On real analysis

Q1 Consider the following sequences of real numbers:

$$e^{-n} \qquad \sin\left(\frac{n\pi}{2}\right) \qquad \frac{(-1)^n}{n} \qquad \sum_{i=1}^{n} 2^{-i} \qquad \sum_{i=1}^{n} \frac{1}{i}.$$

Three of these sequences converge, as $n \to \infty$. Which three? What are the limits in these cases?

Q2 (a) Let (a_n) be a sequence of real numbers, and suppose that $a_n \to a$ and $a_n \to b$ for some $a, b \in \mathbb{R}$. Prove that $a = b$.

(b) Give an example of two sequences (a_n) and (b_n), which converge to the same limit, and for which $a_n < b_n$ for all n.

Q3 Consider the following functions, which are defined : $\mathbb{R} \to \mathbb{R}$.

$$f(x) = |x| \qquad g(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \qquad h(x) = xe^{-x^2}$$

Two of these function are continuous (on $\mathbb{R}$). Which two? Why not the other?

One of these functions is differentiable (on $\mathbb{R}$). Which one? Why not the others?

Q4 Let $p \in \mathbb{R}$. Show that

$$f(p) = \int_1^\infty x^p \, dx = \begin{cases} \frac{1}{p+1} & \text{if } p < -1, \\ +\infty & \text{if } p \geq -1. \end{cases}$$

Sketch the graph of $f(p)$.

On probability and stochastic processes

Q5 Let Y be an exponential random variable with parameter $\lambda > 0$. Calculate $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$. Hence, show that $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

Q6 Let (X_n) be a Markov chain with state space $\mathbb{N}$, and transition probabilities given by $p_{i,i+1} = \frac{1}{2}$ and $p_{i,1} = \frac{1}{2}$ (with all other transitions having zero probability).

(a) Draw a graph of the transitions that (X_n) may make, and annotate the edges of your graph with the transition probabilities.

(b) Is (X_n) transient or recurrent?

(c) Find the stationary distribution of (X_n). Do you recognize this distribution?

Q7 Let (X_n) be a sequence of random variables, with distribution

$$\mathbb{P}[X_n = n^2] = \frac{1}{n} \qquad \mathbb{P}[X_n = 0] = 1 - \frac{1}{n}.$$

Show that $\mathbb{E}[X_n] \to \infty$ as $n \to \infty$, but that $\mathbb{P}[|X_n| > \epsilon] \to 0$ for any $\epsilon > 0$.

Q8 I roll a (fair, six-sided) dice three times. What is the probability that I roll the same number more than once? What about if I roll the dice $n \in \mathbb{N}$ times?