Loops and covers in algebraic topology

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A topological space \((X, \mathcal{T})\) is a set of points \(X\) together with a set \(\mathcal{T}\) of neighbourhoods of each point in \(X\). The set \(\mathcal{T}\) must satisfy the following properties:

1. \(\emptyset\) and \(X\) are in \(\mathcal{T}\)

2. All intersections of a finite number of neighbourhoods are in \(\mathcal{T}\)

3. All unions of an arbitrary number of neighbourhoods are in \(\mathcal{T}\)
Covering spaces

Formally, a covering space of $X$ is a space $\tilde{X}$ together with a continuous map $p: \tilde{X} \to X$ such that for every $x \in X$ there exists an open neighborhood $U$ of $x$ such that $p^{-1}(U)$ is a union of disjoint open sets in $\tilde{X}$, each of which is homeomorphically mapped onto $U$ by $p$. 

[Diagram of a covering space]
A path $f$ is a function $f : [0, 1] \to X$ with $f(0)$ the left end-point and $f(1)$ the right end-point. A homotopy between paths $f$ and $g$ is a continuous family of continuous maps $F : [0, 1] \times [0, 1] \to X$ such that $F(0, t) = f(0) = g(0), F(1, t) = f(1) = g(1), F(x, 0) = f$ and $F(x, 1) = g$. 
The fundamental group

The fundamental group has equivalence classes of loops as elements and the concatenation of paths as its binary operation. Working at the level of homotopy equivalence, we have:

1. closure: for any two loops $f$ and $g$, $f \cdot g$ is also a loop,

2. associativity: for loops $f, g$ and $h$, we have $f \cdot (g \cdot h) = (f \cdot g) \cdot h$,

3. neutral element: the constant loop which stays at $x_0$,

4. inverses: any loop $f$ has an inverse $\bar{f}$ which is the same loop traversed in the opposite direction.
Van Kampen Theorem

\[ X = U_1 \cup U_2 \]

\[ \pi_1(X, x) = \pi_1(U_1, x) \ast \pi_1(U_2, x) \]
Let $X$ be a nice space. For a covering space $\tilde{X} \xrightarrow{p} X$ we can consider the set $\text{Pre}(x_0) = p^{-1}(x_0)$, the set of pre-images of the base-point $x_0$.

The group action of $\pi = \pi_1(X, x_0)$: for $[f] \in \pi$, define $\gamma_f : p^{-1}(x_0) \to p^{-1}(x_0)$ with $\tilde{x}_0 \mapsto \tilde{f}\tilde{x}_0(1)$ where $\tilde{f}$ is the lift of $f$ with $\tilde{f}(0) = \tilde{x}_0$.

$\phi : \pi \to \text{Sym}(p^{-1}(x_0))$ given by $[f] \to \gamma_f$

$\phi([f' \cdot f]) = \gamma_{f' \cdot f} = \phi([f'])\phi([f])$
Define $\mathcal{F} : \text{Cov}(X) \longrightarrow \pi - \text{set}$ given by $\mathcal{F}(p : \tilde{X} \to X) = p^{-1}(x_0)$ with left action as defined previously and $\mathcal{F}(f : \tilde{X}_1 \to \tilde{X}_2) = f|_{p_1^{-1}(x_0)}$

$f\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are both lifts of $\gamma$ so $f(\tilde{\gamma}_1(1)) = \tilde{\gamma}_2(1)$
Functor from $\pi$ – set to $\text{Cov}(X)$

Define $\mathcal{G} : \pi - \text{set} \rightarrow \text{Cov}(x)$ given by $\mathcal{G}(p^{-1}(x_0)) = p$ such that $p : (U \times p^{-1}(x_0))/\sim \rightarrow X$ with $([\gamma], \tilde{x}_0) \mapsto \gamma(1)$ where $([\gamma], \tilde{x}_0)$ is identified with $([\gamma][\tilde{\gamma}], [\tilde{\gamma}][\tilde{x}_0])$. Moreover $\mathcal{G}(f : p^{-1}(x_0) \rightarrow p^{-1}(x_0)) = h$ with $h : (U \times p^{-1}(x_0))/\sim \rightarrow (U \times p^{-1}(x_0))/\sim$ where $([\gamma], \tilde{x}_0) \mapsto ([\gamma], f(\tilde{x}_0))$
\[ F \circ G \text{ is isomorphic to } \text{id}_{\pi\text{-set}} \]

\[ \begin{array}{ccc}
\mathcal{F} \circ \mathcal{G}(p^{-1}(x_0)) & \xrightarrow{\alpha_{p^{-1}(x_0)}} & p^{-1}(x_0) \\
\mathcal{F} \circ \mathcal{G}(f) & \downarrow & \downarrow f \\
\mathcal{F} \circ \mathcal{G}(p^{-1'}(x_0)) & \xrightarrow{\alpha_{p^{-1'}(x_0)}} & p^{-1'}(x_0)
\end{array} \]

\[ \alpha_{p^{-1}(x_0)} : (\pi \times p^{-1}(x_0))/\sim \rightarrow p^{-1}(x_0) \text{ with } ([\gamma], \tilde{x}_0) \mapsto [\gamma]\tilde{x}_0 \]
$G \circ F$ is isomorphic to $\text{id}_{\text{Cov}(X)}$

\[ \begin{array}{c}
G \circ \mathcal{F} (\tilde{X}_1) \xrightarrow{\beta_{\tilde{x}_1}} \tilde{X}_1 \\
\downarrow \ G \circ \mathcal{F} (f) \quad \quad \quad \quad \quad \quad \downarrow f \\
G \circ \mathcal{F} (\tilde{X}_2) \xrightarrow{\beta_{\tilde{x}_2}} \tilde{X}_2
\end{array} \]

$\beta_{\tilde{X}} : (U \times p^{-1}(x_0))/\sim \longrightarrow \tilde{X}$ with $([\gamma], \tilde{x}_0) \mapsto \tilde{\gamma}(1)$
Conclusion