determines the OFRF numerically.

The intention of extending linear system frequency domain analysis and design methods to the nonlinear case has motivated studies of the describing function (DF) (Gelb & Vander Velde, 1968), generalised frequency response functions (GFRFs) (George, 1959), nonlinear output frequency response functions (NOFRFs) (Lang & Billings, 2005), and other concepts (Ruotolo, Surace, Crespo, & Storer, 1996). The distinctive feature of the OFRF proposed in this paper is that the OFRF explicitly reveals how the system output spectrum depends on the system parameters. This will be extremely helpful for the analysis of how parameter changes affect the system behaviours and for the design of the parameters to achieve desired system output frequency responses.

It should be pointed out that Assumptions (i) and (ii) in Section 3.2 are the conditions for the results in the present study to be valid. This implies that systems which exhibit subharmonics and chaos cannot currently be analysed using the OFRF based approach. However, because the basis of all the analysis in the present study is the Volterra series approach which occupies the middle ground in generality and applicability of the theories of nonlinear systems (Rugh, 1981), the results achieved in this paper have a considerable significance for the systematic application of nonlinear system analysis and design in engineering practice.

Although only the Volterra series model over zero equilibrium is considered in this paper, the OFRF based analysis, in principle, can also be applied to nonlinear systems operating over regions about nonzero equilibriums which possess fading memory properties. Further research is being conducted to derive different OFRF representations for the output spectra of nonlinear systems over different regions of operation. The objective is to produce a more comprehensive cover of system behaviours in order to achieve a more effective OFRF based nonlinear system analysis and design methodology.

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Appendix A.

Proof of Proposition 2. For $n = 2$, it is known from Proposition 1 that the proposition holds.

Assume that the proposition also holds for all values of $n$ up to $\pi - 1$ with $\pi \geq 3$, and consider the case of $n = \pi$ below.

Substituting $n = \pi$ into (12) for $n$ yields

$$H_\pi(j\omega_1, \ldots, j\omega_\pi) = \sum_{l_1=0}^{L} c_{01}(l_1) j\omega_1 + \cdots + j\omega_\pi H_\pi(j\omega_1, \ldots, j\omega_\pi)$$

where $H_\pi(j\omega_1, \ldots, j\omega_\pi)$ can be expressed as

$$H_\pi(j\omega_1, \ldots, j\omega_\pi) = \sum_{i=1}^{\pi} \prod_{l_1=0}^{L} H_{l_1}(j\omega_1, \ldots, j\omega_{l_1})$$

with $\omega_{l_1} = \omega_j$ for $l_1 = 1, \ldots, L$.
From (37), it is known that given the frequency variables \( \omega_1, \ldots, \omega_n \), \( H_{\bar{\pi} - i, p}(\cdot) \) and \( H_{p}(\cdot) \) in the second and third terms on the right-hand side of (36) are a linear combination of the terms of the form of \( \prod H_l(\cdot) \) with \( 1 \leq l \leq \bar{\pi} - 1 \). From this analysis and the assumption that Proposition 2 holds for \( n = 2, \ldots, \bar{\pi} - 1, \bar{\pi} \geq 3 \), it is concluded that given \( H_1(\cdot) \) and the frequency variables, \( H_{\bar{\pi}}(\cdot) \) can be described as a polynomial function of the parameters in

\[
\left\{ \begin{array}{l}
c_{ij}(l_1, \ldots, l_p), \\
i = 1, \ldots, \bar{\pi}
\end{array} \right\}
\]

\[
\bigcup \left\{ \begin{array}{l}
c_{ij}(l_1, \ldots, l_p), l_1 = 0, \ldots, L; i = 1, \ldots, p + q \\
q = 1, \ldots, \bar{\pi} - 1 \\
p = 1, \ldots, \bar{\pi} - q
\end{array} \right\}
\]

\[
\bigcup C^{\bar{\pi} - 1}_2
\]

\[
\bigcup \left\{ \begin{array}{l}
c_{ij}(l_1, \ldots, l_p), l_1 = 0, \ldots, L; i = 1, \ldots, p \\
p = 2, \ldots, \bar{\pi}
\end{array} \right\}
\]

\[
= C_\bar{\pi} \bigcup C^{\bar{\pi} - 1}_2 = C^{\bar{\pi} - 1}_2.
\]

(38)

that is, the proposition holds for \( n = \bar{\pi} \). □

**Proof of Proposition 3**. Proposition 2 indicates that

\[
H_n(j\omega_1, \ldots, j\omega_n) = \sum_{(j_1, \ldots, j_n) \in J_n} \gamma^\prime_{j_1, \ldots, j_n}(\omega_1, \ldots, \omega_n) x_{j_1}^1 \ldots x_{j_n}^n,
\]

\( n \geq 2 \)

where \( x_1, \ldots, x_n \) represent all the elements in \( C_2^n \), \( \gamma^\prime_{j_1, \ldots, j_n}(\omega_1, \ldots, \omega_n) \) represents a function of \( \omega_1, \ldots, \omega_n \) and the parameters in \( H_1(\cdot) \), and \( J_n \) is a set of \( s_n \)-dimensional nonnegative integer vectors which contains the exponents of those monomials \( x_{j_1}^1 \ldots x_{j_n}^n \) which are present in the polynomial representation (39).

Substituting (39) into the second equation in (3) yields

\[
Y_n(j\omega) = \sum_{(j_1, \ldots, j_n) \in J_n} \gamma^\prime_{j_1, \ldots, j_n}(\omega) x_{j_1}^1 \ldots x_{j_n}^n, \quad n \geq 2,
\]

\( (j_1, \ldots, j_n) \in J_n \)

\[
\gamma^\prime_{j_1, \ldots, j_n}(\omega) = \frac{1}{(2\pi)^{n-1}} \int_{\omega_1 + \cdots + \omega_n = \omega} \gamma^\prime_{j_1, \ldots, j_n}(\omega_1, \ldots, \omega_n)
\]

\[
\times \prod_{i=1}^{n} U(\omega_i) \, \text{d}\sigma_{\text{N}, \omega}.
\]

(41)

Therefore the output spectrum \( Y(j\omega) \) can be expressed as

\[
Y(j\omega) = H_1(j\omega)U(j\omega) + \sum_{n=2}^{N} \sum_{(j_1, \ldots, j_n) \in J_n} \gamma^\prime_{j_1, \ldots, j_n}(\omega) x_{j_1}^1 \ldots x_{j_n}^n
\]

\[
= H_1(j\omega)U(j\omega) + \sum_{(j_1, \ldots, j_N) \in \mathcal{J}} \gamma_{j_1, \ldots, j_N}(\omega) x_{j_1}^1 \ldots x_{j_N}^n,
\]

(42)

where \( \mathcal{J} \) is a set of \( s_N \)-dimensional nonnegative integer vectors which contains the exponents of those monomials \( x_{j_1}^1 \ldots x_{j_N}^n \) which are present in the polynomial representation (42).

Denote

\[
J = \mathcal{J} \bigcup \left\{ \begin{array}{l}
0, \ldots, 0 \\
s_N
\end{array} \right\}
\]

and

\[
\gamma_{0, \ldots, 0}(\omega) = H_1(j\omega)U(j\omega),
\]

then, from (42), \( Y(j\omega) \) can further be written as (21), that is, the output spectrum is a polynomial function of the parameters in \( C^{s_N}_2 \). □

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